

An Iterative Method for Solving Two Special Cases of Lane-Emden Type Equation

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Abstract

In this work we apply the differential transformation method or DTM for solving some classes of Lane-Emden type equations as a model for the dimensionless density distribution in an isothermal gas sphere $\left(y'' + \frac{2}{x} y' + e^{\pm y} = 0 \right)$ and as a study of the gravitational potential of (white-dwarf) stars $\left(y'' + \frac{2}{x} y' + (y^2 - C)^{3/2} = 0 \right)$, which are nonlinear ordinary differential equations on the semi-infinite domain [1] [2]. The efficiency of the DTM is illustrated by investigating the convergence results for this type of the Lane-Emden equations. The numerical results show the reliability and accuracy of this method.

Keywords

Differential Transformation, Lane-Emden, Isothermal Gas Sphere, White-Dwarfs, Iterative Method

1. Introduction

Other classical nonlinear equation, which has been the object of much study, is Lane-Emden's equation. This equation has the form

$$y'' + \frac{2}{x} y' + g(y) = 0 \quad (1)$$

with $0 < x \leq 1$ and the subject to initial conditions

$$y(0) = \alpha, y'(0) = \beta \quad (2)$$

where α and β are constants and $g(y)$ is a real-valued continuous function where α and β are constants and $g(y)$ is a real-valued continuous function. The Equation (1) was used to model various problems, including the isothermal gas spheres, theory of thermionic currents and the gravitational potential of stars [1] among others.

Let us consider a spherical cloud of gas (see **Figure 1**) and denote its hydrostatic pressure at a distance r_1 from the centre by P . Let $M(r_1)$ be the mass of the spheres of radius r_1 , ϕ the gravitational potential of the gas and g the acceleration of gravity.

Then, we have the following equation

$$g = \frac{GM(r_1)}{r_1^2} = -\phi'(r_1) \tag{3}$$

where G is the gravitational constant.

Thus, three conditions are assumed for the determination of ϕ and P

$$dP = -g\rho dr_1 = \rho d\phi \tag{4}$$

where ρ is the density of the gas.

$$\nabla^2\phi = \phi''(r_1) + \frac{2}{r_1}\phi'(r_1) = -4\pi G\rho \tag{5}$$

and

$$P = K\rho^\gamma \tag{6}$$

where γ and K are arbitrary constants.

Now, solving (4) and (6) with $\phi = 0$ when $\rho = 0$ we have

$$\rho = \phi^{\frac{1}{\gamma-1}} K^{\frac{1}{1-\gamma}} \tag{7}$$

or

$$\rho = L\phi^n \tag{8}$$

where $n = \frac{1}{\gamma-1}$ and $L = K^{-n}$. If this value of ρ is replaced into Equation (5), we obtain

$$\nabla^2\phi = -\delta^2\phi^n \tag{9}$$

where $\delta^2 = 4\pi LG$.

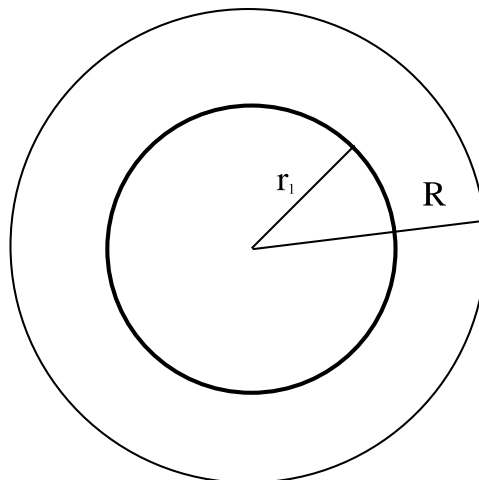


Figure 1. Spherical cloud of gas.

Now, since $\frac{1}{\rho}dP = d\phi$, by integration $\phi = K \log\left(\frac{\rho}{\rho_0}\right)$, that is, $\rho = \rho_0 e^{\frac{\phi}{K}}$. If ρ_0 is the central density, then ϕ_0 must be zero, a change from the condition in the previous case where ϕ was zero only at the boundary of the sphere.

Poisson's equation is now replaced by

$$\nabla^2 \phi = -\delta^2 e^{\frac{\phi}{K}} \tag{10}$$

where $\delta^2 = 4\pi\rho_0 G$, equation which is known as *Liouville's equation*. If we assume symmetry as before, Equation (1) in polar coordinates reduces to the following

$$\phi''(r_1) + \frac{2}{r_1}\phi'(r_1) + \delta^2 e^{\frac{\phi}{K}} = 0 \tag{11}$$

which replaces Equation (9).

If we let $\phi = Ky$ and $r_1 = \frac{\sqrt{K}}{\delta}x$, then (11) becomes

$$y'' + \frac{2}{x}y' + e^y = 0 \tag{12}$$

which is to be solved subject to the boundary conditions $y(0) = 0$ and $y'(0) = 0$. The counterpart [2] of the Equation (12) in which e^y is replaced by e^{-y} appears in *Richardson's theory* of thermionic currents when one seeks to determine the density and electric force of an electron gas in the neighborhood of a hot body in thermal equilibrium.

Finally, now consider $g(y) = (y^2 - C)^{3/2}$, then Equation (1) is turned to the white-dwarf equation, which introduced by [2] in his study of gravitational potential of the degenerate stars. This Equation is defined in the form

$$y'' + \frac{2}{x}y' + (y^2 - C)^{3/2} = 0$$

With $x \in [0, \infty]$ and subject to initial conditions $y(0) = 1$ and $y'(0) = 0$. For instance if $C = 0$, we have Lane-Emden equation of index $m = 3$ [3].

The Differential Transformation Method is a semi-numerical-analytic method for solving ordinary and partial differential equations. Zhou first introduced the concept of DTM in 1986 [4]. This technique constructs an analytical solution in the form of a polynomial. DTM is an alternative procedure for obtaining analytical Taylor series solution of the differential equations. The series often coincides with the Taylor expansion of the true solution at point $x_0 = 0$, in the value case, although the series can be rapidly convergent in a very small region.

Many numerical methods were developed for this type of nonlinear ordinary differential equations, specifically on Lane-Emden type equations such as the Adomian Decomposition Method (ADM) [5], the Homotopy Perturbation Method (HPM) [6] [7], the Homotopy Analysis Method (HAM) [8] and Bernstein Operational Matrix of Integration [9]. In this paper, we show superiority of DTM by applying them on the same type Lane-Emden type equations. The power series solution of the reduced equation transforms into an approximate implicit solution of the original equation. A spectral method (Legendre-Spectral method) was proposed to solve white-dwarf equation; this spectral method provides the most convenient computer implementation [10].

2. Description of DTM

Differential transformation method of the function $y(x)$ is defined as follows

$$Y(k) = \frac{1}{k!} \left[\frac{d^k y(x)}{dx^k} \right]_{x=x_0} \tag{13}$$

In (13), $y(x)$ is the original function and $Y(k)$ is the transformed function and the inverse differential

transformation is defined by

$$y(x) = \sum_{k=0}^{\infty} Y(k)x^k \tag{14}$$

In real applications, function $y(x)$ is expressed by a finite series and Equation (14) can be written as

$$\sum_{k=0}^n Y(k)x^k \tag{15}$$

Equation (15) implies that

$$\sum_{k=n+1}^{\infty} Y(k)x^k$$

The following theorems can be deduced from Equations (13) and (15).

Theorem 1 If $y(x) = f(x) \pm g(x)$, then $Y(k) = F(k) \pm G(k)$.

Theorem 2 If $y(x) = af(x)$, then $Y(k) = aF(k)$, $a \in \mathbb{R}$.

Theorem 3 If $y(x) = \frac{d^n g(x)}{dx^n}$, then $Y(k) = \frac{(k+n)!}{k!} G(k+n)$.

Theorem 4 If $y(x) = g(x)h(x)$, then $Y(k) = \sum_{k_1=0}^k G(k_1)H(k-k_1)$.

Theorem 5 If $y(x) = x^n$, then $Y(k) = \delta(k-n)$, where $\delta(k-n) = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$

Theorem 6 (Cárdenas) If $y(x) = x^n f(x)$, with $m \in N$, then $Y(k) = \begin{cases} 0, & k < m \\ F(k-m), & k \geq m \end{cases}$

The proofs of Theorems are available in [11].

3. Test Problems

To illustrate the ability of DTM for the Lane-Emden type equation, three examples are provided. The results reveal that this method is very effective.

Example 1 Consider the nonlinear initial-value problem $xy'' + \frac{2}{x}y' + e^y = 0$ subject to $y(0) = y'(0) = 0$.

Multiplying both sides by x we obtain

$$xy'' + 2y' + xe^y = 0 \tag{16}$$

Applying theorems 1-6 to Equation (16)

$$Y(k+1) = \frac{-1}{(k+2)(k+1)} \left[\delta(k-1) + Y(k-1) + \frac{1}{2!}S_1 + \frac{1}{3!}S_2 + \frac{1}{4!}S_3 + \dots \right] \tag{17}$$

where

$$S_1 = \sum_{k_1=0}^{k-1} Y(k_1)Y(k-1-k_1) \tag{18}$$

$$S_2 = \sum_{k_2=0}^{k-1} \sum_{k_1=0}^{k_2} Y(k_1)Y(k_2-k_1)Y(k-1-k_2) \tag{19}$$

$$S_3 = \sum_{k_3=0}^{k-1} \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} Y(k_1)Y(k_2-k_1)Y(k_3-k_2)Y(k-1-k_3) \tag{20}$$

for all $k \geq 1$.

Now, from the initial conditions $y(0) = y'(0) = 0$ we can obtain

$$Y(0) = 0 \text{ and } Y(1) = 0 \tag{21}$$

Substituting Equation (21) into Equation (17) and by recursive method, the results are listed as follows.

$$\text{For } k = 1, Y(2) = \frac{-1}{6} [\delta(1-1) + Y(0) + 0 + 0] = \frac{-1}{6}.$$

$$\begin{aligned} \text{For } k = 2, Y(3) &= \frac{-1}{12} \left[\delta(2-1) + Y(1) + \frac{1}{2!} \sum_{k_1=0}^1 Y(k_1) Y(k-1-k_1) \right. \\ &\quad \left. + \frac{1}{3!} \sum_{k_2=0}^1 \sum_{k_1=0}^{k_2} Y(k_1) Y(k_2-k_1) Y(k-1-k_2) + \dots \right] = 0 \end{aligned}$$

and then, $Y(3) = 0$. For $k = 3$ we have:

$$Y(4) = \frac{-1}{5 \times 4} \left[\delta(3-1) + Y(2) + \frac{1}{2!} S_1 + \frac{1}{3!} S_2 + \frac{1}{4!} S_3 + \dots \right]$$

$$\text{Now, as } Y(2) = \frac{-1}{6} \text{ and } S_1 = S_2 = S_3 = \dots = 0, \text{ then } Y(4) = \frac{1}{4 \times 5 \times 6} = \frac{1}{120}$$

For $k = 4$ we have:

$$Y(5) = \frac{-1}{6 \times 5} \left[\delta(4-1) + Y(3) + \frac{1}{2!} S_1 + \frac{1}{3!} S_2 + \frac{1}{4!} S_3 + \dots \right]$$

In this case as $Y(3) = 0$ and $S_1 = S_2 = S_3 = \dots = 0$, then $Y(5) = 0$.

The lector can see that

$$S_1 = \sum_{k_1=0}^3 Y(k_1) Y(k-1-k_1) = Y(0)Y(3) + Y(1)Y(2) + Y(2)Y(1) + Y(3)Y(0) = 0$$

For $k = 5$ we have:

$$\begin{aligned} Y(6) &= \frac{-1}{7 \times 6} \left[\delta(5-1) + Y(4) + \frac{1}{2!} S_1 + \frac{1}{3!} S_2 + \frac{1}{4!} S_3 + \dots \right] \\ &= -\frac{1}{7 \times 6} \left[\frac{1}{120} + \frac{1}{2!} S_1 + \frac{1}{3!} S_2 + \frac{1}{4!} S_3 + \dots \right] \end{aligned}$$

Now, we can see:

$$\begin{aligned} S_1 &= \sum_{k_1=0}^4 Y(k_1) Y(k-1-k_1) = Y(0)Y(4) + Y(1)Y(3) + Y(2)Y(2) + Y(3)Y(1) + Y(4)Y(0) \\ &= \frac{-1}{6} \times -\frac{1}{6} = \frac{1}{36} \end{aligned}$$

$S_2 = S_3 = \dots = 0$ and then

$$Y(6) = \frac{-1}{7 \times 6} \left[\frac{1}{120} + \frac{1}{2!} \times \frac{1}{36} \right] = \frac{-8}{21 \times 6!}$$

For $k = 6$ we have:

$$\begin{aligned} Y(7) &= \frac{-1}{8 \times 7} \left[\delta(6-1) + Y(5) + \frac{1}{2!} S_1 + \frac{1}{3!} S_2 + \frac{1}{4!} S_3 + \dots \right] \\ &= -\frac{1}{8 \times 7} \left[\frac{1}{2!} S_1 + \frac{1}{3!} S_2 + \frac{1}{4!} S_3 + \dots \right] \end{aligned}$$

Here,

$$\begin{aligned} S_1 &= \sum_{k_1=0}^5 Y(k_1) Y(k-1-k_1) = Y(0)Y(5) + Y(1)Y(4) + \\ &\quad Y(2)Y(3) + Y(3)Y(2) + Y(4)Y(1) + Y(5)Y(0) = 0 \end{aligned}$$

and

$$S_2 = \sum_{k_2=0}^5 \sum_{k_1=0}^{k_2} Y(k_1)Y(k_2 - k_1)Y(k - 1 - k_2) = 0$$

and so, $S_2 = S_3 = \dots = 0$. Consequently, $Y(7) = 0$.

For $k = 7$ we have:

$$Y(8) = \frac{-1}{9 \times 8} \left[\delta(7-1) + Y(6) + \frac{1}{2!}S_1 + \frac{1}{3!}S_2 + \frac{1}{4!}S_3 + \dots \right] - \frac{1}{9 \times 8} \left[\frac{-8}{21 \times 6!} + \frac{1}{2!}S_1 + \frac{1}{3!}S_2 + \frac{1}{4!}S_3 + \dots \right]$$

Here

$$S_1 = \sum_{k_1=0}^6 Y(k_1)Y(k - 1 - k_1) = Y(0)Y(6) + Y(1)Y(5) + Y(2)Y(4) + Y(3)Y(3) + Y(4)Y(2) + Y(5)Y(1) + Y(6)Y(0) = \frac{-1}{6} \times \frac{1}{120} + \frac{1}{120} \times -\frac{1}{6} = \frac{-1}{360}$$

and

$$S_2 = \sum_{k_2=0}^6 \sum_{k_1=0}^{k_2} Y(k_1)Y(k_2 - k_1)Y(k - 1 - k_2) = \frac{-1}{216}$$

Consequently, $S_3 = S_4 = \dots = 0$. Finally,

$$Y(8) = \frac{-1}{9 \times 8} \left[\frac{-8}{21 \times 6!} + \frac{1}{2} \times \left(-\frac{1}{360} \right) + \frac{1}{6} \times \left(-\frac{1}{216} \right) \right] = \frac{61}{1632960}$$

Therefore using (15), the closed form of the solution can be easily written as:

$$y(x) = \sum_{k=0}^n Y(k)x^k = Y(0)x^0 + Y(1)x^1 + Y(2)x^2 + Y(3)x^3 + Y(4)x^4 + \dots = -\frac{1}{6}x^2 + \frac{1}{120}x^4 - \frac{1}{21 \times 6!}x^6 + \frac{61}{1632960}x^8 - \dots$$

A series solution obtained by Wazwaz [5] and series expansion respectively is

$$y(x) \cong -\frac{1}{6}x^2 + \frac{1}{5 \times 4!}x^4 - \frac{1}{21 \times 6!}x^6 + \frac{122}{81 \times 8!}x^8 - \frac{61 \times 67}{495 \times 10!}x^{10} + \dots \tag{22}$$

Table 1 shows the comparison of $y(x)$ obtained by the DTM (method proposed in this work) and those obtained by Wazwaz. The resulting graph of the isothermal gas spheres equation in comparison to the present method and those obtained by Wazwaz is shown in **Figure 2**.

Example 2 Consider the following problem $y'' + \frac{2}{x}y' + e^{-y} = 0$ subject to $y(0) = y'(0) = 0$. Multiplying both sides by x

$$xy'' + 2y' + xe^{-y} = 0 \tag{23}$$

As before, using theorems 1-6 we obtain

$$Y(k+1) = \frac{-1}{(k+2)(k+1)} \left[\delta(k-1) - Y(k-1) + \frac{1}{2!}S_1 - \frac{1}{3!}S_2 + \frac{1}{4!}S_3 - \dots \right] \tag{24}$$

where S_1, S_2 and S_3 are as (18), (19) and (20) respectively for all $k \geq 1$. Now, from the initial conditions $y(0) = y'(0) = 0$ we have

$$Y(0) = 0 \text{ and } Y(1) = 0 \tag{25}$$

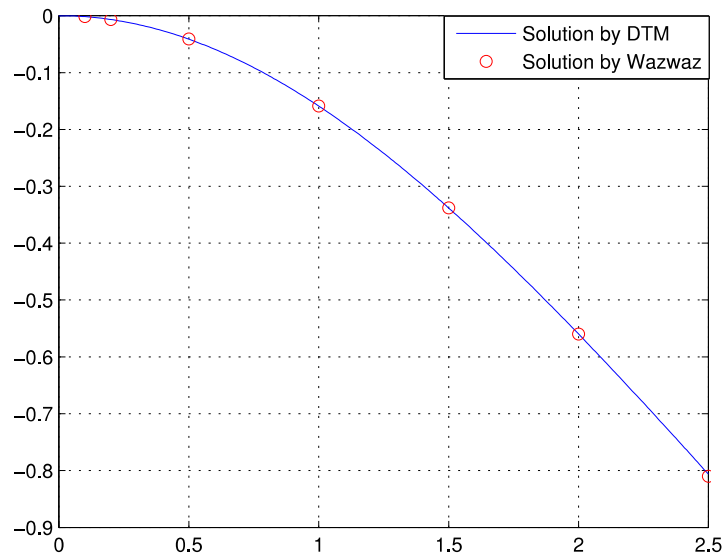


Figure 2. Comparison between DTM and Wazwaz's algorithm.

Table 1. Comparison between DTM and Wazwaz's algorithm.

x	DTM	Wazwaz	Error
0.0	0.0000000000	0.0000000000	0.0000000000
0.1	-0.0016658338	-0.0016658338	0.0000000000
0.2	-0.0066533671	-0.0066533671	0.0000000000
0.5	-0.0411539573	-0.0411539568	0.0000000005
1.0	-0.1588276775	-0.1588273536	0.0000003239
1.5	-0.3380194248	-0.3380131102	0.0000063146
2.0	-0.5598230174	-0.5599626601	0.0001396427
2.5	-0.8063552923	-0.8100196713	0.0036643790

Substituting Equation (25) into Equation (24) and by recursive method, the results are listed as follows. For $k = 1, 2, 3, 4, 5$, we have respectively

$$Y(2) = -\frac{1}{6}, Y(3) = 0, Y(4) = -\frac{1}{120}, Y(5) = 0, Y(6) = -\frac{1}{1890}$$

So on, we can use (15) and the closed form of the solution can be easily written as

$$y(x) = \sum_{k=0}^n Y(k)x^k = Y(0)x^0 + Y(1)x^1 + Y(2)x^2 + Y(3)x^3 + Y(4)x^4 + \dots$$

$$= -\frac{1}{6}x^2 - \frac{1}{120}x^4 - \frac{1}{1890}x^6 - \dots$$

A solution obtained by Yahya [12] by using the power series method is

$$y(x) \cong -\frac{1}{6}x^2 - \frac{1}{120}x^4 - \frac{8}{21 \times 6!}x^6 - \frac{122}{81 \times 8!}x^8 - \dots$$

We can see Figure 3 and compare with [13], the results are very good.

As final example and to illustrate the ability of DTM for white-dwarf equation, the next problem is provided for $c = 0.2$.

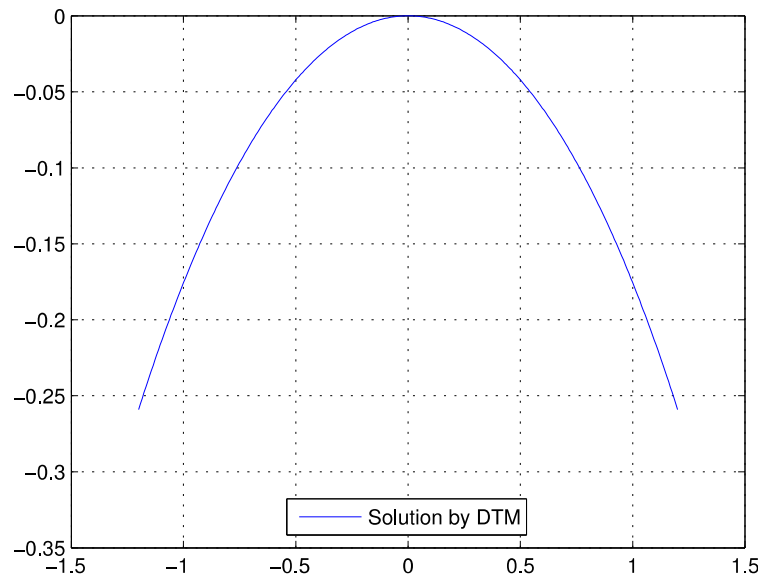


Figure 3. Numerical results by using DTM.

Example 3 Consider the problem $y'' + \frac{2}{x}y' + (y^2 - c)^{3/2} = 0$ subject to $y(0) = 1$ and $y'(0) = 0$. Multiplying both sides by x we obtain

$$xy'' + 2y' + x(y^2 - c)^{3/2} = 0 \tag{27}$$

Here, is easy to verify that the function $g(y) = (y^2 - c)^{3/2}$ has a series expansion

$$g(y) \approx q^3 + 3q(y-1) + \frac{3q^2 + 3}{2q}(y-1)^2 + \frac{9q^2 - 3}{3!q^3}(y-1)^3 + \dots \tag{28}$$

where $q^2 = 1 - c$. Therefore, Equation (27) takes the form

$$xy'' + 2y' + x \left[q^3 + 3q(y-1) + \frac{3q^2 + 3}{2q}(y-1)^2 + \frac{9q^2 - 3}{3!q^3}(y-1)^3 + \dots \right] \tag{29}$$

Using in (29) the above theorems we have the following

$$(k+1)kY(k+1) + 2(k+1)Y(k+1) + \alpha_1\delta(k-1) + \alpha_2Y(k-1) + \frac{\alpha_3}{2!}S_1 + \frac{\alpha_4}{3!}S_2 \dots \tag{30}$$

or

$$Y(k+1) = \frac{1}{(k+2)(k+1)} \left[-\alpha_1\delta(k-1) - \alpha_2Y(k-1) - \frac{\alpha_3}{2!}S_1 - \frac{\alpha_4}{3!}S_2 \dots \right] \tag{31}$$

where

$$\begin{aligned} \alpha_1 &= q^3 - 3q + \frac{3q^2 + 3}{2q} - \frac{9q^2 - 3}{3!q^3} + \dots \\ \alpha_2 &= 3q - 2\frac{3q^2 + 3}{2q} + 3\frac{9q^2 - 3}{3!q^3} + \dots \\ \alpha_3 &= \frac{3q^2 + 3}{2q} - 3\frac{9q^2 - 3}{3!q^3} + \dots \end{aligned}$$

and successively. Also,

$$S_1 = \sum_{k_1=0}^{k-1} Y(k_1)Y(k-1-k_1) \tag{32}$$

$$S_2 = \sum_{k_2=0}^{k-1} \sum_{k_1=0}^{k_2} Y(k_1)Y(k_2-k_1)Y(k-1-k_2) \tag{33}$$

$$S_3 = \sum_{k_3=0}^{k-1} \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} Y(k_1)Y(k_2-k_1)Y(k_3-k_2)Y(k-1-k_3) \tag{34}$$

for all $k \geq 1$. Now, from the initial conditions we have

$$Y(0) = 1 \text{ and } Y(1) = 0 \tag{35}$$

Substituting (35) into Equation (31) and by recursive method, the results are listed as follows.

$$\text{For } k = 1, Y(2) = \frac{1}{6} \left[-\alpha_1 - \alpha_2 - \frac{\alpha_3}{2!} \right] = -\frac{1}{6} q^3 \text{ . or } k = 2, Y(3) = \frac{1}{12} \left[-\alpha_1 \delta(2-1) - \alpha_2 Y(1) - \frac{\alpha_3}{2!} S_1 - \dots \right] = 0$$

$$\text{and then } Y(3) = 0. \text{ for } k = 3, Y(4) = \frac{1}{20} \left[-\alpha_1 \delta(3-1) - \alpha_2 Y(2) - \frac{\alpha_3}{2!} S_1 - \dots \right] \text{ and then } Y(4) = \frac{1}{40} q^4 \text{ . For}$$

$$k = 4, Y(5) = \frac{1}{30} \left[-\alpha_1 \delta(4-1) - \alpha_2 Y(3) - \frac{\alpha_3}{2!} S_1 - \dots \right] \text{ and so } Y(5) = 0 \text{ . For}$$

$$k = 5, Y(6) = \frac{1}{42} \left[-\alpha_1 \delta(5-1) - \alpha_2 Y(4) - \frac{\alpha_3}{2!} S_1 - \dots \right] \text{ therefore } Y(6) = -\frac{5}{5040} q^7 - \frac{14}{5040} q^5 \text{ . Using (15), the}$$

closed form of the solution can be easily written as

$$\begin{aligned} y(x) &= \sum_{k=0}^n Y(k) x^k = Y(0) x^0 + Y(1) x + Y(2) x^2 + Y(3) x^3 + \dots \\ &= 1 - \frac{1}{6} (1-c)^{3/2} x^2 + \frac{1}{40} (1-c)^4 x^4 - \left(\frac{5}{5040} (1-c)^{7/2} + \frac{14}{5040} (1-c)^{5/2} \right) x^6 + \dots \end{aligned}$$

A series solution obtained by Chandrasekhar [2] using series expansion was

$$y(x) \cong 1 - \frac{q^3}{6} x^2 + \frac{q^4}{40} x^4 - \frac{q^5}{7!} (5q^2 + 14) x^6 + \frac{q^6}{3 \times 9!} (339q^2 + 280) x^8 + \dots \tag{36}$$

Table 2 shows the comparison of $y(x)$ obtained by the DTM and those obtained by Parand [14]. The resulting graph of the white-dwarfs equation in comparison to the present method and the obtained by [14] is shown in **Figure 4**.

4. On Convergence of DTM

We can write the DTM as

Table 2. Comparison between DTM and Legendre-Spectral method.

x	DTM	L-S method	Error
0.0	0.000000000	0.000000000	0.000000000
0.5	0.9812035800	0.912034800	0.0000001000
1.0	0.9270041568	0.9270031568	0.0000010000
1.5	0.8439248841	0.8439247581	0.0000001260
2.0	0.7425430743	0.7425430235	0.0000000508
2.5	0.6365969111	0.6365953025	0.0000016086
3.0	0.5410635754	0.5410633690	0.0000002064
3.5	0.4691977489	0.4691238259	0.0000739230

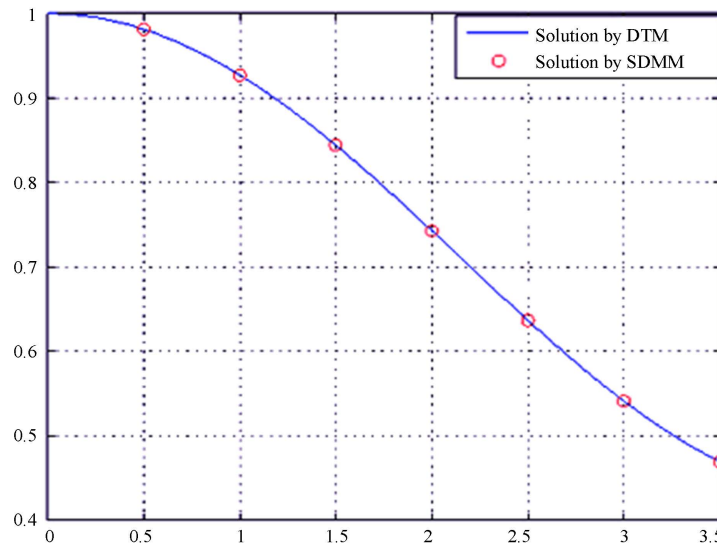


Figure 4. Comparison between DTM and a Legendre-Spectral Method.

$$\sum_{j=0}^k \alpha_j y_{n+j} = h\phi_f(x_n, y_n, \dots, y_{n+k-1}; h), \quad n = 0, \dots, N - k \tag{37}$$

where ϕ_f increase function depends on its arguments through the function f . The method (37) means k steps, needed for early values $k : y_n, \dots, y_{n+k-1}$ to calculate y_{n+k} . It is therefore necessary to have bootstrap values y_0, \dots, y_{k-1} .

The method (37) is said to be convergent if for all IVP has to

$$\lim_{N \rightarrow \infty} \max_{k \leq n \leq N} \|y(x_n - y_n)\| = 0 \quad \text{if} \quad \lim_{N \rightarrow \infty} \max_{0 \leq n \leq k-1} \|y(x_n - y_n)\| = 0$$

Remark. The condition $\lim_{N \rightarrow \infty} \max_{0 \leq n \leq k-1} \|y(x_n - y_n)\| = 0$ on the bootstrap values is equivalent to asking that $\lim_{h \rightarrow 0^+} y_n = y(x_0)$ for $n = 0, \dots, k - 1$. Here, we are asking that bootstrap values $\{y_n\}_{n=0}^{k-1}$ approximate well and the initial data $y(x_0)$; if this is not, then no reason to expect that numerical solution closely matches the theoretical.

Now let us consider the following form of the Equation (1)

$$\mathcal{G}(y(x)) = 0 \tag{38}$$

Here \mathcal{G} is a nonlinear differential operator, which encloses the linear and nonlinear term of the Lane-Emden type equation. Now, the linear term $\mathcal{H} = y^{(n)}$ is always invertible and the nonlinear term is \mathcal{J} . Therefore (38) may be written as

$$\mathcal{H}(y(x)) + \mathcal{J}(y(x)) = 0 \tag{39}$$

or

$$\mathcal{H}(y(x)) = -\mathcal{J}(y(x)) = 0 \tag{40}$$

Applying DTM in (40) we can obtain

$$(k+1)(k+2)\dots(k+n)Y(k+n) = -\mathcal{J}(y(x)) \tag{41}$$

Remember that differential transformation of \mathcal{G} and \mathcal{J} are computed by using theorems 1 - 6. Let us consider the Equation (38) in the following form

$$y(x) = \mathcal{Y}(y(x)) \tag{42}$$

Here, \mathcal{Y} is a nonlinear operator. It is noted that Equation (15) is equivalent to the sequence

$$\sum_{k=0}^n Y(k)x^k = y_0 + y_1 + y_2 + \dots + y_n = T_n \tag{43}$$

This sequence is determined using the iterative scheme

$$T_{n+1} = \mathcal{Y}(T_n) \tag{44}$$

and associated with $T = \mathcal{Y}(T)$.

The following theorem guarantees that the scheme of DTM converges to the solution $y(x)$ of Lane-Emden Equation (1).

Theorem 7 Let \mathcal{Y} be a nonlinear operator from a Banach space \mathcal{X} and $y(x)$ be the solution (exact) of Equation (42). The series solution (14) converges to $y(x)$, if there exists a constant $0 \leq c < 1$ such that $\|y_{k+1}\| \leq c \|y_k\|$ for $k \in \mathbb{Z}^+ \cup \{0\}$.

Proof. We prove that the sequence $\{T_n\}_{n=0}^\infty$ is a Cauchy sequence in \mathcal{X} . Therefore,

$$\|T_n - T_{n+1}\| = \|y_{n+1}\| \leq c \|y_n\| \leq c^2 \|y_{n-1}\| \leq \dots \leq c^{n+1} \|y_0\|$$

Thus, for any $m, n \in \mathbb{Z}^+, m \leq n$,

$$\begin{aligned} \|T_n - T_m\| &\leq \|T_n - T_{n-1}\| + \|T_{n-1} - T_{n-2}\| + \dots + \|T_{m+1} - T_m\| \\ &\leq c^n \|y_0\| + c^{n-1} \|y_0\| + \dots + c^{m+1} \|y_0\| \\ &\leq (c^{m+1} + c^{m+2} + \dots + c^n) \|y_0\| \\ &= c^{m+1} \frac{1 - c^{n-m}}{1 - c} \|y_0\| \end{aligned}$$

so $\|T_n - T_m\| < \frac{c^{m+1}}{1 - c} \|y_0\|$ implying that the sequence $\{T_n\}_{n=0}^\infty$ is Cauchy, i.e. since $0 \leq c < 1$ then

$\lim_{n,m \rightarrow \infty} \|T_n - T_m\| = 0$, therefore there exists $T \in \mathcal{X}$ such that $\lim_{n \rightarrow \infty} T_n = T$, i.e. $T = \sum_{n=0}^\infty y_n$ converges.

Now, we can say too that Equation (42) is similar to solve $T = \mathcal{Y}(T)$, therefore this implies that if \mathcal{Y} is continuous then

$$\mathcal{Y}(T) = \mathcal{Y}\left(\lim_{n \rightarrow \infty} T_n\right) = \lim_{n \rightarrow \infty} \mathcal{Y}(T_n) = \lim_{n \rightarrow \infty} (T_{n+1}) = T$$

i.e. T is a solution of $y(x) = \mathcal{Y}(y(x))$ and this completes the proof.

Figure 5 shows the maximum point-wise error between the numerical solution obtained by using DTM and the Chandrasekhar solution. It is observed that both schemes are almost the same accuracy.

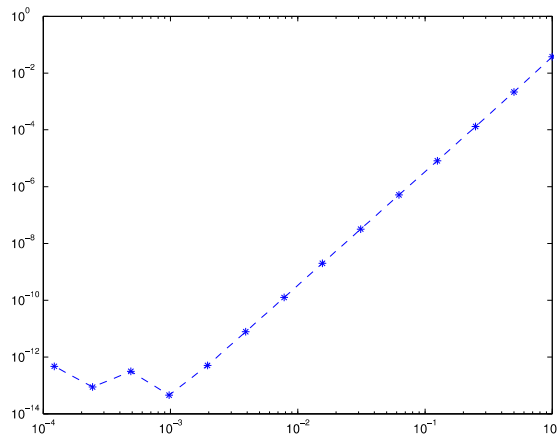


Figure 5. Example 1. $\|e_\infty\|$ error obtained by using DTM.

5. Conclusion

In this work, we presented the definition and handling of one-dimensional differential transformation method. Using DTM, the Lane-Emden equations were transformed into algebraic equations (iterative equations). The new scheme obtained by using DTM yields an analytical solution in the form of a rapidly convergent series. This method makes the solution procedure much more attractive. The figures and tables clearly show the high efficiency of DTM and the convergence of the method for three examples in investigated.

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