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Mean Square Convergent Finite Difference Scheme for Stochastic Parabolic PDEs

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Abstract

Stochastic partial differential equations (SPDEs) describe the dynamics of stochastic processes depending on space-time continuum. These equations have been widely used to model many applications in engineering and mathematical sciences. In this paper we use three finite difference schemes in order to approximate the solution of stochastic parabolic partial differential equations. The conditions of the mean square convergence of the numerical solution are studied. Some case studies are discussed.

Keywords

Stochastic Partial Differential Equations, Mean Square Sense, Second Order Random Variable, Finite Difference Scheme

1. Introduction

Stochastic partial differential equations (SPDEs) frequently arise from applications in areas such as physics, engineering and finance. However, in many cases it is difficult to derive an explicit form of their solution. In recent years, some of the main numerical methods for solving stochastic partial differential equations (SPDEs), like finite difference and finite element schemes, have been considered [1]-[9], e.g. [10]-[12], based on a finite difference scheme in both space and time. It is well known that explicit time discretization via standard methods (e.g., as the Euler-Maruyama method) leads to a time step restriction due to the stiffness originating from the discretization of the diffusion operator (e.g. the Courant-Friedrichs-Lewy (CFL) $\Delta t \le C(\Delta x)^2$, where Δt and Δx are the time and space discretization, respectively). Mohammed [8] discussed stochastic finite difference schemes by three points under the following condition of stability: $0 \le r\beta = \frac{\Delta t}{(\Delta x)^2}\beta \le \frac{1}{2}, \quad \beta > 0$ and five

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points under this condition of stability: $0 \le r\beta = \frac{\Delta t}{(\Delta x)^2}\beta \le \frac{2}{5}, \quad \beta > 0$. Our aim of this paper is to use the stochastic finite difference schemes by seven points that are strong convergences to our problem and much better stability properties than three and five points.

This paper is organized as follows. In Section 2, some important preliminaries are discussed. In Section 3, the Finite Difference Scheme with seven points for solving stochastic parabolic partial differential equation is discussed. In Section 4, some case studies are discussed. The general conclusions are presented in the last section.

2. Preliminaries

In this section we will state some definition from [9] as follow:

Definition 1. A sequence of r.v's $\{X_{nk}, n, k > o\}$ converges in mean square $(m.s)$ to a random variable X if:

$$\lim_{nk \to \infty} \|X_{nk} - X\| = 0 \ \ i.e.\ \ X_{nk} \xrightarrow{m.s} X$$

Definition 2. A stochastic difference scheme $L_k^n u_k^n = G_k^n$ approximating SPDE $Lv = G$ is consistent in mean square at time $t = (n+1)\Delta t$, if for any differentiable function $\Phi = \Phi(x,t)$, we have in mean square:

$$E\left|(L\Phi - G)_k^n - \left(L_k^n \Phi(k\Delta x, n\Delta t) - G_k^n\right)\right|^2 \to 0$$

$$\text{As } k \to \infty,\ n \to \infty \Delta x \to 0,\ \Delta t \to 0 \text{ and } (k\Delta x, n\Delta t) \to (x,t).$$

Definition 3. A stochastic difference scheme is stable in mean square if there exist some positive constants ε, δ and constants *k*, *b* such that:

$$E\left|u_k^{n+1}\right|^2 \le k e^{bt} E\left|u^0\right|^2$$

$$\text{For all } 0 \le t = (n+1)\Delta t,\ 0 \le \Delta x \le \varepsilon \text{ and } 0 \le \Delta t \le \delta.$$

Definition 4. A stochastic difference scheme $L_k^n u_k^n = G_k^n$ approximating SPDE $Lv = G$ is convergent in mean square at time $t = (n+1)\Delta t$, if:

$$E\left|u_k^n - u\right|^2 \to 0$$

$$\text{As } n \to \infty,\ k \to \infty,\ \Delta x \to 0,\ \Delta t \to 0 \text{ and } (k\Delta x, n\Delta t) \to (x,t).$$

3. Stochastic Parabolic Partial Differential Equation (SPPDE)

In this section the stochastic finite difference method is used for solving the SPPDE. Consider the following stochastic parabolic partial differential equation in the form:

$$u_t(x,t) = \beta u_{xx}(x,t) + \sigma u(x,t)\mathrm{d}W(t) \tag{1}$$

$$u(x,0) = u_0(x),\quad x \in [0, X],\quad t \in [0, T] \tag{2}$$

$$u(0,t) = u(X,t) = 0 \tag{3}$$

where $W(t)$ is a white noise stochastic process and β, σ are constants.

3.1. Stochastic Difference Scheme (with Seven Points)

For the Equations (1)-(3) the difference scheme is:

$$u_k^{n+1} = u_k^n + \frac{r\beta}{180}\left(2u_{k-3}^n - 27u_{k-2}^n + 270u_{k-1}^n - 490u_k^n + 270u_{k+1}^n - 27u_{k+2}^n + 2u_{k+3}^n\right) + \sigma\Delta t u_k^n\left(W_k^{n+1} - W_k^n\right), \tag{4}$$

$$u_k^0 = u_0(x_k), \tag{5}$$

$$u_0^n = u_x^n = 0, \tag{6}$$

where $r = \frac{\Delta t}{(\Delta x)^2}$ and it can be written in the form:

$$u_k^{n+1} = \left(1 - \frac{49}{18} r\beta\right) u_k^n + \frac{r\beta}{180}\left(2u_{k-3}^n - 27u_{k-2}^n + 270u_{k-1}^n + 270u_{k+1}^n - 27u_{k+2}^n + 2u_{k+3}^n\right) + \sigma\Delta t u_k^n \left(W_k^{n+1} - W_k^n\right). \quad (7)$$

3.1.1. Consistency

In this subsection we study the consistency in mean square sense of the Equation (7).

Theorem 3.1. The stochastic difference scheme (7) is consistent in mean square sense.

Proof. Assume that $\Phi(x,t)$ be a smooth function then:

$$L(\Phi)_k^n = \Phi(k\Delta x,(n+1)\Delta t) - \Phi(k\Delta x, n\Delta t) - \beta\int_{n\Delta t}^{(n+1)\Delta t} \Phi_{xx}(k\Delta x, s)\mathrm{d}s - \sigma\int_{n\Delta t}^{(n+1)\Delta t} \Phi(k\Delta x, s)\mathrm{d}W(s),$$

$$\begin{aligned} L_k^n\Phi = {} & \Phi(k\Delta x,(n+1)\Delta t) - \Phi(k\Delta x, n\Delta t) - \frac{r\beta}{180}\big(2\Phi((k-3)\Delta x, n\Delta t) - 27\Phi((k-2)\Delta x, n\Delta t) \\ & + 270\Phi((k+1)\Delta x, n\Delta t) - 490\Phi(k\Delta x, n\Delta t) + 270\Phi((k+1)\Delta x, n\Delta t) - 27\Phi((k+2)\Delta x, n\Delta t) \\ & + 2\Phi((k+3)\Delta x, n\Delta t)\big) - \sigma\Delta t\Phi(k\Delta x, s)\big(W((n+1)\Delta t) - W(n\Delta t)\big), \end{aligned}$$

then we have:

$$\begin{aligned} E\left|L(\Phi)_k^n - L_k^n\Phi\right|^2 = {} & E\Big| -\beta\int_{n\Delta t}^{(n+1)\Delta t} \Phi_{xx}(k\Delta x, s)\mathrm{d}s - \sigma\int_{n\Delta t}^{(n+1)\Delta t} \Phi(k\Delta x, s)\mathrm{d}W(s) \\ & + \frac{r\beta}{180}\big(2\Phi((k-3)\Delta x, n\Delta t) - 27\Phi((k-2)\Delta x, n\Delta t) + 270\Phi((k+3)\Delta x, n\Delta t) \\ & - 490\Phi(k\Delta x, n\Delta t) + 270\Phi((k+1)\Delta x, n\Delta t) - 27\Phi((k+2)\Delta x, n\Delta t) + 2\Phi((k+3)\Delta x, n\Delta t)\big) \\ & + \sigma\Delta t\Phi(k\Delta x, s)\big(W((n+1)\Delta t) - W(n\Delta t)\big)\Big|^2 \\ = {} & E\Big| -\beta\Big[\int_{n\Delta t}^{(n+1)\Delta t} \Phi_{xx}(k\Delta x, s)\mathrm{d}s - \frac{r}{180}\big(2\Phi((k-3)\Delta x, n\Delta t) - 27\Phi((k-2)\Delta x, n\Delta t) \\ & + 270\Phi((k+1)\Delta x, n\Delta t) - 490\Phi(k\Delta x, n\Delta t) + 270\Phi((k+1)\Delta x, n\Delta t) \\ & - 27\Phi((k+2)\Delta x, n\Delta t) + 2\Phi((k+3)\Delta x, n\Delta t)\big)\Big] \\ & - \sigma\Big[\int_{n\Delta t}^{(n+1)\Delta t} \Phi(k\Delta x, s)\mathrm{d}W(s) - \Delta t\Phi(k\Delta x, s)\big(W((n+1)\Delta t) - W(n\Delta t)\big)\Big]\Big|^2 \end{aligned}$$

since:

$$E|X+Y|^S \le k\left(E|X|^s + E|Y|^s\right) \text{ for } k = \begin{cases} 1 & s \le 1 \\ 2^{s-1} & s \ge 1 \end{cases},$$

then:

$$\begin{aligned} E\left|L(\Phi)_k^n - L_k^n\Phi\right|^2 \le {} & 2\beta^2 E\Big|\Big[\int_{n\Delta t}^{(n+1)\Delta t} \Phi_{xx}(k\Delta x, s)\mathrm{d}s - \frac{\Delta t}{180\Delta x^2}\big(2\Phi((k-3)\Delta x, n\Delta t) \\ & - 27\Phi((k-2)\Delta x, n\Delta t) + 270\Phi((k+1)\Delta x, n\Delta t) - 490\Phi(k\Delta x, n\Delta t) \\ & + 270\Phi((k+1)\Delta x, n\Delta t) - 27\Phi((k+2)\Delta x, n\Delta t) + 2\Phi((k+3)\Delta x, n\Delta t)\big)\Big]\Big|^2 \\ & + 2\sigma^2 E\left|\int_{n\Delta t}^{(n+1)\Delta t} \left[\Phi(k\Delta x, s) - \Phi(k\Delta x, s)\right]\mathrm{d}W(s)\right|^2. \end{aligned}$$

and from the inequality

$$E\left|\int_{t_0}^{t} f(s,w)\,\mathrm{d}W_s\right|^{2n} \le (t-t_0)^{n-1}\left[n(2n-1)\right]^n \int_{t_0}^{t} E\left[\left|f(s,w)\right|^{2n}\right]\mathrm{d}s.$$

Applying in the part

$$E\left|\int_{n\Delta t}^{(n+1)\Delta t}\left[\Phi(k\Delta x,s)-\Phi(k\Delta x,s)\right]\mathrm{d}W(s)\right|^2 \le (n+1-n)\Delta t\left[1(2-1)\right]\int_{n\Delta t}^{(n+1)\Delta t} E\left[\left|\Phi(k\Delta x,s)-\Phi(k\Delta x,s)\right|^2\right]\mathrm{d}s$$
$$= \Delta t\int_{n\Delta t}^{(n+1)\Delta t} E\left[\left|\Phi(k\Delta x,s)-\Phi(k\Delta x,s)\right|^2\right]\mathrm{d}s,$$

and $\Phi(x,t)$ is deterministic function we have:

$$E\left|L(\Phi)_k^n - L_k^n\Phi\right|^2 \le 2\beta^2 E\Bigg\|\Bigg[\int_{n\Delta t}^{(n+1)\Delta t}\Phi_{xx}(k\Delta x,s)\,\mathrm{d}s-\frac{\Delta t}{180\Delta x^2}\Big(2\Phi\big((k-3)\Delta x,n\Delta t\big)$$
$$-27\Phi\big((k-2)\Delta x,n\Delta t\big)+270\Phi\big((k+1)\Delta x,n\Delta t\big)-490\Phi\big(k\Delta x,n\Delta t\big)$$
$$+270\Phi\big((k+1)\Delta x,n\Delta t\big)-27\Phi\big((k+2)\Delta x,n\Delta t\big)+2\Phi\big((k+3)\Delta x,n\Delta t\big)\Big)\Bigg]\Bigg\|^2$$
$$+2\sigma^2\Delta t\int_{n\Delta t}^{(n+1)\Delta t} E\left[\left|\Phi(k\Delta x,s)-\Phi(k\Delta x,s)\right|^2\right]\mathrm{d}s$$

as time $t=(n+1)\Delta t$, $\Delta t\to 0$, $\Delta x\to 0$, and $(k\Delta x,n\Delta t)\to(x,t)$, then:

$$E\left|L(\Phi)_k^n - L_k^n\Phi(k\Delta x,n\Delta t)\right|^2 \to 0,$$

hence the stochastic difference scheme (7) is consistent in mean square sense.

3.1.2. Stability

Theorem 3.2. The stochastic difference scheme (7) is stable in mean square sense.

Proof. Since

$$u_k^{n+1}=\left(1-\frac{49}{18}r\beta\right)u_k^n+\frac{r\beta}{180}\left(2u_{k-3}^n-27u_{k-2}^n+270u_{k-1}^n+270u_{k+1}^n-27u_{k+2}^n+2u_{k+3}^n\right)+\sigma\Delta t u_k^n\left(W_k^{n+1}-W_k^n\right),$$

then:

$$E\left|u_k^{n+1}\right|^2=E\left|\left(1-\frac{49}{18}r\beta\right)u_k^n+\frac{r\beta}{180}\left(2u_{k-3}^n-27u_{k-2}^n+270u_{k-1}^n+270u_{k+1}^n-27u_{k+2}^n+2u_{k+3}^n\right)+\sigma\Delta t u_k^n\left(W_k^{n+1}-W_k^n\right)\right|^2.$$

Since $\{W(.,t)-W(.,s)\}$ is normally distributed with mean zero and variance $t-s$ increments of the winner process are independent u_k^n and

$$\left(W_k^{n+1}-W_k^n\right)=\left(W\left(k\Delta x,(n+1)\Delta t\right)-W\left(k\Delta x,n\Delta t\right)\right).$$

Then we have:

$$E\left|u_k^{n+1}\right|^2=\left(1-\frac{49}{18}r\beta\right)^2 E\left|u_k^n\right|^2+\left|\frac{r\beta}{90}\right|\left|1-\frac{49}{18}r\beta\right|E\left|u_k^n\left(2u_{k-3}^n-27u_{k-2}^n+270u_{k-1}^n+270u_{k+1}^n-27u_{k+2}^n+2u_{k+3}^n\right)\right|$$
$$+\left(\frac{r\beta}{180}\right)^2 E\left|2\left(u_{k-3}^n+u_{k+3}^n\right)-27\left(u_{k-2}^n+u_{k+2}^n\right)+270\left(u_{k-1}^n+u_{k+1}^n\right)\right|^2+\sigma^2(\Delta t)^2 E\left|u_k^n\right|^2$$

$$E\left|u_k^{n+1}\right|^2=\left(1-\frac{49}{18}r\beta\right)^2+\left|\frac{r\beta}{90}\left(1-\frac{49}{18}r\beta\right)\right|\left(2E\left|u_k^n\left(u_{k-3}^n+u_{k+3}^n\right)\right|-27E\left|u_k^n\left(u_{k-2}^n+u_{k+2}^n\right)\right|+270E\left|u_k^n\left(u_{k-1}^n+u_{k+1}^n\right)\right|\right)$$
$$+\left(\frac{r\beta}{180}\right)^2 E\left|2\left(u_{k-3}^n+u_{k+3}^n\right)-27\left(u_{k-2}^n+u_{k+2}^n\right)+270\left(u_{k-1}^n+u_{k+1}^n\right)\right|^2+\sigma^2(\Delta t)^2 E\left|u_k^n\right|^2,$$

since:

$$E\left|X+Y\right|^{S}\le k\left(E\left|X\right|^{s}+E\left|Y\right|^{s}\right),\quad k=\begin{cases}1 & s\le 1\\ 2^{s-1} & s\ge 1\end{cases}$$

then:

$$\begin{aligned}
&\frac{1}{(180)^2}E\left|2\left(u_{k-3}^n+u_{k+3}^n\right)-27\left(u_{k-2}^n+u_{k+2}^n\right)+270\left(u_{k-1}^n+u_{k+1}^n\right)\right|^2\\
&=\frac{1}{(180)^2}\Big[E\Big(4\left(u_{k-3}^n+u_{k+3}^n\right)^2+729\left(u_{k-2}^n+u_{k+2}^n\right)^2+72900\left(u_{k-1}^n+u_{k+1}^n\right)^2-54\left(u_{k-3}^n+u_{k+3}^n\right)\left(u_{k-2}^n+u_{k+2}^n\right)\\
&\quad+540\left(u_{k-3}^n+u_{k+3}^n\right)\left(u_{k-1}^n+u_{k+1}^n\right)-7290\left(u_{k-2}^n+u_{k+2}^n\right)\left(u_{k-1}^n+u_{k+1}^n\right)\Big)\Big]\\
&\le\frac{1}{(180)^2}E\Big[4\left(\left(u_{k-3}^n\right)^2+\left(u_{k+3}^n\right)^2\right)+729\left(\left(u_{k-2}^n\right)^2+\left(u_{k+2}^2\right)^2\right)+72900\left(\left(u_{k-1}^n\right)^2+\left(u_{k+1}^2\right)^2\right)\\
&\quad-54\left(\left(u_{k-3}^nu_{k-2}^n\right)+\left(u_{k-3}^nu_{k+2}^n\right)+\left(u_{k+3}^nu_{k-2}^n\right)+\left(u_{k+3}^nu_{k+2}^n\right)\right)+540\left(\left(u_{k-3}^nu_{k-1}^n\right)+\left(u_{k-3}^nu_{k+1}^n\right)+\left(u_{k+3}^nu_{k-1}^n\right)\right.\\
&\quad\left.+\left(u_{k+3}^nu_{k+1}^n\right)\right)-7290\left(\left(u_{k-2}^nu_{k-1}^n\right)+\left(u_{k-2}^nu_{k+1}^n\right)+\left(u_{k+2}^nu_{k-1}^n\right)+\left(u_{k+2}^nu_{k+1}^n\right)\right)\Big]\\
&\le\frac{1}{(180)^2}\Big[4\left(E\left(u_{k-3}^n\right)^2+E\left(u_{k+3}^n\right)^2\right)+729\left(E\left(u_{k-2}^n\right)^2+E\left(u_{k+2}^n\right)^2\right)+72900\left(E\left(u_{k-1}^n\right)^2+E\left(u_{k+1}^n\right)^2\right)\\
&\quad-54\left(E\left(u_{k-3}^nu_{k-2}^n\right)+E\left(u_{k-3}^nu_{k+2}^n\right)+E\left(u_{k+3}^nu_{k-2}^n\right)+E\left(u_{k+3}^nu_{k+2}^n\right)\right)+540\left(E\left(u_{k-3}^nu_{k-1}^n\right)+E\left(u_{k-3}^nu_{k+1}^n\right)\right.\\
&\quad\left.+E\left(u_{k+3}^nu_{k-1}^n\right)+E\left(u_{k+3}^nu_{k+1}^n\right)\right)-7290\left(E\left(u_{k-2}^nu_{k-1}^n\right)+E\left(u_{k-2}^nu_{k+1}^n\right)+E\left(u_{k+2}^nu_{k-1}^n\right)+E\left(u_{k+2}^nu_{k+1}^n\right)\right)\Big]\\
&=\frac{1}{8100}E\left(u_{k-3}^n\right)^2+\frac{1}{8100}E\left(u_{k+3}^n\right)^2+\frac{9}{400}\left(E\left(u_{k-2}^n\right)^2+E\left(u_{k+2}^n\right)^2\right)+\frac{9}{4}\left(E\left(u_{k-1}^n\right)^2+E\left(u_{k+1}^n\right)^2\right)\\
&\quad-\frac{1}{600}\left(E\left(u_{k-3}^nu_{k-2}^n\right)+E\left(u_{k-3}^nu_{k+2}^n\right)+E\left(u_{k+3}^nu_{k-2}^n\right)+E\left(u_{k+3}^nu_{k+2}^n\right)\right)+\frac{1}{60}\left(E\left(u_{k-3}^nu_{k-1}^n\right)+E\left(u_{k-3}^nu_{k+1}^n\right)\right.\\
&\quad\left.+E\left(u_{k+3}^nu_{k-1}^n\right)+E\left(u_{k+3}^nu_{k+1}^n\right)\right)-\frac{9}{40}\left(E\left(u_{k-2}^nu_{k-1}^n\right)+E\left(u_{k-2}^nu_{k+1}^n\right)+E\left(u_{k+2}^nu_{k-1}^n\right)+E\left(u_{k+2}^nu_{k+1}^n\right)\right)
\end{aligned}$$

then:

$$\begin{aligned}
E\left|u_k^{n+1}\right|^2&\le\left(1-\frac{49}{18}r\beta\right)^2E\left|u_k^n\right|^2+\left|\frac{r\beta}{90}\left(1-\frac{49}{18}r\beta\right)\right|\left(2\left(E\left|u_k^nu_{k-3}^n\right|+E\left|u_k^nu_{k+3}^n\right|\right)-27\left(E\left|u_k^nu_{k-2}^n\right|+E\left|u_k^nu_{k+2}^n\right|\right)\right.\\
&\quad\left.+270\left(E\left|u_k^nu_{k-1}^n\right|+E\left|u_k^nu_{k+1}^n\right|\right)\right)+(r\beta)^2\Big[\frac{1}{8100}E\left(u_{k-3}^n\right)^2+\frac{1}{8100}E\left(u_{k+3}^n\right)^2\\
&\quad+\frac{9}{400}\left(E\left(u_{k-2}^n\right)^2+E\left(u_{k+2}^n\right)^2\right)+\frac{9}{4}\left(E\left(u_{k-1}^n\right)^2+E\left(u_{k+1}^n\right)^2\right)\\
&\quad-\frac{1}{600}\left(E\left(u_{k-3}^nu_{k-2}^n\right)+E\left(u_{k-3}^nu_{k+2}^n\right)+E\left(u_{k+3}^nu_{k-2}^n\right)+E\left(u_{k+3}^nu_{k+2}^n\right)\right)\\
&\quad+\frac{1}{60}\left(E\left(u_{k-3}^nu_{k-1}^n\right)+E\left(u_{k-3}^nu_{k+1}^n\right)+E\left(u_{k+3}^nu_{k-1}^n\right)+E\left(u_{k+3}^nu_{k+1}^n\right)\right)-\frac{9}{40}\left(E\left(u_{k-2}^nu_{k-1}^n\right)\right.\\
&\quad\left.+E\left(u_{k-2}^nu_{k+1}^n\right)+E\left(u_{k+2}^nu_{k-1}^n\right)+E\left(u_{k+2}^nu_{k+1}^n\right)\right)\Big]+\sigma^2(\Delta t)^2E\left|u_k^n\right|^2\\
&\le\left(1-\frac{49}{18}r\beta\right)^2\sup_kE\left|u_k^n\right|^2+\left|\frac{r\beta}{90}\left(1-\frac{49}{18}r\beta\right)\right|\left(2\left(\sup_kE\left|u_k^n\right|^2+\sup_kE\left|u_k^n\right|^2\right)\right.
\end{aligned}$$

$$-27\left(\sup_k E\left|u_k^n\right|^2+\sup_k E\left|u_k^n\right|^2\right)+270\left(\sup_k E\left|u_k^n\right|^2+\sup_k E\left|u_k^n\right|^2\right)\bigg)+\left(r\beta\right)^2\bigg[\frac{1}{8100}\sup_k E\left|u_k^n\right|^2$$
$$+\frac{1}{8100}\sup_k E\left|u_k^n\right|^2+\frac{9}{400}\left(\sup_k E\left|u_k^n\right|^2+\sup_k E\left|u_k^n\right|^2\right)+\frac{9}{4}\left(\sup_k E\left|u_k^n\right|^2+\sup_k E\left|u_k^n\right|^2\right)$$
$$-\frac{1}{600}\left(\sup_k E\left|u_k^n\right|^2+\sup_k E\left|u_k^n\right|^2+\sup_k E\left|u_k^n\right|^2+\sup_k E\left|u_k^n\right|^2\right)$$
$$+\frac{1}{60}\left(\sup_k E\left|u_k^n\right|^2+\sup_k E\left|u_k^n\right|^2+\sup_k E\left|u_k^n\right|^2+\sup_k E\left|u_k^n\right|^2\right)$$
$$-\frac{9}{40}\left(\sup_k E\left|u_k^n\right|^2+\sup_k E\left|u_k^n\right|^2+\sup_k E\left|u_k^n\right|^2+\sup_k E\left|u_k^n\right|^2\right)\bigg]+\sigma^2\left(\Delta t\right)^2\sup_k E\left|u_k^n\right|^2$$
$$\leq\left(1-\frac{49}{18}r\beta\right)^2\sup_k E\left|u_k^n\right|^2+\left|\frac{r\beta}{90}\left(1-\frac{49}{18}r\beta\right)\right|\left(490\sup_k E\left|u_k^n\right|^2\right)$$
$$+\left(r\beta\right)^2\left(\frac{2401}{648}\sup_k E\left|u_k^n\right|^2\right)+\sigma^2\left(\Delta t\right)^2\sup_k E\left|u_k^n\right|^2$$
$$=\left(1-\frac{49}{18}r\beta\right)^2\sup_k E\left|u_k^n\right|^2+\frac{49}{9}\left|r\beta\left(1-\frac{49}{18}r\beta\right)\right|\sup_k E\left|u_k^n\right|^2$$
$$+\frac{2401}{648}\left(r\beta\right)^2\sup_k E\left|u_k^n\right|^2+\sigma^2\Delta t\sup_k E\left|u_k^n\right|^2$$
$$=\left[\left(1-\frac{49}{18}r\beta\right)^2+\frac{49}{9}\left|r\beta\left(1-\frac{49}{18}r\beta\right)\right|+\frac{2401}{648}\left(r\beta\right)^2\right]\sup_k E\left|u_k^n\right|^2+\sigma^2\left(\Delta t\right)^2\sup_k E\left|u_k^n\right|^2$$
$$=\left[\left(1-\frac{49}{18}r\beta\right)^2+\frac{49}{9}\left|r\beta\left(1-\frac{49}{18}r\beta\right)\right|+\frac{2401}{324}\left(r\beta\right)^2\right]\sup_k E\left|u_k^n\right|^2$$
$$+\frac{2401}{324}\left(r\beta\right)^2\sup_k E\left|u_k^n\right|^2+\sigma^2\left(\Delta t\right)^2\sup_k E\left|u_k^n\right|^2.$$

Now with:

$$0\leq r\beta=\frac{\Delta t}{\left(\Delta x\right)^2}\beta\leq\frac{18}{49},$$

then:

$$\left|1-\frac{49}{18}r\beta\right|=1-\frac{49}{18}\left|r\beta\right|.$$

Hence:

$$E\left|u_k^{n+1}\right|^2\leq\left[\left(1-\frac{49}{18}r\beta\right)+\frac{49}{18}r\beta\right]^2E\left|u_k^n\right|^2+\frac{2401}{324}\left(r\beta\right)^2E\left|u_k^n\right|^2+\sigma^2\Delta tE\left|u_k^n\right|^2$$
$$=E\left|u_k^n\right|^2+\frac{2401}{324}\left(r\beta\right)^2E\left|u_k^n\right|^2+\sigma^2\left(\Delta t\right)^2E\left|u_k^n\right|^2$$
$$=\left(1+\frac{2401}{324}\left|r\beta\right|^2+\sigma^2\left(\Delta t\right)^2\right)E\left|u_k^n\right|^2$$

It is enough to select λ such that: $\frac{2401}{324}\left|r\beta\right|^2+\sigma^2\left(\Delta t\right)^2\leq\lambda^2\Delta t$ for all k then we put $\Delta t=\frac{t}{n+1}$ to get:

$$E\left|u_k^{n+1}\right|^2\leq\left(1+\frac{\lambda^2t}{n+1}\right)^{n+1}E\left|u_k^0\right|^2=\mathrm{e}^{\lambda^2t}E\left|u_k^0\right|^2. \tag{8}$$

Hence the scheme is conditionally stable with $k=1$ and $b=\lambda^2$ in mean square sense with the condition:

$$0 \le r\beta = \frac{\Delta t}{(\Delta x)^2}\beta \le \frac{18}{49} \quad \text{for} \quad \beta > 0.$$

3.1.3. Convergence

Theorem 3.3. The random difference scheme (7) is convergent in mean square sense

Proof.

$$E\left|u_k^n - u\right|^2 = E\left|\left(L_k^n\right)^{-1}\left(L_k^n u_k^n - L_k^n u\right)\right|^2.$$

Since the scheme is consistent then we have $L_k^n u_k^n \xrightarrow{ms} Lu$ then we obtain

$$E\left|\left(L_k^n\right)^{-1}\left(L_k^n u_k^n - L_k^n u\right)\right|^2 \to 0,$$

as $\Delta t \to 0$, $\Delta x \to 0$ and $(k\Delta x, n\Delta t) \to (x,t)$ and since the scheme is stable then $\left(L_k^n\right)^{-1}$ is bounded hence:

$$E\left|u_k^n - u\right|^2 \to 0 \quad \text{as} \quad \Delta t \to 0, \quad \Delta x \to 0,$$

then the random difference scheme (7) is convergent in mean square sense.

4. Applications

Consider the linear heat equation with multiplicative noise in this form

$$u_t(x,t) = 0.001 u_{xx} - u(x,t)\,\mathrm{d}w(t), \quad t \in [0,1], \; x \in [0,1]$$
$$u_0(x) = x^2(1-x)^2$$
$$u(0,t) = u(1,t) = 0$$

The SFDS with seven points is:

$$u_k^{n+1} = \left(1 - \frac{49}{18} r\beta\right) u_k^n + \frac{r\beta}{180}\left(2u_{k-3}^n - 27u_{k-2}^2 + 270u_{k-1}^n + 270u_{k+1}^n - 27u_{k+2}^2 + 2u_{k+3}^2\right) - \Delta t u_k^n \left(W_k^{n+1} - W_k^n\right).$$

Since the stability condition from Theorem 3.2 is:

$$0 \le r\beta = \frac{\Delta t}{(\Delta x)^2}\beta \le \frac{18}{49}, \quad \beta = 0.001$$

let $M = 150$ then $\Delta x = \frac{1}{150}$ therefore:

$$0 \le \frac{\Delta t}{\left(\frac{1}{150}\right)^2} 0.001 \le \frac{18}{49}$$

then we have:

$$0 \le \Delta t \le \frac{4}{245} \Rightarrow N \ge 62$$

In Theorem 3.8, we assumed that:

$$\frac{2401}{324}\left|r\beta\right|^2 + \sigma^2 (\Delta t)^2 \le \lambda^2 \Delta t.$$

Therefore, for different amount N, we can derive low boundary of λ^2 (see **Table 1**).

Table 1. λ^2 for stability.

N	62	200	500	1000	1500	6000
λ^2	60.52520161	18.7628125	7.505125	3.7525625	2.501708333	0.62542

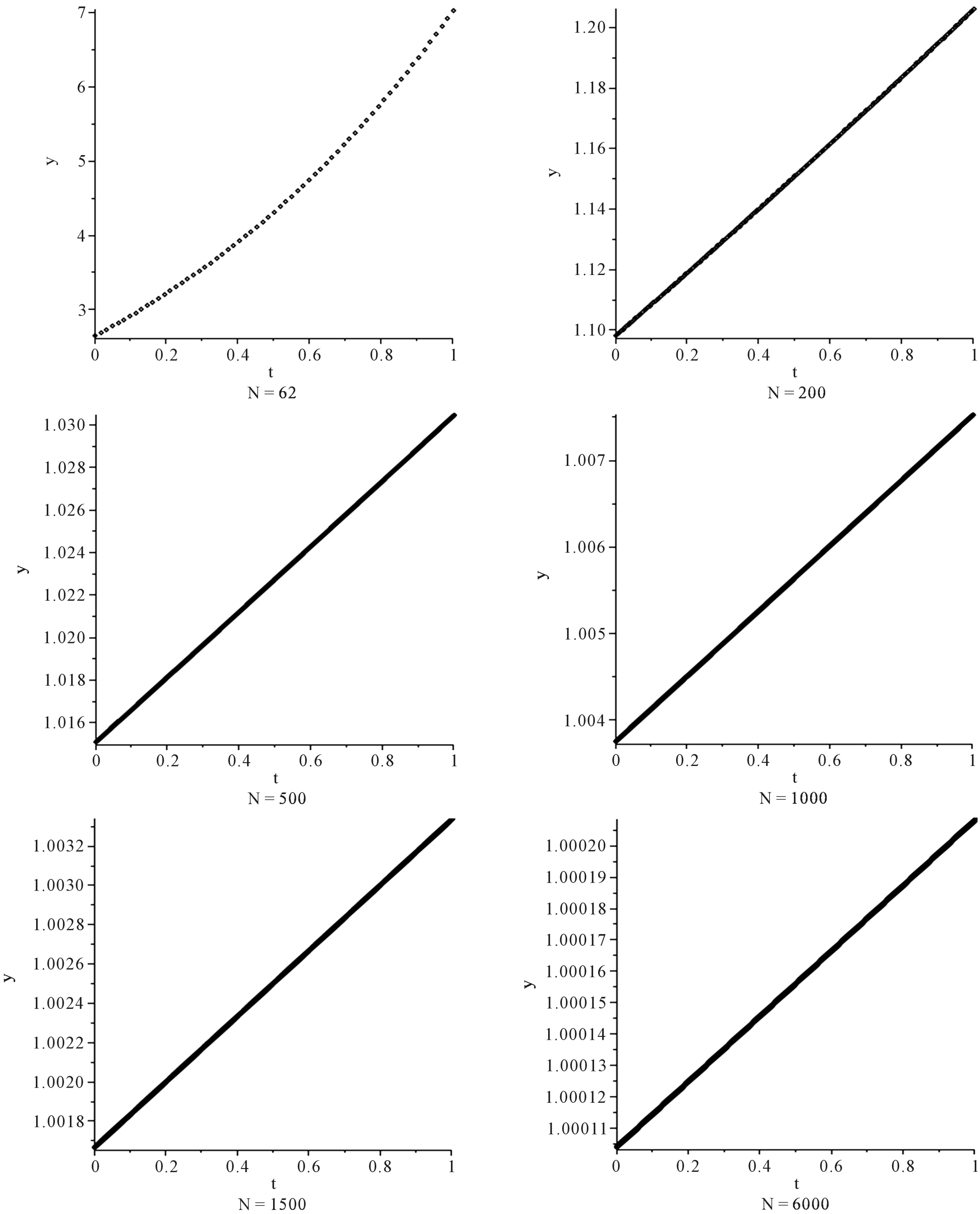

Figure 1. Stability.

On the other hand, in Equation (8), we had

$$E\left|u_k^{n+1}\right|^2 \le e^{\lambda^2 t} E\left|u_k^0\right|^2 \Rightarrow \frac{E\left|u_k^{n+1}\right|^2}{E\left|u_k^0\right|^2} \le e^{\lambda^2 t} \Rightarrow y = \frac{E\left|u_k^{n+1}\right|^2}{E\left|u_k^0\right|^2} \le e^{\lambda^2 t}, \quad \Delta t = \frac{t}{n+1},$$

and for stability, we represent y for different N (see **Figure 1**).

5. Conclusion and Future Works

The stochastic parabolic partial differential equations can be solved numerically using the stochastic difference (with seven points) method in mean square sense. More complicated problems in linear stochastic parabolic partial differential equations can be studied using finite difference method in mean square sense. The techniques of solving nonlinear stochastic partial differential equations using the finite difference method with the aid of mean square calculus can enhance greatly the treatment of stochastic partial differential equations in mean square sense.

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