

# Nonlinear Interaction of $N$ Conservative Waves in Two Dimensions

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## Abstract

Kinematic Fourier (KF) structures, exponential kinematic Fourier (KEF) structures, dynamic exponential (DEF) Fourier structures, and KEF-DEF structures with constant and space-dependent structural coefficients are developed in the current paper to treat kinematic and dynamic problems for nonlinear interaction of  $N$  conservative waves in the two-dimensional theory of the Newtonian flows with harmonic velocity. The computational method of solving partial differential equations (PDEs) by decomposition in invariant structures, which continues the analytical methods of undetermined coefficients and separation of variables, is extended by using an experimental and theoretical computation in Maple™. For internal waves vanishing at infinity, the Dirichlet problem is formulated for kinematic and dynamics systems of the vorticity, continuity, Helmholtz, Lamb-Helmholtz, and Bernoulli equations in the upper and lower domains. Exact solutions for upper and lower cumulative flows are discovered by the experimental computing, proved by the theoretical computing, and verified by the system of Navier-Stokes PDEs. The KEF and KEF-DEF structures of the cumulative flows are visualized by instantaneous surface plots with isocurves. Modeling of a deterministic wave chaos by aperiodic flows in the KEF, DEF, and KEF-DEF structures with  $5N$  parameters is considered.

## Keywords

Structures, Waves, Computation, Experiment, Theory

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## 1. Introduction

The two-dimensional (2d) Navier-Stokes system of partial differential equations (PDEs) for a Newtonian fluid with a constant density  $\rho$  and a constant kinematic viscosity  $\nu$  in a gravity field  $\mathbf{g}$  is

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p_t + \nu \Delta \mathbf{v} + \mathbf{g}, \quad \nabla \cdot \mathbf{v} = 0, \quad (1-2)$$

where  $\mathbf{v} = (u, 0, w)$  is a vector field of the flow velocity,  $\mathbf{g} = (0, 0, -g_z)$  is a vector field of the gravitational acceleration,  $p_t$  is a scalar field of the total pressure,  $\nabla = (\partial/\partial x, 0, \partial/\partial z)$  and  $\Delta = \partial^2/\partial x^2 + \partial^2/\partial z^2$  are the gradient and the Laplacian in the 2d Cartesian coordinate system  $\mathbf{x} = (x, 0, z)$  of the three-dimensional (3d) space with unit vectors  $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ , respectively, and  $t$  is time.

By a flow vorticity  $\boldsymbol{\omega} = (0, \nu, 0)$  of the velocity field

$$\nabla \times \mathbf{v} = \boldsymbol{\omega}, \quad (3)$$

Equation (1) may be written into the Lamb-Pozrikidis form [1] [2]

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left( \frac{1}{2} \mathbf{v} \cdot \mathbf{v} + \frac{p_t}{\rho} - \mathbf{g} \cdot \mathbf{x} \right) + \boldsymbol{\omega} \times \mathbf{v} + \nu \nabla \times \boldsymbol{\omega} = \mathbf{0}, \quad (4)$$

which sets a dynamic balance of inertial, potential, vortical, and viscous forces, respectively.

Using a dynamic pressure per unit mass [3]

$$p_d = \frac{p_t - p_0}{\rho} - \mathbf{g} \cdot \mathbf{x}, \quad (5)$$

where  $p_0$  is a reference pressure, a kinetic energy per unit mass  $k_e = \mathbf{v} \cdot \mathbf{v}/2$ , the 2d Helmholtz decomposition [4] of the velocity field

$$\mathbf{v} = \nabla \phi + \nabla \times \boldsymbol{\psi} \quad (6)$$

and the vortex force

$$\boldsymbol{\omega} \times \mathbf{v} = \nabla d + \nabla \times \mathbf{a}, \quad (7)$$

Equation (4) is reduced to the Lamb-Helmholtz PDE

$$\nabla b_e + \nabla \times \mathbf{h}_e = 0 \quad (8)$$

for a scalar Bernoulli potential

$$b_e = \frac{\partial \phi}{\partial t} + p_d + k_e + d \quad (9)$$

and a vector Helmholtz potential

$$\mathbf{h}_e = \frac{\partial \boldsymbol{\psi}}{\partial t} + \nu \boldsymbol{\omega} + \mathbf{a}, \quad (10)$$

where  $\phi$  and  $d$  are scalar potentials,  $\boldsymbol{\psi} = (0, \eta, 0)$  and  $\mathbf{a} = (0, b, 0)$  are vector potentials,  $\eta$  and  $b$  are pseudovector potentials of  $\mathbf{v}$  and  $\boldsymbol{\omega} \times \mathbf{v}$ , respectively. The Lamb-Helmholtz PDE (8) means a dynamic balance between potential and vortical forces of the Navier-Stokes PDE (1), which are separated completely.

A linear part of the kinematic problem for free-surface waves of the theory of the ideal fluid with  $\nu = 0$  implies the exponential Fourier eigenfunctions [5], which are obtained by the classical method of separation of variables of the 2d Laplace Equation in [4] and [1]. This analytical method was recently developed into the computational method of solving PDEs by decomposition into invariant structures. In [3], the Boussinesq-Rayleigh-Taylor structures were developed for topological flows away from boundaries. The trigonometric Taylor structures and the trigonometric-hyperbolic structures [6] were used to describe spatiotemporal cascades of exposed and hidden perturbations of the Couette flow, respectively. In [7], the theory of the invariant trigonometric, hyperbolic, and elliptic structures was constructed and applied for modeling dual perturbations of the Poiseuille-Hagen flow.

To treat linear and nonlinear parts of kinematic and dynamic problems for 2d internal waves in the theory of Newtonian flows with harmonic velocity, kinematic Fourier (KF) structures, exponential kinematic Fourier (KEF) structures, dynamic exponential Fourier (DEF) structures, and KEF-DEF structures with constant structural coefficients are developed in the current paper. The structure of this paper is as follows. In section 2, the kinematic problems for velocity components and dual potentials of the velocity field are formulated in upper and

lower domains and treated in the KF and KEF structures. To compute and explore Jacobian determinants (JDs) of the velocity field, the DEF structure is also constructed in this section. In section 3, the dynamic problems for the Bernoulli potential and the total pressure are formulated and computed in the KF, KEF, and KEF-DEF structures. The Navier-Stokes system of PDEs is employed for verification of experimental and theoretical solutions for cumulative upper and lower flows in this section, as well. Visualization and discussion of the developed structures and fluid-dynamic variables is given in section 4, which is followed by a summary of main results in Section 5.

## 2. Kinematic Problems for Conservative Flows

The following solutions and admissible boundary conditions for the kinematic problems of section 2 in the KF and DEF structures were primarily computed experimentally in Maple™ by programming with lists of equations and expressions in the virtual environment of a global variable Eqs with 29 procedures of 670 code lines.

### 2.1. Formulation of Theoretical Kinematic Problems for Velocity Components

Theoretical kinematic problems for harmonic velocity components  $u = u(x, z, t)$  and  $w = w(x, z, t)$  of a cumulative flow  $\mathbf{v} = u\mathbf{i} + w\mathbf{k}$  of a Newtonian fluid are given by vanishing the  $y$ -component of the vorticity Equation (3) and the continuity Equation (2), respectively,

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0, \quad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \quad (11-12)$$

To consider nonlinear interaction of  $N$  internal, conservative waves with a harmonic velocity field, the cumulative flow is decomposed into a superposition of local flows

$$u = \sum_{n=1}^N u_n(x, z, t), \quad w = \sum_{n=1}^N w_n(x, z, t), \quad (13)$$

such that the local vorticity and continuity equations are

$$\frac{\partial u_n}{\partial z} - \frac{\partial w_n}{\partial x} = 0, \quad \frac{\partial u_n}{\partial x} + \frac{\partial w_n}{\partial z} = 0, \quad (14-15)$$

where  $n = 1, 2, \dots, N$ . If Equations (14)-(15) for the local flows are fulfilled, then substitution of superpositions (13) into (11)-(12) and changing order of summation and differentiation yield that Equations (11)-(12) for the cumulative flow are also satisfied.

Upper flows are specified by the Dirichlet condition in the KF structure on a lower boundary  $z = 0$  of an upper domain  $x \in (-\infty, \infty)$  and  $z \in [0, \infty)$  (see **Figure 1**)

$$w_n|_{z=0} = Fw_n ca_n + Gw_n sa_n \quad (16)$$

and a vanishing condition as  $z \rightarrow \infty$

$$w_n|_{z=\infty} = 0. \quad (17)$$

Lower flows are identified by the Dirichlet condition on a lower boundary  $z = 0$  of a lower domain  $x \in (-\infty, \infty)$  and  $z \in (-\infty, 0]$  (see **Figure 1**)

$$w_n|_{z=0} = Fw_n ca_n + Gw_n sa_n \quad (18)$$

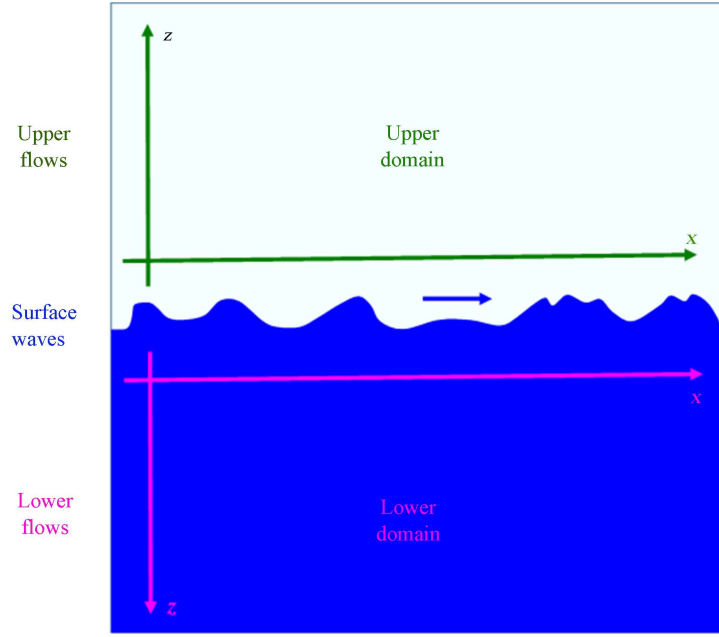
and a vanishing condition as  $z \rightarrow -\infty$

$$w_n|_{z=-\infty} = 0. \quad (19)$$

Thus, an effect of surface waves on the internal waves is described by the Dirichlet conditions (16) and (18). Here, a structural notation

$$ca_n = \cos(\alpha_n), \quad sa_n = \sin(\alpha_n), \quad (20)$$

is used for kinematic structural functions  $ca_n$  and  $sa_n$ , where  $Fw_n$  and  $Gw_n$  are boundary coefficients,  $\alpha_n = \rho_n X_n$  is an argument of the kinematic and dynamic structural functions,  $X_n = x - Cx_n t - Xa_n$  is a propa-



**Figure 1.** Configuration of upper and lower domains for internal, conservative waves.

gation coordinate,  $\rho_n$  is a wavenumber,  $Cx_n$  is a celerity, and  $Xa_n$  is an initial coordinate for all  $n$ .

As we will see later, boundary conditions for  $u_n$  are then redundant since boundary parameters of  $u_n$

$$u_n|_{z=0} = -Gw_nca_n + Fw_nsa_n, \quad u_n|_{z=0} = Gw_nca_n - Fw_nsa_n \quad (21-22)$$

for the upper and lower flows, respectively, depend on boundary parameters of  $w_n$ . Similarly to  $w_n$ ,  $u_n$  vanishes as  $z \rightarrow \pm\infty$

$$u_n|_{z=\infty} = 0, \quad u_n|_{z=-\infty} = 0, \quad (23-24)$$

for the upper and lower flows, respectively.

Thus, the  $x$ - and  $z$ - components of the velocity field of the cumulative flows are expanded in the KF structures with constant structural coefficients

$$u|_{z=0} = \sum_{n=1}^N (-Gw_nca_n + Fw_nsa_n), \quad w|_{z=0} = \sum_{n=1}^N (Fw_nca_n + Gw_nsa_n), \quad (25)$$

$$u|_{z=0} = \sum_{n=1}^N (Gw_nca_n - Fw_nsa_n), \quad w|_{z=0} = \sum_{n=1}^N (Fw_nca_n + Gw_nsa_n), \quad (26)$$

and the velocity components vanish as  $z \rightarrow \pm\infty$

$$u|_{z=\infty} = 0, \quad w|_{z=\infty} = 0, \quad (27)$$

$$u|_{z=-\infty} = 0, \quad w|_{z=-\infty} = 0, \quad (28)$$

for the upper and lower cumulative flows, respectively.

## 2.2. Theoretical Solutions for the Velocity Field

Theoretical solutions of kinematic problems (11)-(28) are constructed in the KF structure  $p(x, z, t)$  of two spatial variables  $x, z$ , and time  $t$  with a general term  $p_n$ , which in the structural notation may be written as

$$p(x, z, t) = \sum_{n=1}^N p_n(x, z, t) = \sum_{n=1}^N (fp_n(z)ca_n + gp_n(z)sa_n), \quad (29)$$

where first letters  $f$  and  $g$  of structural coefficients  $fp_n(z)$  and  $gp_n(z)$  refer to the kinematic structural functions  $ca_n, sa_n$  and a second letter to the expanded variable  $p$ . Thus, general terms of the velocity components of the local flows in the structural notation become

$$u_n = fu_n(z)ca_n + gu_n(z)sa_n, \quad w_n = fw_n(z)ca_n + gw_n(z)sa_n. \quad (30-31)$$

It may be shown that spatial derivatives of  $p_n$  are

$$\frac{\partial p_n}{\partial x} = \rho_n (gp_n(z)ca_n - fp_n(z)sa_n), \quad \frac{\partial p_n}{\partial z} = \frac{dfp_n}{dz}ca_n + \frac{dgp_n}{dz}sa_n. \quad (32-33)$$

Application of (32)-(33) to (30)-(31), substitution in (14)-(15), and collection of the structural functions reduce the vorticity and continuity PDEs to the following system of two vorticity and continuity ordinary differential equations (ODEs) in the KF structures:

$$\left( \frac{dfu_n}{dz} - \rho_n gw_n \right) ca_n + \left( \rho_n fw_n + \frac{dgu_n}{dz} \right) sa_n = 0, \quad \left( \frac{dfw_n}{dz} + \rho_n gu_n \right) ca_n + \left( -\rho_n fu_n + \frac{dgw_n}{dz} \right) sa_n = 0. \quad (34-35)$$

For Equations (34)-(35) to be satisfied exactly for all variables, parameters, and functions of the local flows:  $x, z, t, \alpha_n, \rho_n, fu_n, gu_n, fw_n$ , and  $gw_n$ , all coefficients of two kinematic structural functions must vanish. Thus, two ODEs (34)-(35) are reduced to two systems of ODEs for  $fu_n, gw_n$  and  $fw_n, gu_n$ , respectively:

$$\frac{dfu_n}{dz} - \rho_n gw_n = 0, \quad -\rho_n fu_n + \frac{dgw_n}{dz} = 0, \quad (36-37)$$

$$\rho_n fw_n + \frac{dgu_n}{dz} = 0, \quad \frac{dfw_n}{dz} + \rho_n gu_n = 0. \quad (38-39)$$

Since boundary conditions (25)-(26) are expanded in the KF structure exactly, remainders of structural approximations (34)-(35) vanish, and exact solutions of ODEs (36)-(39) produce exact solutions of vorticity and continuity PDEs (14)-(15). If (25)-(26) are replaced with series approximations, then their remainders constitute errors of the series approximations.

Solutions of ODEs for structural coefficients (36)-(39) are constructed in an exponential structure

$$(fu_n, gu_n, fw_n, gw_n) = (Fu_n, Gu_n, Fw_n, Gw_n)e^{c_n z}, \quad (40)$$

where  $Fu_n, Gu_n, Fw_n, Gw_n$ , and  $c_n$  are structural coefficients. Substitution of exponential structure (40) in Equations (36) and (38) reduces these ODEs to algebraic equations (AEs) for structural parameters:

$$Fu_n = \frac{\rho_n Gw_n}{c_n}, \quad Gu_n = -\frac{\rho_n Fw_n}{c_n}. \quad (41)$$

Substitution of (40) and (41) in (37) and (39) reduces these ODEs to AEs for admissible values of the structural coefficient  $c_n$  with the following solutions for the upper and lower flows, respectively:

$$c_n = -\rho_n, \quad c_n = \rho_n. \quad (42)$$

Since the admissible values of  $c_n$  coincide for Equations (37) and (39), ODEs for structural coefficients (36)-(39) are compatible both for the upper and lower flows.

Finally, substitutions of (40)-(42) in (30)-(31) and (13) yield the velocity components in the KEF structures for the upper cumulative flow

$$u(x, z, t) = \sum_{n=1}^N (-Gw_n ca_n + Fw_n sa_n) e^{-\rho_n z}, \quad w(x, z, t) = \sum_{n=1}^N (Fw_n ca_n + Gw_n sa_n) e^{-\rho_n z}, \quad (43-44)$$

and the lower cumulative flow

$$u(x, z, t) = \sum_{n=1}^N (Gw_n ca_n - Fw_n sa_n) e^{\rho_n z}, \quad w(x, z, t) = \sum_{n=1}^N (Fw_n ca_n + Gw_n sa_n) e^{\rho_n z}, \quad (45-46)$$

while boundary conditions (16)-(19) and (21)-(28) are obviously satisfied.

### 2.3. The DEF structure and Theoretical Jacobian Determinants of the Velocity Components

Define two KEF structures  $l(x, z, t)$  and  $h(x, z, t)$  with general terms  $l_n$  and  $h_m$  by using a generalized Einstein notation for summation, which is extended for exponents,

$$l(x, z, t) = \sum_{n=1}^N l_n = (Fl_n ca_n + Gl_n sa_n) e^{\mp \rho_n z}, \quad h(x, z, t) = \sum_{m=1}^N h_m = (Fh_m ca_m + Gl_m sa_m) e^{\mp \rho_m z}. \quad (47)$$

Computation of a general term  $p_{n,n} = l_n h_n$  by summation of diagonal terms yields

$$p_{n,n} = (Fl_n Fh_n + Gl_n Gh_n + (Fl_n Fh_n - Gl_n Gh_n) Cas_{n,n} + (Fl_n Gh_n + Fh_n Gl_n) Sas_{n,n}) e^{\mp 2 \rho_n z} / 2. \quad (48)$$

Trigonometric structural functions  $Cas_{n,m}$ ,  $Cad_{n,m}$ ,  $Sas_{n,m}$ , and  $Sad_{n,m}$  of the DEF structure are defined by the following expressions:

$$Cad_{n,m} = \cos(\alpha_n - \alpha_m), \quad Cas_{n,m} = \cos(\alpha_n + \alpha_m), \quad Sad_{n,m} = \sin(\alpha_n - \alpha_m), \quad Sas_{n,m} = \sin(\alpha_n + \alpha_m), \quad (49)$$

where capital letters  $C$  and  $S$  stand for dynamic structural functions cosine and sine, letter  $a$  for arguments  $\alpha_n, \alpha_m$ , and letters  $s$  and  $d$  for sum and difference of arguments  $\alpha_n$  and  $\alpha_m$ .

A general term  $p_{n,m} = l_n h_m$  computed by rectangular summation of non-diagonal terms becomes

$$p_{n,m} = ((Fl_n Fh_m + Gl_n Gh_m) Cad_{n,m} + (Fl_n Fh_m - Gl_n Gh_m) Cas_{n,m} + (-Fl_n Gh_m + Fh_m Gl_n) Sad_{n,m} + (Fl_n Gh_m + Fh_m Gl_n) Sas_{n,m}) e^{\mp (\rho_n + \rho_m) z} / 2. \quad (50)$$

By triangular summation,  $p_{n,m}$  is reduced to

$$p_{n,m} = e^{\mp (\rho_n + \rho_m) z} / 2 \times ((Fl_n Fh_m + Fl_m Fh_n + Gl_n Gh_m + Gl_m Gh_n) Cad_{n,m} + (Fl_n Fh_m + Fl_m Fh_n - Gl_n Gh_m - Gl_m Gh_n) Cas_{n,m} + (-Fl_n Gh_m + Fl_m Gh_n - Fh_n Gl_m + Fh_m Gl_n) Sad_{n,m} + (Fl_n Gh_m + Fl_m Gh_n + Fh_n Gl_m + Fh_m Gl_n) Sas_{n,m}). \quad (51)$$

Using general terms (48) and (51), summation formula for the product of the KEF structures is written as the DEF structure

$$p(x, z, t) = l(x, z, t) h(x, z, t) = \frac{1}{2} \sum_{n=1}^N (Fdp_{n,n} + Fsp_{n,n} Cas_{n,n} + Gsp_{n,n} Sas_{n,n}) e^{\mp 2 \rho_n z} + \frac{1}{2} \sum_{n=1}^{N-1} \sum_{m=n+1}^N (Fdp_{n,m} Cad_{n,m} + Fsp_{n,m} Cas_{n,m} + Gdp_{n,m} Sad_{n,m} + Gsp_{n,m} Sas_{n,m}) e^{\mp (\rho_n + \rho_m) z} \quad (52)$$

with the following structural coefficients:

$$\begin{aligned} Fdp_{n,m} &= Fl_n Fh_n + Gl_n Gh_n, & Fsp_{n,n} &= Fl_n Fh_n - Gl_n Gh_n, & Gsp_{n,n} &= Fl_n Gh_n + Fh_n Gl_n, \\ Fdp_{n,m} &= Fl_n Fh_m + Fl_m Fh_n + Gl_n Gh_m + Gl_m Gh_n, & Fsp_{n,m} &= Fl_n Fh_m + Fl_m Fh_n - Gl_n Gh_m - Gl_m Gh_n, \\ Gdp_{n,m} &= -Fl_n Gh_m + Fl_m Gh_n - Fh_n Gl_m + Fh_m Gl_n, & Gsp_{n,m} &= Fl_n Gh_m + Fl_m Gh_n + Fh_n Gl_m + Fh_m Gl_n, \end{aligned} \quad (53)$$

where first two letters  $Fd$ ,  $Fs$ ,  $Gd$ , and  $Gs$  of structural coefficients  $Fdp_{n,m}$ ,  $Fsp_{n,m}$ ,  $Gdp_{n,m}$ , and  $Gsp_{n,m}$  stand for dynamic structural functions  $Cad_n$ ,  $Cas_n$ ,  $Sad_n$ , and  $Sas_n$ , respectively, and a third letter for variable  $p$ .

Computation of local JDs for the velocity components of the upper and lower flow, respectively, yields

$$\frac{\partial u_n}{\partial x} \frac{\partial w_n}{\partial z} - \frac{\partial u_n}{\partial z} \frac{\partial w_n}{\partial x} = -(Fw_n^2 + Gw_n^2) \rho_n^2 e^{\mp 2 \rho_n z}. \quad (54)$$

Thus, velocity components  $u_n$  and  $w_n$  are independent for non-trivial structural coefficients  $Fw_n$  and  $Gw_n$  since the local JDs vanish when  $Fw_n^2 + Gw_n^2 = 0$ .

Computation of a global JD by using (52)-(53) for velocity components of the upper and lower cumulative flows (43)-(46) with slant internal waves gives

$$\begin{aligned}
J_g &= \frac{\partial u}{\partial x} \frac{\partial w}{\partial z} - \frac{\partial u}{\partial z} \frac{\partial w}{\partial x} = -\sum_{n=1}^N (Fw_n^2 + Gw_n^2) \rho_n^2 e^{\mp 2\rho_n z} \\
&+ 2 \sum_{n=1}^{N-1} \sum_{m=n+1}^N \left( -(Fw_n Fw_m + Gw_n Gw_m) Cad_{n,m} \right. \\
&\left. + (Fw_n Gw_m - Fw_m Gw_n) Sad_{n,m} \right) \rho_n \rho_m e^{\mp(\rho_n + \rho_m)z}.
\end{aligned} \tag{55}$$

So,  $J_g$  is a superposition of a propagation JD with general term  $Jc_{n,n}$  proportional to  $Cad_{n,n} \equiv 1$ , an interaction JD with  $Jc_{n,m}$  proportional to  $Cad_{n,m}$ , and an interaction JD with  $Js_{n,m}$  proportional to  $Sad_{n,m}$ , which describe interaction between parallel and orthogonal internal waves, respectively.

$Jc_{n,n}$  coincides with (54). They describe propagation of internal waves and vanish only for internal waves with  $Fw_n^2 + Gw_n^2 = 0$ .  $Js_{n,m}$  vanishes for parallel waves with

$$\frac{Fw_m}{Fw_n} = \frac{Gw_m}{Gw_n} = A_{n,m}. \tag{56}$$

Global JD (55) then becomes

$$J_p = -\sum_{n=1}^N (Fw_n^2 + Gw_n^2) \rho_n^2 e^{\mp 2\rho_n z} - 2 \sum_{n=1}^{N-1} \sum_{m=n+1}^N A_{n,m} (Fw_n^2 + Gw_n^2) \rho_n \rho_m Cad_{n,m} e^{\mp(\rho_n + \rho_m)z}. \tag{57}$$

Thus, the global JD does not vanish for parallel waves with non-vanishing  $Fw_n^2 + Gw_n^2$ .  $Jc_{n,m}$  vanishes for orthogonal waves with

$$\frac{Fw_m}{Gw_n} = -\frac{Gw_m}{Fw_n} = B_{n,m}. \tag{58}$$

In this case, global JD (55) is reduced to

$$J_o = -\sum_{n=1}^N (Fw_n^2 + Gw_n^2) \rho_n^2 e^{\mp 2\rho_n z} - 2 \sum_{n=1}^{N-1} \sum_{m=n+1}^N B_{n,m} (Fw_n^2 + Gw_n^2) \rho_n \rho_m Sad_{n,m} e^{\mp(\rho_n + \rho_m)z}. \tag{59}$$

Thus, the global JD does not vanish also for orthogonal waves with non-vanishing  $Fw_n^2 + Gw_n^2$ . In the general case (55) of slant internal waves, both  $Js_{n,m}$  and  $Jc_{n,m}$  are non-vanishing. So, both propagating and interacting waves are independent for structural coefficients with  $Fw_n^2 + Gw_n^2 \neq 0$  for all  $n$ .

## 2.4. Theoretical Solutions for the Pseudovector and Scalar Potentials in the KEF Structures

Theoretical kinematic problems for cumulative pseudo-vector potential  $\eta(x, z, t)$  and cumulative scalar potential  $\phi(x, z, t)$  of  $\mathbf{v}$  are set by the global Helmholtz PDEs (6)

$$\frac{\partial \eta}{\partial z} + u = 0, \quad \frac{\partial \eta}{\partial x} - w = 0, \tag{60}$$

$$\frac{\partial \phi}{\partial x} - u = 0, \quad \frac{\partial \phi}{\partial z} - w = 0, \tag{61}$$

since the potential-vortical duality the velocity field admits two presentations:  $\mathbf{v} = \nabla \phi$  for  $\boldsymbol{\psi} = \mathbf{0}$  and  $\mathbf{v} = \nabla \times \boldsymbol{\psi}$  for  $\phi = 0$ . The cumulative kinematic potentials are decomposed into a superposition of local kinematic potentials

$$\eta = \sum_{n=1}^N \eta_n(x, z, t), \quad \phi = \sum_{n=1}^N \phi_n(x, z, t), \tag{62}$$

such that the local Helmholtz PDEs are

$$\frac{\partial \eta_n}{\partial z} + u_n = 0, \quad \frac{\partial \eta_n}{\partial x} - w_n = 0, \tag{63}$$

$$\frac{\partial \phi_n}{\partial x} - u_n = 0, \quad \frac{\partial \phi_n}{\partial z} - w_n = 0, \tag{64}$$

where  $n = 1, 2, \dots, N$ . The boundary conditions for  $\eta_n$  and  $\phi_n$  and redundant when the problem is formulated in the KF structures.

Construct general terms of the kinematic potentials of the local flows in the KF structure with space-dependent coefficients

$$\eta_n = fe_n(z)ca_n + ge_n(z)sa_n, \quad \phi_n = fp_n(z)ca_n + gp_n(z)sa_n. \quad (65-66)$$

Application of (32)-(33) to (65)-(66), substitution in (63)-(64), and collection of the structural functions reduce four Helmholtz PDEs to the following system of two Helmholtz ODEs and two Helmholtz AEs for the upper flows

$$\left(\frac{dfe_n}{dz} - Gw_n e^{-\rho_n z}\right)ca_n + \left(\frac{dge_n}{dz} + Fw_n e^{-\rho_n z}\right)sa_n = 0, \quad (\rho_n ge_n - Fw_n e^{-\rho_n z})ca_n - (\rho_n fe_n + Gw_n e^{-\rho_n z})sa_n = 0, \quad (67-68)$$

$$(\rho_n gp_n + Gw_n e^{-\rho_n z})ca_n - (\rho_n fp_n + Fw_n e^{-\rho_n z})sa_n = 0, \quad \left(\frac{dfp_n}{dz} - Fw_n e^{-\rho_n z}\right)ca_n + \left(\frac{dgp_n}{dz} - Gw_n e^{-\rho_n z}\right)sa_n = 0, \quad (69-70)$$

and the lower flows

$$\left(\frac{dfe_n}{dz} + Gw_n e^{\rho_n z}\right)ca_n + \left(\frac{dge_n}{dz} - Fw_n e^{\rho_n z}\right)sa_n = 0, \quad (\rho_n ge_n - Fw_n e^{\rho_n z})ca_n - (\rho_n fe_n + Gw_n e^{\rho_n z})sa_n = 0, \quad (71-72)$$

$$(\rho_n gp_n - Gw_n e^{\rho_n z})ca_n - (\rho_n fp_n - Fw_n e^{\rho_n z})sa_n = 0, \quad \left(\frac{dfp_n}{dz} - Fw_n e^{\rho_n z}\right)ca_n + \left(\frac{dgp_n}{dz} - Gw_n e^{\rho_n z}\right)sa_n = 0. \quad (73-74)$$

For Equations (67)-(74) to be satisfied exactly for all variables, parameters, and functions of the upper and lower flows:  $x, z, t, \alpha_n, \rho_n, fe_n, ge_n, fp_n, gp_n, Fw_n$ , and  $Gw_n$ , all coefficients of structural functions  $ca_n$  and  $sa_n$  must vanish. Thus, two Helmholtz ODEs and two Helmholtz AEs are reduced to the following four AEs and four ODEs with respect to  $fe_n$ ,  $ge_n$ ,  $fp_n$ , and  $gp_n$  for the upper flows

$$\rho_n fe_n + Gw_n e^{-\rho_n z} = 0, \quad \rho_n ge_n - Fw_n e^{-\rho_n z} = 0, \quad \rho_n fp_n + Fw_n e^{-\rho_n z} = 0, \quad \rho_n gp_n + Gw_n e^{-\rho_n z} = 0, \quad (75)$$

$$\frac{dfe_n}{dz} - Gw_n e^{-\rho_n z} = 0, \quad \frac{dge_n}{dz} + Fw_n e^{-\rho_n z} = 0, \quad \frac{dfp_n}{dz} - Fw_n e^{-\rho_n z} = 0, \quad \frac{dgp_n}{dz} - Gw_n e^{-\rho_n z} = 0, \quad (76)$$

and the lower flows

$$\rho_n fe_n + Gw_n e^{\rho_n z} = 0, \quad \rho_n ge_n - Fw_n e^{\rho_n z} = 0, \quad \rho_n fp_n - Fw_n e^{\rho_n z} = 0, \quad \rho_n gp_n - Gw_n e^{\rho_n z} = 0, \quad (77)$$

$$\frac{dfe_n}{dz} + Gw_n e^{\rho_n z} = 0, \quad \frac{dge_n}{dz} - Fw_n e^{\rho_n z} = 0, \quad \frac{dfp_n}{dz} - Fw_n e^{\rho_n z} = 0, \quad \frac{dgp_n}{dz} - Gw_n e^{\rho_n z} = 0. \quad (78)$$

Since general terms of remainders of structural approximations (67)-(74) vanish, exact solutions of AEs and ODEs (75)-(78) produce exact solutions of the Helmholtz PDEs (63)-(64).

Solving AEs (75) and (77) with respect to  $fe_n$ ,  $ge_n$ ,  $fp_n$ , and  $gp_n$  gives for the upper flows

$$fe_n = -\frac{Gw_n}{\rho_n} e^{-\rho_n z}, \quad ge_n = \frac{Fw_n}{\rho_n} e^{-\rho_n z}, \quad fp_n = -\frac{Fw_n}{\rho_n} e^{-\rho_n z}, \quad gp_n = -\frac{Gw_n}{\rho_n} e^{-\rho_n z}, \quad (79)$$

and the lower flows

$$fe_n = -\frac{Gw_n}{\rho_n} e^{\rho_n z}, \quad ge_n = \frac{Fw_n}{\rho_n} e^{\rho_n z}, \quad fp_n = \frac{Fw_n}{\rho_n} e^{\rho_n z}, \quad gp_n = \frac{Gw_n}{\rho_n} e^{\rho_n z}. \quad (80)$$

Substitution of solutions (79)-(80) in ODEs (76) and (78) reduces them to identities.

Substitution of structural coefficients (79)-(80) in the KF structures (65)-(66) and super positions (62) returns the cumulative pseudo vector and scalar potentials in the KEF structures for the upper cumulative flow

$$\eta(x, z, t) = \sum_{n=1}^N \frac{1}{\rho_n} (-Gw_n ca_n + Fw_n sa_n) e^{-\rho_n z}, \quad \phi(x, z, t) = -\sum_{n=1}^N \frac{1}{\rho_n} (Fw_n ca_n + Gw_n sa_n) e^{-\rho_n z}, \quad (81-82)$$



and the lower cumulative flow

$$\eta(x, z, t) = \sum_{n=1}^N \frac{1}{\rho_n} (-Gw_n c a_n + Fw_n s a_n) e^{\rho_n z}, \quad \phi(x, z, t) = \sum_{n=1}^N \frac{1}{\rho_n} (Fw_n c a_n + Gw_n s a_n) e^{\rho_n z}. \quad (83-84)$$

## 2.5. Harmonic Relationships for the Velocity Components and the Kinematic Potentials

Comparison of solutions for  $u_n$  and  $w_n$  with spatial derivatives in  $x$  of  $w_n$  and  $u_n$  shows that they are directly proportional to each other, respectively, for the upper flows

$$u_n = -\frac{1}{\rho_n} \frac{\partial w_n}{\partial x}, \quad w_n = \frac{1}{\rho_n} \frac{\partial u_n}{\partial x}, \quad (85)$$

and the lower flows

$$u_n = \frac{1}{\rho_n} \frac{\partial w_n}{\partial x}, \quad w_n = -\frac{1}{\rho_n} \frac{\partial u_n}{\partial x}. \quad (86)$$

In fluid dynamics, these connections mean that a non-uniform vertical flow generates a horizontal flow and a non-uniform horizontal flow produces a vertical flow.

Similarly, comparison of solutions for  $\eta_n$  and  $\phi_n$  with solutions for  $u_n$  and  $w_n$  shows that they are also directly proportional, respectively, for the upper flows

$$\eta_n = \frac{u_n}{\rho_n}, \quad \phi_n = -\frac{w_n}{\rho_n}, \quad (87)$$

and the lower flows

$$\eta_n = -\frac{u_n}{\rho_n}, \quad \phi_n = \frac{w_n}{\rho_n}. \quad (88)$$

Finally, comparison of solutions for  $\eta_n$  and  $\phi_n$  with spatial derivatives in  $x$  of  $\phi_n$  and  $\eta_n$  shows that they are proportional to each other, respectively, for the upper flows

$$\eta_n = \frac{1}{\rho_n} \frac{\partial \phi_n}{\partial x}, \quad \phi_n = -\frac{1}{\rho_n} \frac{\partial \eta_n}{\partial x}, \quad (89)$$

and the lower flows

$$\eta_n = -\frac{1}{\rho_n} \frac{\partial \phi_n}{\partial x}, \quad \phi_n = \frac{1}{\rho_n} \frac{\partial \eta_n}{\partial x}. \quad (90)$$

Connections (85)-(90) between solutions in the KEF structures are available since there are only two independent combinations of trigonometric structural functions  $Fw_n c a_n + Gw_n s a_n$  and  $Gw_n c a_n - Fw_n s a_n$ .

Computation of  $\nabla \eta_n \cdot \nabla \phi_n$  by using (81)-(84) both for the upper and lower flows gives

$$\frac{\partial \eta_n}{\partial x} \frac{\partial \phi_n}{\partial x} + \frac{\partial \eta_n}{\partial z} \frac{\partial \phi_n}{\partial z} = 0. \quad (91)$$

Thus, local isocurves of  $\eta_n$  and  $\phi_n$  remain orthogonal for all times in agreement with the Helmholtz Equations (63)-(64). Similarly, local isocurves of  $u_n$  and  $w_n$  remain orthogonal since both for the upper and lower flows

$$\frac{\partial u_n}{\partial x} \frac{\partial w_n}{\partial x} + \frac{\partial u_n}{\partial z} \frac{\partial w_n}{\partial z} = 0, \quad (92)$$

in agreement with the local vorticity and continuity Equations (14)-(15).

Computation of  $\nabla \eta \cdot \nabla \phi$  by (52)-53) and (81)-(84) both for the upper and lower cumulative flows gives

$$\frac{\partial \eta}{\partial x} \frac{\partial \phi}{\partial x} + \frac{\partial \eta}{\partial z} \frac{\partial \phi}{\partial z} = 0. \quad (93)$$

Thus, global isocurves of  $\eta$  and  $\phi$  also remain orthogonal for all times in agreement with the cumulative Helmholtz Equations (60)-(61). Finally, global isocurves of  $u$  and  $w$  remain orthogonal since both for the

upper and lower cumulative flows

$$\frac{\partial u}{\partial x} \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \frac{\partial w}{\partial z} = 0, \quad (94)$$

in agreement with the cumulative vorticity and continuity Equations (11)-(12).

It is a straightforward matter to show that for the KEF structure  $p(x, z, t)$  with a general term  $p_n$

$$p(x, z, t) = \sum_{n=1}^N p_n = (Fp_n c a_n + Gp_n s a_n) e^{\mp \rho_n z}, \quad (95)$$

spatial derivatives of second order in the  $x$ - and  $z$ - directions are

$$\frac{\partial^2 p_n}{\partial x^2} = -\rho_n^2 (Fp_n c a_n + Gp_n s a_n) e^{\mp \rho_n z}, \quad \frac{\partial^2 p_n}{\partial z^2} = \rho_n^2 (Fp_n c a_n + Gp_n s a_n) e^{\mp \rho_n z}, \quad (96-97)$$

and the Laplacian of  $p_n$  vanishes. Thus, the KEF structure is an invariant, harmonic structure both for the upper and lower flows.

Application of (96)-(97) to (43)-(46) shows that  $u_n$  and  $w_n$  are conjugate harmonic functions since

$$\frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n}{\partial z^2} = 0, \quad \frac{\partial^2 w_n}{\partial x^2} + \frac{\partial^2 w_n}{\partial z^2} = 0 \quad (98)$$

both for the upper and lower flows, in agreement with vector identity  $\Delta \mathbf{v}_n = -\nabla \times \boldsymbol{\omega}_n + \nabla(\nabla \cdot \boldsymbol{\omega}_n) = 0$ . By Equations (13),  $u$  and  $w$  are also conjugate harmonic functions

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} = 0, \quad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} = 0, \quad (99)$$

both for the upper and lower cumulative flows, in agreement with vector identity  $\Delta \mathbf{v} = -\nabla \times \boldsymbol{\omega} + \nabla(\nabla \cdot \boldsymbol{\omega}) = 0$ .

Similarly, applying (96)-(97) to (81)-(84) shows that  $\eta_n$  and  $\phi_n$  are conjugate harmonic functions as

$$\frac{\partial^2 \eta_n}{\partial x^2} + \frac{\partial^2 \eta_n}{\partial z^2} = 0, \quad \frac{\partial^2 \phi_n}{\partial x^2} + \frac{\partial^2 \phi_n}{\partial z^2} = 0 \quad (100)$$

both for the upper and lower flows, in agreement with  $\nabla \cdot \mathbf{v}_n = \nabla \cdot \nabla \phi_n = \Delta \phi_n = 0$  and  $\nabla \times \mathbf{v}_n = \nabla \times (\nabla \times \boldsymbol{\psi}_n) = \nabla(\nabla \cdot \boldsymbol{\psi}_n) - \Delta \boldsymbol{\psi}_n = -\Delta \boldsymbol{\psi}_n = 0$ . By Equation (62),  $\eta$  and  $\phi$  are also conjugate harmonic functions

$$\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial z^2} = 0, \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad (101)$$

both for the upper and lower cumulative flows, in agreement with vector identities  $\nabla \cdot \mathbf{v} = \nabla \cdot \nabla \phi = \Delta \phi = 0$  and

$$\nabla \times \mathbf{v} = \nabla \times (\nabla \times \boldsymbol{\psi}) = \nabla(\nabla \cdot \boldsymbol{\psi}) - \Delta \boldsymbol{\psi} = -\Delta \boldsymbol{\psi} = 0.$$

The theoretical solutions in the KEF and DEF structures for the kinematic problems of section 2 were computed theoretically in Maple™ by programming with symbolic general terms in the virtual environment of a global variable Equation with 26 procedures of 591 code lines. The theoretical solutions for velocity components (43)-(46), the products of the KEF structures (52)-(53), and the kinematic potentials (81)-(84) of the upper and lower cumulative flows were justified by the correspondent experimental solutions for  $N = 1, 3, 10$ .

### 3. Dynamic Problems for Conservative Flows

The following solutions for the dynamic problems of section 3 in the KF, DEF, and KEF-DEF structures were primarily computed experimentally by programming with lists of equations and expressions in the virtual environment of the global variable Equations with 19 procedures of 472 code lines.

#### 3.1. Theoretical Solutions for the Helmholtz and Bernoulli Potentials in the KEF Structures

Theoretical dynamic problems in the KF structures for the Helmholtz and Bernoulli potentials of the cumu-

lative flows are set by the Lamb-Helmholtz PDEs (8)

$$\frac{\partial b_e}{\partial x} - \frac{\partial h_e}{\partial z} = 0, \quad \frac{\partial b_e}{\partial z} + \frac{\partial h_e}{\partial x} = 0, \quad (102-103)$$

while (10) for the vortical presentation with  $\phi = 0$  is reduced to

$$h_e = \frac{\partial \eta}{\partial t}. \quad (104)$$

Equations (102-104) are complemented by the local Lamb-Helmholtz PDEs

$$\frac{\partial b e_n}{\partial x} - \frac{\partial h e_n}{\partial z} = 0, \quad \frac{\partial b e_n}{\partial z} + \frac{\partial h e_n}{\partial x} = 0, \quad (105-106)$$

where

$$h e_n = \frac{\partial \eta_n}{\partial t}, \quad (107)$$

since the cumulative dynamic potentials are again decomposed into the local dynamic potentials as follows:

$$h_e = \sum_{n=1}^N h e_n(x, z, t), \quad b_e = \sum_{n=1}^N b e_n(x, z, t). \quad (108)$$

Boundary conditions are again redundant because the problem is formulated in the KF structures.

Construct a general term of the Bernoulli potential of the local flows in the KF structure with space-dependent coefficients

$$b e_n = f b_n(z) c a_n + g b_n(z) s a_n. \quad (109)$$

Computation of the temporal derivative of  $\eta_n$ , application of (32)-(33), substitution in (105)-(106), and collection of the structural functions reduce two Lamb-Helmholtz PDEs to the following system of the Lamb-Helmholtz AE and ODE for the upper flows

$$\begin{aligned} \rho_n (g b_n - C x_n F w_n e^{-\rho_n z}) c a_n - \rho_n (f b_n + C x_n G w_n e^{-\rho_n z}) s a_n &= 0, \\ \left( \frac{d f b_n}{d z} - \rho_n C x_n G w_n e^{-\rho_n z} \right) c a_n + \left( \frac{d g b_n}{d z} + \rho_n C x_n F w_n e^{-\rho_n z} \right) s a_n &= 0, \end{aligned} \quad (110)$$

and the lower flows

$$\begin{aligned} \rho_n (g b_n + C x_n F w_n e^{\rho_n z}) c a_n - \rho_n (f b_n - C x_n G w_n e^{\rho_n z}) s a_n &= 0, \\ \left( \frac{d f b_n}{d z} - \rho_n C x_n G w_n e^{\rho_n z} \right) c a_n + \left( \frac{d g b_n}{d z} + \rho_n C x_n F w_n e^{\rho_n z} \right) s a_n &= 0. \end{aligned} \quad (111)$$

For Equations (110)-(111) to be satisfied exactly for all  $x, z, t, \alpha_n, \rho_n, C x_n, f b_n, g b_n, F w_n$ , and  $G w_n$  all coefficients of structural functions  $c a_n$  and  $s a_n$  must vanish. Thus, the Lamb-Helmholtz AE and ODE are reduced to the following two AEs and two ODEs for space-dependent structural coefficients  $f b_n$  and  $g b_n$  for the upper flows

$$\begin{aligned} f b_n + C x_n G w_n e^{-\rho_n z} &= 0, \quad g b_n - C x_n F w_n e^{-\rho_n z} = 0, \\ \frac{d f b_n}{d z} - \rho_n C x_n G w_n e^{-\rho_n z} &= 0, \quad \frac{d g b_n}{d z} + \rho_n C x_n F w_n e^{-\rho_n z} = 0, \end{aligned} \quad (112-113)$$

and the lower flows

$$\begin{aligned} f b_n - C x_n G w_n e^{\rho_n z} &= 0, \quad g b_n + C x_n F w_n e^{\rho_n z} = 0, \\ \frac{d f b_n}{d z} - \rho_n C x_n G w_n e^{\rho_n z} &= 0, \quad \frac{d g b_n}{d z} + \rho_n C x_n F w_n e^{\rho_n z} = 0. \end{aligned} \quad (114-115)$$

Since general terms of remainders of structural approximations (110)-(111) vanish, exact solutions of (112)-(115) produce exact solutions of (105)-(106).

Solving AEs (112) and (114) for structural coefficients  $fb_n$  and  $gb_n$  yields for the upper flows

$$fb_n = -Cx_n Gw_n e^{-\rho_n z}, \quad gb_n = Cx_n Fw_n e^{-\rho_n z}, \quad (116)$$

and the lower flows

$$fb_n = Cx_n Gw_n e^{\rho_n z}, \quad gb_n = -Cx_n Fw_n e^{\rho_n z}. \quad (117)$$

Substitution of solutions (116)-(117) in ODEs (113) and (115) reduced them to identities.

Substitution of structural coefficients (116)-(117) in super positions (108) and the KF structure (109) gives the cumulative Helmholtz and Bernoulli potentials in the KEF structures for the upper cumulative flow

$$h_e(x, z, t) = -\sum_{n=1}^N Cx_n (Fw_n ca_n + Gw_n sa_n) e^{-\rho_n z}, \quad b_e(x, z, t) = \sum_{n=1}^N Cx_n (-Gw_n ca_n + Fw_n sa_n) e^{-\rho_n z}, \quad (118-119)$$

and the lower cumulative flow

$$h_e(x, z, t) = -\sum_{n=1}^N Cx_n (Fw_n ca_n + Gw_n sa_n) e^{\rho_n z}, \quad b_e(x, z, t) = \sum_{n=1}^N Cx_n (Gw_n ca_n - Fw_n sa_n) e^{\rho_n z}. \quad (120-121)$$

Similar to the kinematic potentials (87)-(88), the dynamic potentials and the velocity components are directly proportional both for the upper and lower flows

$$he_n = -Cx_n w_n, \quad be_n = Cx_n u_n. \quad (122)$$

Like in (89)-(90), the Helmholtz and Bernoulli potentials and derivatives of the Bernoulli and Helmholtz potentials in  $x$  are directly proportional to each other both for the upper flows

$$he_n = -\frac{1}{\rho_n} \frac{\partial be_n}{\partial x}, \quad be_n = \frac{1}{\rho_n} \frac{\partial he_n}{\partial x}, \quad (123)$$

and the lower flows

$$he_n = \frac{1}{\rho_n} \frac{\partial be_n}{\partial x}, \quad be_n = -\frac{1}{\rho_n} \frac{\partial he_n}{\partial x}. \quad (124)$$

Analogous to (91)-(94), isocurves of  $he_n, be_n$  and global isocurves of  $h_e, b_e$  are orthogonal for all times

$$\frac{\partial he_n}{\partial x} \frac{\partial be_n}{\partial x} + \frac{\partial he_n}{\partial z} \frac{\partial be_n}{\partial z} = 0, \quad \frac{\partial h_e}{\partial x} \frac{\partial b_e}{\partial x} + \frac{\partial h_e}{\partial z} \frac{\partial b_e}{\partial z} = 0. \quad (125-126)$$

in agreement with the Lamb-Helmholtz Equations (105)-(106) and (102)-(103). For the same reason,  $he_n, be_n$  and  $h_e, b_e$  are local and global conjugate harmonic functions as

$$\frac{\partial^2 he_n}{\partial x^2} + \frac{\partial^2 he_n}{\partial z^2} = 0, \quad \frac{\partial^2 be_n}{\partial x^2} + \frac{\partial^2 be_n}{\partial z^2} = 0, \quad \frac{\partial^2 h_e}{\partial x^2} + \frac{\partial^2 h_e}{\partial z^2} = 0, \quad \frac{\partial^2 b_e}{\partial x^2} + \frac{\partial^2 b_e}{\partial z^2} = 0. \quad (127-128)$$

### 3.2. Theoretical Solutions for the Total Pressure in the KEF-DEF Structures

Theoretical dynamic problems in the KEF-DEF structures for the kinetic energy per unit mass  $k_e$ , the dynamic pressure per unit mass  $p_d$ , and the total pressure  $p_t$  of the cumulative flows are formulated by definition

$$k_e(x, z, t) = \frac{1}{2} (u(x, z, t)^2 + w(x, z, t)^2), \quad (129)$$

the Bernoulli Equation (9) with  $\phi = 0$

$$p_d(x, z, t) = b_e(x, z, t) - k_e(x, z, t), \quad (130)$$

and the hydrostatic Equation (5)

$$p_t(x, z, t) = p_0 - \rho g_z z + \rho p_d(x, z, t), \quad (131)$$

where  $p_0$  is the reference pressure at  $z = 0$ .

Computation of  $k_e$  by (52)-(53) and (43)-(46) returns

$$k_e(x, z, t) = \frac{1}{2} \sum_{n=1}^N (Fw_n^2 + Gw_n^2) e^{\mp 2\rho_n z} + \sum_{n=1}^{N-1} \sum_{m=n+1}^N \left( (Fw_n Fw_m + Gw_n Gw_m) Cad_{n,m} + (Fw_m Gw_n - Fw_n Gw_m) Sad_{n,m} \right) e^{\mp(\rho_n + \rho_m)z} \quad (132)$$

for the upper and lower cumulative flows, respectively. Substitution of (119), (121), and (132) in (131) yields

$$p_t(x, z, t) = p_0 - \rho g_z z + \rho \left[ \mp \sum_{n=1}^N Cx_n (Gw_n ca_n - Fw_n sa_n) e^{\mp \rho_n z} - \frac{1}{2} \sum_{n=1}^N (Fw_n^2 + Gw_n^2) e^{\mp 2\rho_n z} - \sum_{n=1}^{N-1} \sum_{m=n+1}^N \left( (Fw_n Fw_m + Gw_n Gw_m) Cad_{n,m} + (Fw_m Gw_n - Fw_n Gw_m) Sad_{n,m} \right) e^{\mp(\rho_n + \rho_m)z} \right] \quad (133)$$

for the upper and lower cumulative flows, respectively.

### 3.3. Theoretical Verification by the System of Navier-Stokes PDEs

The system of the Navier-Stokes PDEs (1)-(2) in the scalar notation becomes

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p_t}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right), \quad \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p_t}{\partial z} + \nu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \right) - g_z, \quad (134-135)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \quad (136)$$

Computation of spatial derivatives of (43)-(46) by (32)-(33) immediately reduces (136) to identity. Temporal derivatives of  $\mathbf{v}$  in the KEF structures for the upper and lower cumulative flows, respectively, are

$$\begin{aligned} \frac{\partial u}{\partial t} &= \mp \sum_{n=1}^N Cx_n (Fw_n ca_n + Gw_n sa_n) \rho_n e^{\mp \rho_n z}, \\ \frac{\partial w}{\partial t} &= \sum_{n=1}^N Cx_n (-Gw_n ca_n + Fw_n sa_n) \rho_n e^{\mp \rho_n z}. \end{aligned} \quad (137-138)$$

The directional derivatives of (134)-(135) computed by (52)-(53) in the DEF structures for the upper and lower cumulative flows, respectively, become

$$(\mathbf{v} \cdot \nabla) u = \sum_{n=1}^{N-1} \sum_{m=n+1}^N \left( (Fw_n Gw_m - Fw_m Gw_n) Cad_{n,m} + (Fw_n Fw_m + Gw_n Gw_m) Sad_{n,m} \right) (\rho_m - \rho_n) e^{\mp(\rho_n + \rho_m)z}, \quad (139)$$

$$(\mathbf{v} \cdot \nabla) w = \mp \sum_{n=1}^N (Fw_n^2 + Gw_n^2) \rho_n e^{\mp 2\rho_n z} \mp \sum_{n=1}^{N-1} \sum_{m=n+1}^N \left( (Fw_n Fw_m + Gw_n Gw_m) Cad_{n,m} + (Fw_m Gw_n - Fw_n Gw_m) Sad_{n,m} \right) (\rho_m + \rho_n) e^{\mp(\rho_n + \rho_m)z}. \quad (140)$$

By using (32) and (33), components of the gradient of (133) may be written in the KEF-DEF structures for the upper and lower cumulative flows, respectively, as

$$\begin{aligned} \frac{\partial p_t}{\partial x} &= \rho \left[ \pm \sum_{n=1}^N Cx_n (Fw_n ca_n + Gw_n sa_n) \rho_n e^{\mp \rho_n z} - \sum_{n=1}^{N-1} \sum_{m=n+1}^N \left( (Fw_n Gw_m - Fw_m Gw_n) Cad_{n,m} + (Fw_n Fw_m + Gw_n Gw_m) Sad_{n,m} \right) (\rho_m - \rho_n) e^{\mp(\rho_n + \rho_m)z} \right], \end{aligned} \quad (141)$$

$$\begin{aligned}
\frac{\partial p_t}{\partial z} = & \rho \left[ -g_z + \sum_{n=1}^N Cx_n (Gw_n ca_n - Fw_n sa_n) \rho_n e^{\mp \rho_n z} \pm \sum_{n=1}^N (Fw_n^2 + Gw_n^2) \rho_n e^{\mp 2\rho_n z} \right. \\
& \pm \sum_{n=1}^{N-1} \sum_{m=n+1}^N ((Fw_n Fw_m + Gw_n Gw_m) Cad_{n,m} \\
& \left. + (Fw_m Gw_n - Fw_n Gw_m) Sad_{n,m}) (\rho_m + \rho_n) e^{\mp (\rho_n + \rho_m) z} \right].
\end{aligned} \tag{142}$$

Substitution of Equations (137)-(142) and (99) in (134)-(135) reduces then to identities. Thus, Equations (43)-(46) and (133) constitute exact solutions in the KEF, DEF, and KEF-DEF structures for interaction of  $N$  internal waves both in the upper and lower domains.

The theoretical solutions in the KEF, DEF, and KEF-DEF structures for the dynamic problems of section 3 were computed theoretically by programming with symbolic general terms in the virtual environment of the global variable Equation with 15 procedures of 405 code lines. The theoretical solutions for the Helmholtz and Bernoulli potentials (118)-(121), the total pressure (133), the temporal derivatives (137)-(138), the directional derivatives (139)-(140), and the pressure gradient (141)-(142) of the upper and lower cumulative flows were justified by the correspondent experimental solutions for  $N = 1, 3, 10$ .

#### 4. Visualization and Discussion

The Fourier series with eigenfunctions  $\cos(n\lambda x)$  and  $\sin(n\lambda x)$ , where  $n = 1, 2, \dots, N$  is an integer, model a periodic function with a constant period  $P_x$  and a wavenumber  $\lambda = 2\pi/P_x$  [4]. The trigonometric structural functions  $ca_n$  and  $sa_n$  of the KF, KEF, DEF, and KEF-DEF structures coincide with the Fourier eigenfunctions if  $\rho_n = n\lambda$ . When  $\rho_n = \lambda/p_n$ , where  $p_n = 2, 3, 5, 7, 11, \dots$  is a prime number,  $ca_n$  and  $sa_n$  model a function with a period approaching infinity as  $n \rightarrow \infty$  [6]. For instance, if a sequence of  $\rho_n$  is

$$1/2, 1/3, 1/5, 1/7, 1/11, 1/13, 1/17, \tag{143}$$

local periods of the structural functions grow as  $2\pi/\rho_n$ :

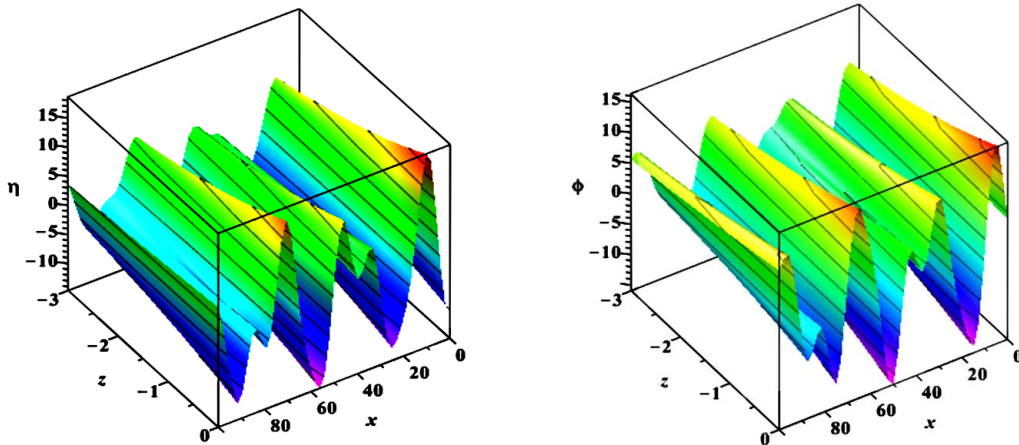
$$4\pi, 6\pi, 10\pi, 14\pi, 22\pi, 26\pi, 34\pi, \tag{144}$$

and a global period of the interaction solution (43)-(46) increases as  $2\pi \prod_{n=1}^N \rho_n^{-1}$ :

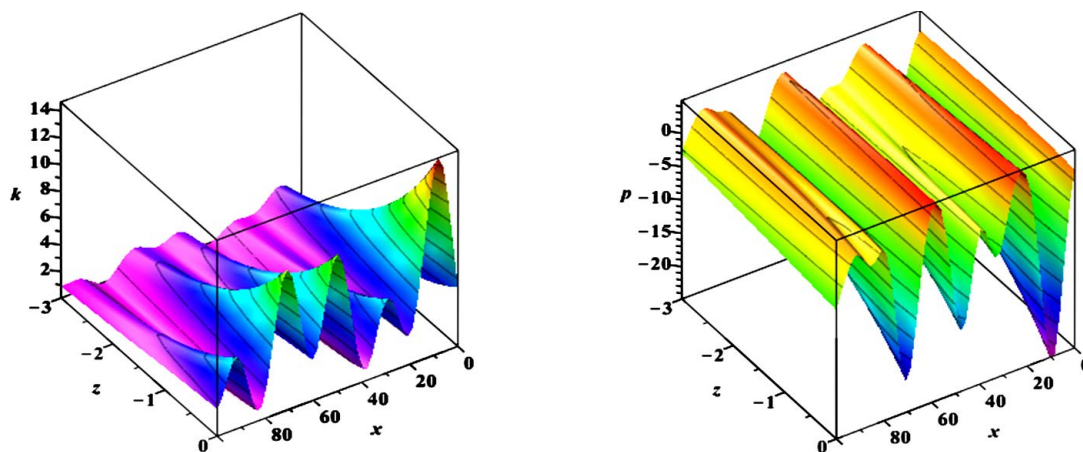
$$4\pi, 12\pi, 60\pi, 420\pi, 4620\pi, 60060\pi, 1021020\pi. \tag{145}$$

The KEF structures of conjugate harmonic solutions are visualized in **Figure 2** by instantaneous 3d surface plots with isocurves for  $\eta(83)$  and  $\phi(84)$ , for  $N = 3$ ;  $\rho_n = 1/2, 1/3, 1/5$ ;  $Cx_n = 1, 2, 3$ ;  $Xa_n = 3, 2, 1$ ;  $Fw_n = 1.2, 1.4, 1.6$ ; and  $Gw_n = 1.1, 1.3, 1.5$  at  $t = 114.2$ . In two dimensions, the pseudovector potential coincides with the stream function and isocurves of  $\eta$  coincides with streamlines [2].

The DEF and KEF-DEF structures are visualized in **Figure 3** by instantaneous 3d surface plots with isocurves



**Figure 2.** Pseudovector potential  $\eta$  and scalar potential  $\phi$  of the lower cumulative flow.



**Figure 3.** Kinetic energy (left) and dynamic pressure (right) of the lower cumulative flow.

for  $k_e$  (132) and  $p_d = g_z z + (p_t - p_0)/\rho$ , where  $p_t$  is given by (133), for  $N = 3$ ;  $\rho_n = 1/2, 1/3, 1/5$ ;  $Cx_n = 1, 2, 3$ ;  $Xa_n = 3, 2, 1$ ;  $Fw_n = 1.2, 1.4, 1.6$ ; and  $Gw_n = 1.1, 1.3, 1.5$  at  $t = 114.2$ . In agreement with the Bernoulli Equation [1], local maximums of the DEF structure for  $k_e$  correspond to local minimums of the KEF-DEF structure for  $p_d$ .

The rate of vanishing of the DEF structure is larger than that of the KEF structure. Animations of  $\eta, \varphi, k_e$ , and  $p_d$  show a transitional behavior of these variables that approach a deterministic chaos, which is determined by  $5N$  parameters:  $\rho_n, Cx_n, Xa_n, Fw_n$ , and  $Gw_n$ , as  $N \rightarrow \infty$ .

## 5. Conclusions

The analytical methods of undetermined coefficients and separation of variables are extended by the computational method of solving 2d PDEs by decomposition in invariant structures. The method is developed by the experimental computing with lists of equations and expressions and the theoretical computing with symbolic general terms. The experimental computing of the kinematic and dynamic problems is implemented by 48 procedures of 1142 code lines and the theoretical computing by 41 procedures of 996 code lines.

To compute the upper and cumulative flows for nonlinear interaction of  $N$  internal waves in the KF structures, the KEF, DEF, and KEF-DEF structures were treated both experimentally and theoretically. These structures with constant and space-dependent structural coefficients are invariant with respect to various differential and algebraic operations. The structures continue the Fourier series for linear and nonlinear problems with solutions vanishing at infinity and model flows of a deterministic wave chaos with the period that approaches infinity.

The exact solutions of the Navier-Stokes PDEs for the nonlinear interaction of  $N$  conservative waves are computed in the upper and lower domains by formulating and solving the Dirichlet problem for the vorticity, continuity, Helmholtz, Lamb-Helmholtz, and Bernoulli equations. The conservative waves are not affected by dissipation since they are derived in the class of flows with the harmonic velocity field. The harmonic relationships between fluid-dynamic variables and their spatial derivatives with respect to  $x$  both for upper and lower flows are obtained.

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