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A New Eighth Order Implicit Block Algorithms for the Direct Solution of Second Order Ordinary Differential Equations

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Abstract

This paper focuses on derivation of a uniform order 8 implicit block method for the direct solution of general second order differential equations through continuous coefficients of Linear Multistep Method (LMM). The continuous formulation and its first derivatives were evaluated at some selected grid and off grid points to obtain our proposed method. The superiority of the method over the existing methods is established numerically.

Keywords

Uniform Order, Second Order Initial Value Problem, Implicit Block Algorithms, Zero Stable

1. Introduction

In the past, efforts have been made by many researchers to develop an efficient algorithm for solving second order differential equations of the form

$$y'' = f(x, y, y'),\ \ y(0) = \alpha,\ \ y'(0) = \beta \tag{1.0}$$

directly through the interpolation and collocation points (see [1]-[4] to mention a few). Since many numerical techniques are available for the solution of higher order initial value problems (IVPs) and these techniques depend on many factors such as speed of convergence, computational expenses, data storage requirement and accuracy.

This paper aimed to address all these factors in the process of derivation and the implementation of this new

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method. Seven-point solutions are obtained from the block at once which speed up the computational processes; the method is self-starting and we obtained better accuracy over the existing methods.

The Equation (1.0) where *f* is a continuous function, is conventionally solved by first reducing it to a system of first order differential equations and then applying the various first order methods available for their solutions. This approach is extensively discussed and established by some of the following researchers such as ([5] [6]). Also [7] [8] and [9] showed that this approach was associated with certain drawbacks. Due to the dimension of the problem after it has been reduced to a system of first order ordinary differential equations (ODEs), also the reduced systems of ODEs are not well posed unlike the given problem. The approach wastes a lot of computer time and human efforts, hence there is a great need to develop new and efficient algorithms to handle problem (1.0) directly without any reduction to its equivalent system of first order ODEs.

Several authors have also solved problem (1.0) through predictor corrector mode (PC) of implementations; among them are [10] and [11]. Although the implementation of the methods in a PC mode yields good accuracy, the approach is more costly to implement, for instance PC routines are very complicated to write, since they require special techniques for supplying starting values and also predicting all the off grid points present in the method which leads to longer computer time and human efforts to handle their approach.

In our new algorithms, we take great advantage of this approach by exploring its continuous formulation nature to obtain some discrete schemes when evaluated at some $x_{n+j}, j=[0,k]$ to form our block method; schemes are equally obtained from the derivative of the continuous formula.

Definition 1.0

A linear multi-step method is said to be zero-stable if the roots $R_j, j=1(1)k$ of the first characteristic polynomials

$$\rho(R)=\det\left[\sum_{i=0}^{k} A_i R^{k-i}\right]=0,\ \ A_0=-1,\ \text{satisfies}\ |R_j|\le 1.$$

If one of the roots is +1, we call this the principal root of $\rho(R)$ (see [12]).

Definition 1.1

A linear multi-step method

$$y(x)=\sum_{j=0}^{k}\alpha_j(x)y_{n+j}=h^2\sum_{j=0}^{k}\beta_j(x)f_{n+j}. \tag{1.2}$$

We associate the linear differential operator

$$L[y(x);h]=\sum_{j=0}^{k}\left[\alpha_j;y(x+jh)-h^2\beta_j y''(x;jh)\right] \tag{1.3}$$

where $y(x)$ is an arbitrary function, continuity differentiable on $[a,b]$.

Expanding the test function $y(x+jh)$ as Taylor series about *x* and collecting terms gives

$$L[y(x);h]=C_0 y(x)+C_1 hy'(x)+\cdots+C_q h^q y^q(x)+\cdots$$

where $C_0,\cdots,C_q$ are constants.

A simple calculation yields the following formulae for the constants C_q in terms of the coefficients α_j,β_j.

$$C_0=\alpha_0+\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_k$$
$$C_1=\alpha_1+2\alpha_2+3\alpha_3+\cdots+k\alpha_k$$
$$C_2=\frac{1}{2!}\left(\alpha_1+2^2\alpha_2+3^2\alpha_3+\cdots+k^2\alpha_k\right)-\left(\beta_0+\beta_1+\beta_2+\cdots+\beta_k\right)$$
$$\vdots$$
$$C_q=\frac{1}{q!}\left(\alpha_1+2^q\alpha_2+3^q\alpha_3+\cdots+k^q\alpha_k\right)-\frac{1}{(q-2)!}\left(\beta_1+2^{q-2}\beta_2+\cdots+k^{q-2}\beta_k\right)\quad q=2,3,\cdots.$$

Hence, we say that the method has order *P* if $C_0=C_1=C_2=C_p=C_{p+1}=0$, but $C_{p+2}\ne 0$. Then C_{p+2} is the error constant and $C_{p+2}h^{p+2}y^{p+2}(x_n)$ is the principal local truncated error at the point x_n.

2. Theory of Block Methods for Second Order Initial Value Problems

Within the r-vectors y_m and f_m (for $m = nr, n = 0,1,\cdots$) $y_m = \left(y_{n+1}, y_{n+2}, y_{n+3}, \cdots, y_{n+r}\right)^{\mathrm{T}}$

$$f_m = \left(f_{n+1}, f_{n+2}, f_{n+3}, \cdots, f_{n+r}\right)^{\mathrm{T}}.$$

The S block r-point methods for

$$\sum_{j=0}^{k} \alpha_j(x) y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j(x) f_{n+j}$$

are given by the matrix finite difference equation.

$$A^0 y_m = \sum_{j=0}^{k} A^{(i)} y_{m-i} + h^2 \sum_{j=0}^{k} B^{(i)} f_{m-i} \qquad (2.0)$$

where $A^{(i)}, B^{(i)}, i = 0,(0)$ are $r \times r$ matrices respectively with element $a_{ij}^{(i)}, b_{ij}^{(i)}$, for $ij = 0(1)r$.

The block scheme (2.0) is explicit if the coefficient matrix B^0 is a null matrix.

Let

$$Z_n = \begin{pmatrix} y(x_{n+1}) \\ y(x_{n+2}) \\ \vdots \\ y(x_{n+r}) \end{pmatrix},$$

be respectively the theoretical solution to Equation (1.0) (see [12] [13]).

3. Specification of the Method

We consider a power series of single variables x in the form

$$P(x) = \sum_{j=0}^{\infty} \alpha_j x^j \qquad (3.0)$$

which is used as the basis or trial function to produce our approximate solution to (1.0) as

$$P(x) = \sum_{j=0}^{m+t-1} \alpha_j x^j \qquad (3.1)$$

$$P'(x) = \sum_{j=1}^{m+t-1} j \alpha_j x^{j-1} \qquad (3.2)$$

$$P''(x) = \sum_{j=2}^{m+t-1} j(j-1) \alpha_j x^{j-2} = f(x, y, y') \qquad (3.3)$$

where a_j are the parameters to be determined, t and m are point of interpolation and collocation points. The Equation (3.3) is collocated at $x = x_{n+j}$, $j = (0,k)$ and interpolating (3.1) at $x = x_{n+j}\, j = 0, \frac{1}{2}$, with this method $k = 3$ and specifically gives the following system of non linear equations of the form h

$$\sum_{j=0}^{m+t-1} \alpha_j x^j = y_{n+i}, \quad i = \left[0, \frac{1}{2}\right] \qquad (3.4)$$

$$\sum_{j=2}^{m+t-1} j(j-1) \alpha_j x^{j-2} = f_{n+i}, \quad i = \left[0, \frac{1}{2}, \frac{3}{4}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\right]. \qquad (3.5)$$

The continuous formulation of the method will be of the form

$$y(x)=\alpha_n y_n+\alpha_{n+\frac{1}{2}}y_{n+\frac{1}{2}} = h^2\left[\beta_n f_n+\beta_{n+\frac{1}{2}}f_{n+\frac{1}{2}}+\beta_{n+\frac{3}{4}}f_{n+\frac{3}{4}}+\beta_{n+1}f_{n+1}+\beta_{n+\frac{3}{2}}f_{n+\frac{3}{2}}+\beta_{n+2}f_{n+2}+\beta_{n+\frac{5}{2}}f_{n+\frac{5}{2}}+\beta_{n+3}f_{n+3}\right]. \tag{3.6}$$

When using Maple 17 mathematical software to obtain the values of $\alpha_j s$ in (3.4), (3.5) and substituting the values in Equation (3.0) to obtain our continuous formulation in the new method as

$$\begin{aligned}
y(x)=&\frac{\left(h-2(x-x_n)\right)}{h}y_n+\frac{2(x-x_n)}{h}y_{n+\frac{1}{2}}+\left[-\frac{1115837h}{10886400}(x-x_n)+\frac{1}{2}(x-x_n)^2-\frac{1683}{1620h}(x-x_n)^3\right.\\
&+\frac{420}{324h^2}(x-x_n)^4-\frac{5453}{5400h^3}(x-x_n)^5+\frac{339}{810h^4}(x-x_n)^6\\
&\left.-\frac{413}{2835h^5}(x-x_n)^7+\frac{45}{1890h^6}(x-x_n)^8-\frac{2}{1215h^6}(x-x_n)^9\right]f_n\\
&+\left[-\frac{313243h}{604800}(x-x_n)+\frac{270}{45h}(x-x_n)^3-\frac{1143}{90h^2}(x-x_n)^4+\frac{957}{75h^3}(x-x_n)^5\right.\\
&\left.-\frac{325}{45h^4}(x-x_n)^6+\frac{740}{315h^5}(x-x_n)^7-\frac{43}{105h^6}(x-x_n)^8+\frac{4}{135h^6}(x-x_n)^9\right]f_{n+\frac{1}{2}}\\
&+\left[\frac{96496h}{127575}(x-x_n)-\frac{92160}{8505h}(x-x_n)^3+\frac{32256}{1215h^2}(x-x_n)^4-\frac{59392}{2025h^3}(x-x_n)^5\right.\\
&\left.+\frac{21504}{1215h^4}(x-x_n)^6-\frac{51200}{8505h^5}(x-x_n)^7+\frac{3072}{2835h^6}(x-x_n)^8-\frac{2048}{25515h^6}(x-x_n)^9\right]f_{n+\frac{3}{4}}\\
&+\left[-\frac{117415h}{241920}(x-x_n)+\frac{270}{36h}(x-x_n)^3-\frac{1413}{72h^2}(x-x_n)^4+\frac{2787}{120h^3}(x-x_n)^5\right.\\
&\left.-\frac{1333}{90h^4}(x-x_n)^6+\frac{331}{63h^5}(x-x_n)^7-\frac{41}{42h^6}(x-x_n)^8+\frac{2}{27h^6}(x-x_n)^9\right]f_{n+1}\\
&+\left[\frac{73279h}{544320}(x-x_n)-\frac{180}{81h}(x-x_n)^3+\frac{501}{81h^2}(x-x_n)^4-\frac{1066}{135h^3}(x-x_n)^5\right.\\
&\left.+\frac{2214}{405h^4}(x-x_n)^6-\frac{1184}{567h^5}(x-x_n)^7+\frac{78}{189h^6}(x-x_n)^8-\frac{8}{243h^6}(x-x_n)^9\right]f_{n+\frac{3}{2}}\\
&+\left[-\frac{53323h}{1209600}(x-x_n)+\frac{135}{180h}(x-x_n)^3-\frac{774}{360h^2}(x-x_n)^4+\frac{1713}{600h^3}(x-x_n)^5\right.\\
&\left.-\frac{187}{90h^4}(x-x_n)^6+\frac{265}{315h^5}(x-x_n)^7-\frac{37}{210h^6}(x-x_n)^8+\frac{2}{135h^6}(x-x_n)^9\right]f_{n+2}\\
&+\left[\frac{5989h}{6048000}(x-x_n)-\frac{54}{315h}(x-x_n)^3+\frac{45}{90h^2}(x-x_n)^4-\frac{51}{75h^3}(x-x_n)^5\right.\\
&\left.+\frac{23}{45h^4}(x-x_n)^6-\frac{68}{315h^5}(x-x_n)^7+\frac{5}{105h^6}(x-x_n)^8-\frac{4}{9455h^6}(x-x_n)^9\right]f_{n+\frac{5}{2}}\\
&+\left[-\frac{34543h}{32659200}(x-x_n)+\frac{90}{4860h}(x-x_n)^3-\frac{531}{9720h^2}(x-x_n)^4+\frac{1223}{16200h^3}(x-x_n)^5\right.\\
&\left.-\frac{141}{2430h^4}(x-x_n)^6+\frac{215}{8505h^5}(x-x_n)^7-\frac{33}{5670h^6}(x-x_n)^8+\frac{2}{3645h^6}(x-x_n)^9\right]f_{n+3}
\end{aligned} \tag{3.7}$$

Evaluating (3.7) at $x=x_{n+j}$ $j=\frac{3}{4},1,\frac{3}{2},2,\frac{5}{2},3$ and the first derivative of Equation (3.7) at $x=x_n$, to obtain the following discrete schemes to form our block method.

$$
\begin{aligned}
& y_{n+\frac{3}{4}} - \frac{3}{2} y_{n+\frac{1}{2}} + \frac{1}{2} y_n = \frac{329909}{4644860} h^2 f_n + \frac{1464259}{10321920} h^2 f_{n+\frac{1}{2}} - \frac{125471}{1088640} h^2 f_{n+\frac{3}{4}} + \frac{774913}{10321920} h^2 f_{n+1} \\
& \qquad - \frac{941669}{46448640} h^2 f_{n+\frac{3}{2}} + \frac{16889}{2580480} h^2 f_{n+2} - \frac{15049}{10321920} h^2 f_{n+\frac{5}{2}} + \frac{43163}{278691840} h^2 f_{n+3} \\
& y_{n+1} - 2y_{n+\frac{1}{2}} + y_n = \frac{2057}{145152} h^2 f_n + \frac{111659}{40320} h^2 f_{n+\frac{1}{2}} - \frac{304}{1701} h^2 f_{n+\frac{3}{4}} + \frac{4177}{26880} h^2 f_{n+1} \\
& \qquad - \frac{7339}{181440} h^2 f_{n+\frac{3}{2}} + \frac{1051}{80640} h^2 f_{n+2} - \frac{13}{4480} h^2 f_{n+\frac{5}{2}} + \frac{671}{2177280} h^2 f_{n+3} \\
& y_{n+\frac{3}{2}} - 3y_{n+\frac{1}{2}} + 2y_n = \frac{20341}{725760} h^2 f_n + \frac{23861}{40320} h^2 f_{n+\frac{1}{2}} - \frac{2792}{8505} h^2 f_{n+\frac{3}{4}} + \frac{39343}{80640} h^2 f_{n+1} \\
& \qquad - \frac{8437}{181440} h^2 f_{n+\frac{3}{2}} + \frac{16889}{2580480} h^2 f_{n+2} - \frac{29}{5760} h^2 f_{n+\frac{5}{2}} + \frac{1187}{2177280} h^2 f_{n+3} \\
& y_{n+2} - 4y_{n+\frac{1}{2}} + 3y_n = \frac{3049}{72576} h^2 f_n + \frac{3599}{4032} h^2 f_{n+\frac{1}{2}} - \frac{4064}{8505} h^2 f_{n+\frac{3}{4}} + \frac{34163}{40320} h^2 f_{n+1} \\
& \qquad + \frac{13333}{90720} h^2 f_{n+\frac{3}{2}} + \frac{2299}{40320} h^2 f_{n+2} - \frac{181}{20160} h^2 f_{n+\frac{5}{2}} + \frac{991}{1088640} h^2 f_{n+3} \\
& y_{n+\frac{5}{2}} - 5y_{n+\frac{1}{2}} + 4y_n = \frac{20149}{362880} h^2 f_n + \frac{8087}{6720} h^2 f_{n+\frac{1}{2}} - \frac{5584}{8505} h^2 f_{n+\frac{3}{4}} + \frac{9955}{8064} h^2 f_{n+1} \\
& \qquad + \frac{6367}{18144} h^2 f_{n+\frac{3}{2}} + \frac{4057}{13440} h^2 f_{n+2} + \frac{197}{20160} h^2 f_{n+\frac{5}{2}} + \frac{611}{1088640} h^2 f_{n+3} \\
& y_{n+3} - 6y_{n+\frac{1}{2}} + 5y_n = \frac{52103}{725760} h^2 f_n + \frac{58657}{40320} h^2 f_{n+\frac{1}{2}} - \frac{5584}{8505} h^2 f_{n+\frac{3}{4}} + \frac{23173}{16128} h^2 f_{n+1} \\
& \qquad + \frac{23987}{36288} h^2 f_{n+\frac{3}{2}} + \frac{40657}{80640} h^2 f_{n+2} + \frac{10529}{40320} h^2 f_{n+\frac{5}{2}} + \frac{36637}{2177280} h^2 f_{n+3} \\
& y'_n = -\frac{1}{32659200h} \Big[65318400 y_n - 65318400 y_{n+\frac{1}{2}} + 3347511 h^2 f_n + 15851025 h^2 f_{n+1} + 1439721 h^2 f_{n+2} \\
& \qquad + 34543 h^2 f_{n+3} + 16915122 h^2 f_{n+\frac{1}{2}} - 4396740 h^2 f_{n+\frac{3}{2}} - 24702976 h^2 f_{n+\frac{3}{4}} - 323406 h^2 f_{n+\frac{5}{2}} \Big]
\end{aligned} \tag{3.8}
$$

Equation (3.8) is our proposed uniform eighth order block method with the error constants exhibited in **Table 1**.

Table 1. Order and error constants of schemes (3.8).

Schemes	Order	Error constants
$y_{n+\frac{3}{4}}$	8	$-\frac{1367003}{5073430118400}$
y_{n+1}	8	$-\frac{569}{1061683200}$
$y_{n+\frac{3}{2}}$	8	$\frac{-703}{707788800}$
y_{n+2}	8	$-\frac{631}{412876800}$
$y_{n+\frac{5}{2}}$	8	$-\frac{2671}{1486356480}$
y_{n+3}	8	$-\frac{1859}{495452160}$
y'_n	8	$\frac{126297}{2048}$

Also the first derivative of (3.7) is evaluated at $x = x_{n+j}$ $j = \frac{1}{2}, \frac{3}{4}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$ as follows

$$\begin{aligned}
y'_{n+\frac{1}{2}} &= \frac{1}{32659200h}\Big[-65318400y_n + 65318400y_{n+\frac{1}{2}} + 939729h^2 f_n + 10585755h^2 f_{n+1} + 899559h^2 f_{n+2} \\
&\quad + 21197h^2 f_{n+3} + 16257618h^2 f_{n+\frac{1}{2}} - 2802180h^2 f_{n+\frac{3}{2}} - 17537024h^2 f_{n+\frac{3}{4}} - 199854h^2 f_{n+\frac{5}{2}}\Big] \\
y'_{n+\frac{3}{4}} &= \frac{1}{4180377600h}\Big[-8360755200y_n + 8360755200y_{n+\frac{1}{2}} + 117608847h^2 f_n + 1161584685h^2 f_{n+1} \\
&\quad + 104563737h^2 f_{n+2} + 2485771h^2 f_{n+3} + 2489303394h^2 f_{n+\frac{1}{2}} \\
&\quad - 321829620h^2 f_{n+\frac{3}{2}} - 1440166912h^2 f_{n+\frac{3}{4}} - 23361102h^2 f_{n+\frac{5}{2}}\Big] \\
y'_{n+1} &= \frac{1}{32659200h}\Big[-65318400y_n + 65318400y_{n+\frac{1}{2}} + 927849h^2 f_n + 13321935h^2 f_{n+1} + 880119h^2 f_{n+2} \\
&\quad + 20657h^2 f_{n+3} + 19061838h^2 f_{n+\frac{1}{2}} - 2768700h^2 f_{n+\frac{3}{2}} - 6754304h^2 f_{n+\frac{3}{4}} - 194994h^2 f_{n+\frac{5}{2}}\Big] \\
y'_{n+\frac{3}{2}} &= \frac{1}{32659200h}\Big[-65318400y_n + 65318400y_{n+\frac{1}{2}} + 880689h^2 f_n + 27057915h^2 f_{n+1} + 134919h^2 f_{n+2} \\
&\quad + 9197h^2 f_{n+3} + 20482578h^2 f_{n+\frac{1}{2}} + 4950780h^2 f_{n+\frac{3}{2}} - 12621824h^2 f_{n+\frac{3}{4}} - 70254h^2 f_{n+\frac{5}{2}}\Big] \\
y'_{n+2} &= \frac{1}{32659200h}\Big[-65318400y_n + 65318400y_{n+\frac{1}{2}} + 949449h^2 f_n + 19814895h^2 f_{n+1} + 7373079h^2 f_{n+2} \\
&\quad + 42257h^2 f_{n+3} + 18750798h^2 f_{n+\frac{1}{2}} + 17483460h^2 f_{n+\frac{3}{2}} - 6754304h^2 f_{n+\frac{3}{4}} - 506034h^2 f_{n+\frac{5}{2}}\Big] \\
y'_{n+\frac{5}{2}} &= \frac{1}{32659200h}\Big[-65318400y_n + 65318400y_{n+\frac{1}{2}} + 801489h^2 f_n + 31736475h^2 f_{n+1} + 22050279h^2 f_{n+2} \\
&\quad - 117043h^2 f_{n+3} + 22167378h^2 f_{n+\frac{1}{2}} + 8671740h^2 f_{n+\frac{3}{2}} - 17537024h^2 f_{n+\frac{3}{4}} + 5709906h^2 f_{n+\frac{5}{2}}\Big] \\
y'_{n+3} &= \frac{1}{32659200h}\Big[-65318400y_n + 65318400y_{n+\frac{1}{2}} + 1434729h^2 f_n - 12701745h^2 f_{n+1} + 1709559h^2 f_{n+2} \\
&\quad + 4747697h^2 f_{n+3} + 8279118h^2 f_{n+\frac{1}{2}} + 36122820h^2 f_{n+\frac{3}{2}} + 24702976h^2 f_{n+\frac{3}{4}} + 25517646h^2 f_{n+\frac{5}{2}}\Big]
\end{aligned} \tag{3.9}$$

Equation (3.9) has the following order and error constants in **Table 2**.

4. Implementation Strategies

Equation (3.9) is substituted in Equation (3.8) when applying to Equation (1.0) directly at $n = 0$, simultaneously produces solutions at the point $y_{\frac{1}{2}}$, $y_{\frac{3}{4}}$, y_1, $y_{\frac{3}{2}}$, y_2, $y_{\frac{5}{2}}$, y_3 at once without any recourse to special predictor

Table 2. Order and error constants of schemes (3.9).

Schemes	Order	Error constants
$y'_{n+\frac{1}{2}}$	8	$-\frac{150147}{4096}$
$y'_{n+\frac{3}{4}}$	8	$-\frac{35647065}{8192}$
y'_{n+1}	8	$-\frac{72999}{2048}$
$y'_{n+\frac{3}{2}}$	8	$-\frac{95139}{4096}$
y'_{n+2}	8	$-\frac{99495}{2048}$
$y'_{n+\frac{5}{2}}$	8	$\frac{89469}{4096}$
y'_{n+3}	8	$-\frac{713511}{2048}$

for y'_n present in the method. For the advancement in the integration processes we used schemes derived at $y_{n+\frac{3}{4}}$, $y_{n+\frac{5}{2}}$, y_{n+3} together as $n = 1, 2, \cdots$. This new method is demonstrated on linear and non linear problems to ascertain their degree of accuracy with the existing methods.

5. Numerical Experiments

Three numerical experiments of two linear and one non linear problem were used to ascertain the efficiency of the method.

Example 1

$$y'' + \frac{6}{x} y' + \frac{4}{x^2} y = 0$$

$$y(1) = 1, \;\; y'(1) = 1, \;\; h = \frac{0.1}{32} \quad x > 0 .$$

Theoretical solution is $y(x) = \frac{5}{3x} - \frac{2}{3x^4}$.

Example 2

$$y'' - 3y' = 8\mathrm{e}^{2x}$$

$$y(0) = 1, \;\; y'(0) = 1, \;\; h = 0.005 .$$

Theoretical solution is $y(x) = -4\mathrm{e}^{2x} + 3\mathrm{e}^{3x} + 2$.

Example 3

$$y'' - 3y' = 8\mathrm{e}^{2xy}$$

$$y(0) = 1, \;\; y'(0) = 1, \;\; h = 0.005 .$$

No theoretical solution.

6. Conclusion

We want to re-emphasize the claim made by [14] for first order schemes that when the derived schemes for various values of k are of the same order the block scheme gotten from the minimal value of k performed excellently well and compared favourably with the exact solutions. This has also been established for second order schemes derived from various values of k which are of the same order with three different numerical experiments tested (see **Figure 1**, **Figure 2** and **Figure 3**).

Table 3 and **Table 4** also display the numerical result of problem 1 and absolute errors by using various block methods of $k = 4$, $k = 5$ together with the new block method at $k = 3$. **Table 5** and **Table 6** display the numerical result of problem 2 and absolute errors by using various block methods of $k = 4$, $k = 5$ together with the new block method at $k = 3$.

Table 7 displays the approximate solution of example 3 with block methods of $k = 4$, $k = 5$ together with the

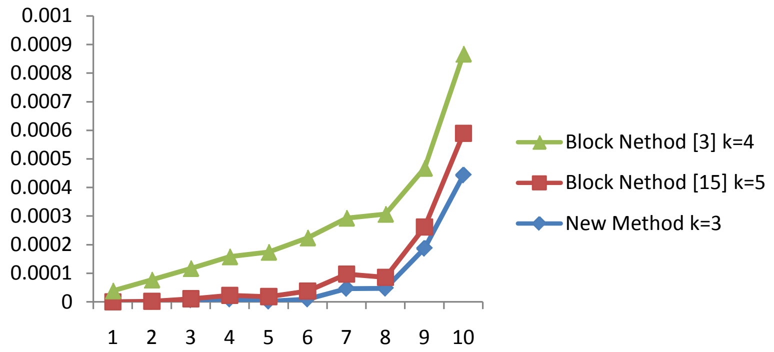

Figure 1. Error graph of problem 1.

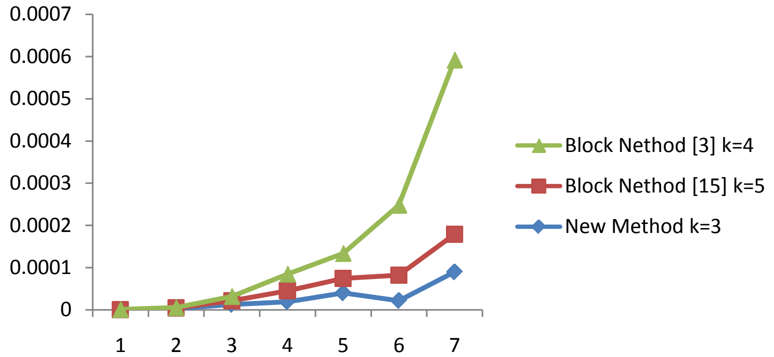

Figure 2. Error graph of problem 2.

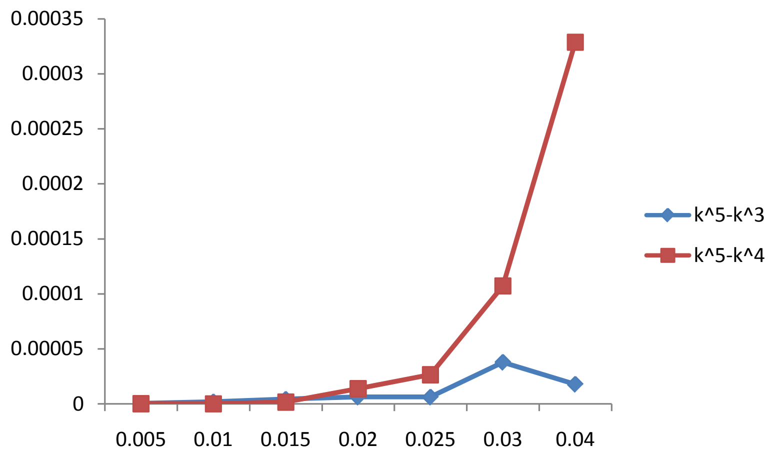

Figure 3. Error graph of problem 3.

Table 3. Table of result for example 1.

x	Theoretical solution	Block method [14] $k = 4$	Block method [15] $k = 5$	New block method $k = 3$
1.003125	1.003076526	1.003114880	1.0030766905	1.0030764430
1.00625	1.006057503	1.006132507	1.00605684265	1.006055854
1.009375	1.008944993	1.009050907	1.0089405789	1.008938355
1.0125	1.011741018	1.011876494	1.01172802434	1.011731527
1.015625	1.014447543	1.014603110	1.014431165439	1.014480080
1.01875	1.017066494	1.017252866	1.017038197167	1.017057078
1.021875	1.019599755	1.019795810	1.01954923805	1.019553250
1.025	1.022049164	1.022270209	1.02201055468	1.0220996286
1.028125	1.024416519	1.024622147	1.02434160973	1.0242295932
1.03125	1.026703578	1.026981486	1.026557694498	1.027146899

Table 4. Absolute error of problem 1.

Block method [3] $k = 4$	Block method [15] $k = 5$	New method $k = 3$
3.8354E(−05)	1.645E(−07)	8.3E(−08)
7.5004E(−05)	6.6035E(−07)	1.16E(−06)
1.05926E(−04)	4.4141E(−06)	6.638E(−06)
1.35476E(−04)	1.299366E(−05)	9.491E(−06)
1.55567E(−04)	1.6377561E(−05)	1.9535E(−06)
1.863726E(−04)	2.8296833E(−05)	9.416E(−06)
1.96055E(−04)	5.051695E(−05)	4.6505E(−05)
2.21045E(−04)	3.860932E(−05)	4.7122E(−05)
2.0562E(−04)	7.490927E(−05)	1.86926E(−04)
2.77908E(−04)	1.458835E(−04)	4.43321E(−04)

Table 5. Table of result for example 2.

x	Theoretical solution	Block method [3] $k = 4$	Block method [15] $k = 5$	New block method $k = 3$
0.005	1.005138526	1.005139114	1.0051388419	1.005138368
0.01	1.010558242	1.010557205	1.0105569711	1.010555066
0.015	1.016265444	1.016255068	1.0162567886	1.016252503
0.02	1.022266643	1.022226977	1.0222407282	1.022247320
0.025	1.028568067	1.028508035	1.02853411642	1.028527886
0.03	1.035176665	1.035010659	1.03511676083	1.035154590
0.04	1.049342284	1.048928801	1.04925342567	1.049432200

Table 6. Absolute error of problem 2.

Block method [3] $k = 4$	Block method [15] $k = 5$	New block method $k = 3$
5.8849E(−07)	3.159E(−07)	1.58E(−07)
1.03675E(−06)	1.2709E(−06)	3.176E(−06)
1.03759E(−05)	8.6554E(−06)	1.2941E(−05)
3.95659E(−05)	2.59148E(−05)	1.9323E(−05)
5.97171E(−05)	3.395058E(−05)	4.0181E(−05)
1.66006E(−04)	5.990417E(−05)	2.2075E(−05)
4.13483E(−04)	8.885833E(−05)	8.9916E(−05)

Table 7. Table of result for example 3.

x	Block method [3] $k = 4$	Block method [15] $k = 5$	New block method $k = 3$
0.005	1.005139120	1.0051388451	1.005138369
0.01	1.010557226	1.0105569851	1.010555080
0.015	1.016255105	1.01625686111	1.016252575
0.02	1.02222703	1.0222409615	1.022247553
0.025	1.028508106	1.0285346996	1.028528475
0.03	1.035010745	1.035118000	1.035155832
0.04	1.048928909	1.049257509	1.049436332

Table 8. Global error of problem 3.

$\left\|y^5 - y^4\right\|$	$\left\|y^5 - y^3\right\|$
2.749E(−07)	4.761E(−07)
2.409E(−07)	1.9051E(−06)
1.75611E(−05)	4.28611E(−06)
1.39315E(−05)	6.5915E(−06)
2.65936E(−05)	6.2246E(−06)
1.07255E(−04)	3.7832E(−05)
3.286E(−04)	1.78823E(−04)

Where $|y^5 - y^4|$ = the absolute difference between approximate solution of $k = 5$ and $k = 4$; $|y^5 - y^3|$ = the absolute difference between approximate solution of $k = 5$ and $k = 3$.

new block method at $k = 3$ while **Table 8** is the global or approximate error of problem 3, since this problem has no theoretical solution.

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