

Bifurcations of Travelling Wave Solutions for the B(m,n) Equation

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Abstract

Using the bifurcation theory of dynamical systems to a class of nonlinear fourth order analogue of the B(m,n) equation, the existence of solitary wave solutions, periodic cusp wave solutions, compactons solutions, and uncountably infinite many smooth wave solutions are obtained. Under different parametric conditions, various sufficient conditions to guarantee the existence of the above solutions are given. Some exact explicit parametric representations of the above waves are determined.

Keywords

Solitary Wave Solution; Periodic Cusp Wave Solution; Periodic Wave Solution; Smoothness of Wave; B(m,n) Equation

1. Introduction

Recently, Song and Shao [1] employed bifurcation method of dynamical systems to investigate bifurcation of solitary waves of the following generalized (2 + 1)-dimensional Boussinesq equation

$$u_{tt} - \alpha u_{xx} - \beta u_{yy} - \gamma (u^2)_{xx} - \delta u_{xxxx} = 0, \quad (1.1)$$

where α, β, γ and δ are arbitrary constants with $\gamma\delta \neq 0$. Chen and Zhang [2] obtained some double periodic and multiple soliton solutions of Equation (1.1) by using the generalized Jacobi elliptic function method. Further, Li [3] studied the generalized Boussinesq equation:

$$\zeta_t + [(1 + \zeta_t)u]_x = au_{xxxx} \quad (1.2)$$

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by using bifurcation method. In this paper, we shall employ bifurcation method of dynamical systems [4]-[11] to investigate bifurcation of solitary waves of the following equation:

$$(u^m)_{tt} = (u^n)_{xx} + (u^n)_{yy} + a(u^2)_{xx} + \delta(u^n)_{xxxx}, \tag{1.3}$$

Numbers of solitary waves are given for each parameter condition. Under some parameter conditions, exact solitary wave solutions will be obtained. It is very important to consider the dynamical bifurcation behavior for the travelling wave solutions of (1.3). In this paper, we shall study all travelling wave solutions in the parameter space of this system. Let $u(x, y, t) = \phi(x + y - ct) = \phi(\xi)$, where c is the wave speed. Then (1.3) becomes to

$$c^2(\phi^m)'' = 2(\phi^n)'' + a(\phi^2)'' + b(u^n)^{(4)}, \tag{1.4}$$

where “'” is the derivative with respect to ξ . Integrating Equation (1.4) twice, using the constants of integration to be zero we find

$$q\phi^m + p\phi^n + \phi^2 + r[n(n-1)\phi^{n-2}(\phi')^2 + n\phi^{n-1}\phi''] = 0, \tag{1.5}$$

where $p = \frac{2}{a}, q = -\frac{c^2}{a}, r = \frac{b}{a}$. Equation (1.5) is equivalent to the two-dimensional systems as follows

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = -\frac{q\phi^m + p\phi^n + \phi^2 + rn(n-1)\phi^{n-2}y^2}{m\phi^{n-1}} \tag{1.6}$$

with the first integral

$$H(\phi, y) = \frac{1}{2}m\phi^{2(n-1)} + \phi^{n+2} \left[\frac{q}{n+m}\phi^{m-2} + \phi^2 + \frac{p}{2n}\phi^{n-2} + \frac{1}{2+n} \right] = h. \tag{1.7}$$

System (1.6) is a 5-parameter planar dynamical system depending on the parameter group (m, n, p, q, r) . For different m, n and a fixed r , we shall investigate the bifurcations of phase portraits of System (1.6) in the phase plane (ϕ, y) as the parameters p, q are changed. Here we are considering a physical model where only bounded travelling waves are meaningful. So we only pay attention to the bounded solutions of System (1.6).

2. Bifurcations of Phase Portraits of (1.6)

In this section, we study all possible periodic annuluses defined by the vector fields of (1.6) when the parameters p, q are varied.

Let $d\xi = m\phi^{n-1}d\zeta$, Then, except on the straight lines $\phi = 0$, the system (1.6) has the same topological phase portraits as the following system

$$\frac{d\phi}{d\zeta} = m\phi^{n-1}, \quad \frac{dy}{d\zeta} = -[p\phi^n + \phi^2 + rn(n-1)\phi^{n-2}y^2] \tag{2.1}$$

Now, the straight lines $\phi = 0$ is an integral invariant straight line of (2.1).

Denote that

$$f(\phi) = 1 + q\phi^{m-2} + p\phi^{n-2}, \quad f'(\phi) = \phi^{n-3}[q(M-2)\phi^{m-n} + p(n-2)] \tag{2.2}$$

For $m - n = 2l (l \in Z^+), m - 1 = 2m_1 - 1, n - 1 = 2n_1 - 1,$

When $\phi = \phi_0 = \left[-\frac{p(n-2)}{q(m-2)} \right]^{\frac{1}{m-n}}, f'(\pm\phi_0) = 0.$

We have $f(\pm\phi_0) = 1 + q \left[-\frac{p(n-2)}{q(m-2)} \right]^{\frac{m-2}{m-n}} + p \left[-\frac{p(n-2)}{q(m-2)} \right]^{\frac{n-2}{m-n}}$

and which imply respectively the relations in the (p, q) -parameter plane

$$L_a : q = -\frac{n-2}{m-2} p^{\frac{m-2}{n-2}} \left(\frac{m-n}{m-2}\right)^{\frac{m-n}{n-2}}, p > 0, q < 0,$$

$$L_b : q = \frac{n-2}{m-2} (-p)^{\frac{m-2}{n-2}} \left(\frac{m-n}{m-2}\right)^{\frac{m-n}{n-2}}, p < 0, q > 0,$$

For $m-n=2l$ ($l \in Z^+$), $m-1=2m_1$, $n-1=2n_1-1$, when $\phi = \phi_0 = \left[-\frac{p(n-2)}{q(m-2)}\right]^{\frac{1}{m-n}}$, $f'(\phi_0)=0$. We have

$f(\phi_0) = 1 + q \left[-\frac{p(n-2)}{q(m-2)}\right]^{\frac{m-2}{m-n}} + p \left[-\frac{p(n-2)}{q(m-2)}\right]^{\frac{n-2}{m-n}}$ and which imply respectively the relations in the (p, q) -parameter plane

$$L_b : q = \frac{n-2}{m-2} (-p)^{\frac{m-2}{n-2}} \left(\frac{m-n}{m-2}\right)^{\frac{m-n}{n-2}}, p < 0, q > 0,$$

$$L_c : q = -\frac{n-2}{m-2} (-p)^{\frac{m-2}{n-2}} \left(\frac{m-n}{m-2}\right)^{\frac{m-n}{n-2}}, p < 0, q < 0,$$

For $m-n=2l-1$ ($l \in Z^+$), $m-1=2m_1-1$, $n-1=2n_1$, when $\phi = \phi_0 = \left[-\frac{p(n-2)}{q(m-2)}\right]^{\frac{1}{m-n}}$, $f'(\phi_0)=0$. We have

$f(\phi_0) = 1 + q \left[-\frac{p(n-2)}{q(m-2)}\right]^{\frac{m-2}{m-n}} + p \left[-\frac{p(n-2)}{q(m-2)}\right]^{\frac{n-2}{m-n}}$, which imply respectively the relations in the (p, q) -parameter plane

$$L_d : q = -\frac{n-2}{m-2} p^{\frac{m-2}{n-2}} \left(\frac{m-n+4}{m-2}\right)^{\frac{m-n}{n-2}}.$$

For $m-n=2l-1$ ($l \in Z^+$), $m-1=2m_1$, $n-1=2n_1$, when $\phi = \phi_0 = \left[-\frac{p(n-2)}{q(m-2)}\right]^{\frac{1}{m-n}}$, $f'(\pm\phi_0)=0$. We have

$f(\phi_0) = 1 + q \left[-\frac{p(n-2)}{q(m-2)}\right]^{\frac{m-2}{m-n}} + p \left[-\frac{p(n-2)}{q(m-2)}\right]^{\frac{n-2}{m-n}}$ and $f(-\phi_0) = 1 - q \left[-\frac{p(n-2)}{q(m-2)}\right]^{\frac{m-2}{m-n}} - p \left[-\frac{p(n-2)}{q(m-2)}\right]^{\frac{n-2}{m-n}}$,

which imply respectively the relations in the (p, q) -parameter plane

$$L_e : q = -\frac{n-2}{m-2} p^{\frac{m-2}{n-2}} \left(\frac{m-n}{m-2}\right)^{\frac{m-n}{n-2}}, pq < 0.$$

Let $M(\phi_e, y_e)$ be the coefficient matrix of the linearized system of (2.1) at an equilibrium point (ϕ_e, y_e) . Then, we have

$$J(\phi_e, 0) = \det(M(\phi_e, 0)) = rn\phi_e^{n-3} (q(m-2)\phi_e^{m-3} + p(n-2)\phi_e^{n-3}).$$

By the theory of planar dynamical systems, we know that for an equilibrium point of a planar integrable system, if $J < 0$ then the equilibrium point is a saddle point; if $J > 0$ and $Trace(M(\phi_e, y_e)) = 0$ then it is a center point; if $J > 0$ and $(Trace(M(\phi_e, y_e)))^2 - 4J(\phi_e, y_e) > 0$, then it is a node; if $J = 0$ and the index of the equilibrium point is 0 then it is a cusp, otherwise, it is a high order equilibrium point. For the function defined by (1.7), we denote that

$$h_i = H(\phi_i, 0) = \phi_i^{n+2} \left[\frac{m-2}{(n-2)(m+n)} + \frac{p(m-n)}{2n(m+n)} \phi_i^{n-2} \right], i = 1-4.$$

We next use the above statements to consider the bifurcations of the phase portraits of (2.1). In the (p, q) parameter plane, the curves partition it into 4 regions for $m-n=2l-1, m-n=2l$ shown in **Figure 1** (1-1), (1-2), (1-3), and (1-4), respectively.

1) The case $q \neq 0$, We use **Figure 2, Figure 3, Figure 4, and Figure 5** to show the bifurcations of the phase portraits of (2.1).

2) The case $q = 0$. We consider the system

$$\frac{d\phi}{d\zeta} = m\phi^{n-1}, \quad \frac{dy}{d\zeta} = -[p\phi^n + \phi^2 + m(n-1)\phi^{n-2}y^2] \tag{2.3}$$

with the first integral

$$H(\phi, y) = -\frac{1}{2}m\phi^{2(n-1)}y^2 - \phi^{n+2} \left[\frac{p}{2n}\phi^{n-2} + \frac{1}{2+n} \right] = h. \tag{2.4}$$

Figure 6 and Figure 7 show respectively the phase portraits of (2.3) for $n = 2n_1$ and $n = 2n_1 + 1$.

3. Exact Explicit Parametric Representations of Traveling Wave Solutions of (1.6)

In this section, we give some exact explicit parametric representations of periodic cusp wave solutions.

1). Suppose that $n = 4, m = 6, r < 0, (p, q) \in A_4$, In this case, we have the phase portrait of (2.1) shown in **Figure 2**

(2-5). Corresponding to the orbit defined by $H(\phi, y) = 0$ to the equilibrium point $S_{\pm} \left(\pm \sqrt{\frac{-p + \sqrt{p^2 - 4q}}{2q}}, 0 \right)$, the

arch curve has the algebraic equation

$$y^2 = \frac{1}{4(-r)} \left[\phi^2 - \frac{-5p - 5\sqrt{\frac{p^2 - 4q}{16} - \frac{4q}{15}}}{2q} \right] \left[\phi^2 - \frac{5p - 5\sqrt{\frac{p^2 - 4q}{16} - \frac{4q}{15}}}{2q} \right]. \tag{3.1}$$

Thus, by using the first Equation of (1.6) and (3.1), we obtain the parametric representation of this arch as follows:

$$\phi(\xi) = \pm \sqrt{\frac{-5p - 5\sqrt{\frac{p^2 - 4q}{16} - \frac{4q}{15}}}{2q}} cn^{-1}(\Omega_1 \xi, k_1), \tag{3.2}$$

where $\Omega_1 = \sqrt{\frac{5\sqrt{\frac{p^2 - 4q}{16} - \frac{4q}{15}}}{4qr}}, k_1 = \sqrt{\frac{\frac{p}{4} - \sqrt{\frac{p^2 - 4q}{16} - \frac{4q}{15}}}{-2\sqrt{\frac{p^2 - 4q}{16} - \frac{4q}{15}}}}$.

We will show in Section 4 that (3.10) gives rise to two periodic cusp wave solutions of peak type and valley type of (1.3).

2). Suppose that $n = 2, m = 4, r > 0, (p, q) \in A_3$, In this case, we have the phase portrait of (2.1) shown in **Figure 2** (2-4). corresponding to the orbit defined by $H(\phi, y) = 0$ to the equilibrium point $A(0, 0)$, the arch curve has the algebraic equation

$$y^2 = \frac{1}{r} \phi^2 \left(\frac{q}{6} \phi^2 + \frac{p+1}{4} \right), \tag{3.3}$$

Thus, by using the first equation of (1.6) and (3.3), we obtain the parametric representation of this arch as follows:

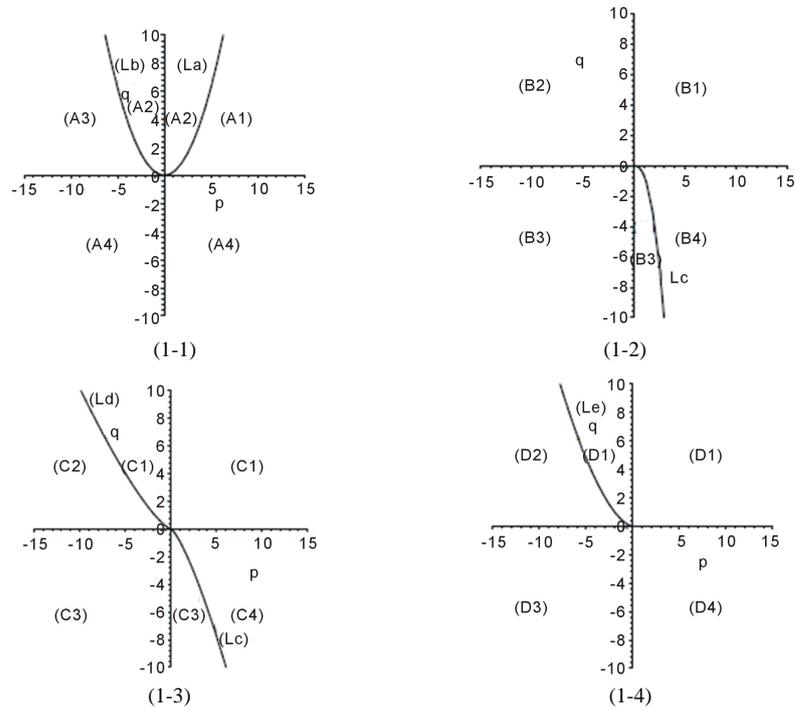


Figure 1. (1-1) $m - n = 2l, n = 2n_1$; (1-2) $m - n = 2l - 1, n = 2n_1$; (1-3) $m - n = 2l, n = 2n_1 + 1$; (1-4) $m - n = 2l - 1, n = 2n_1 + 1$.

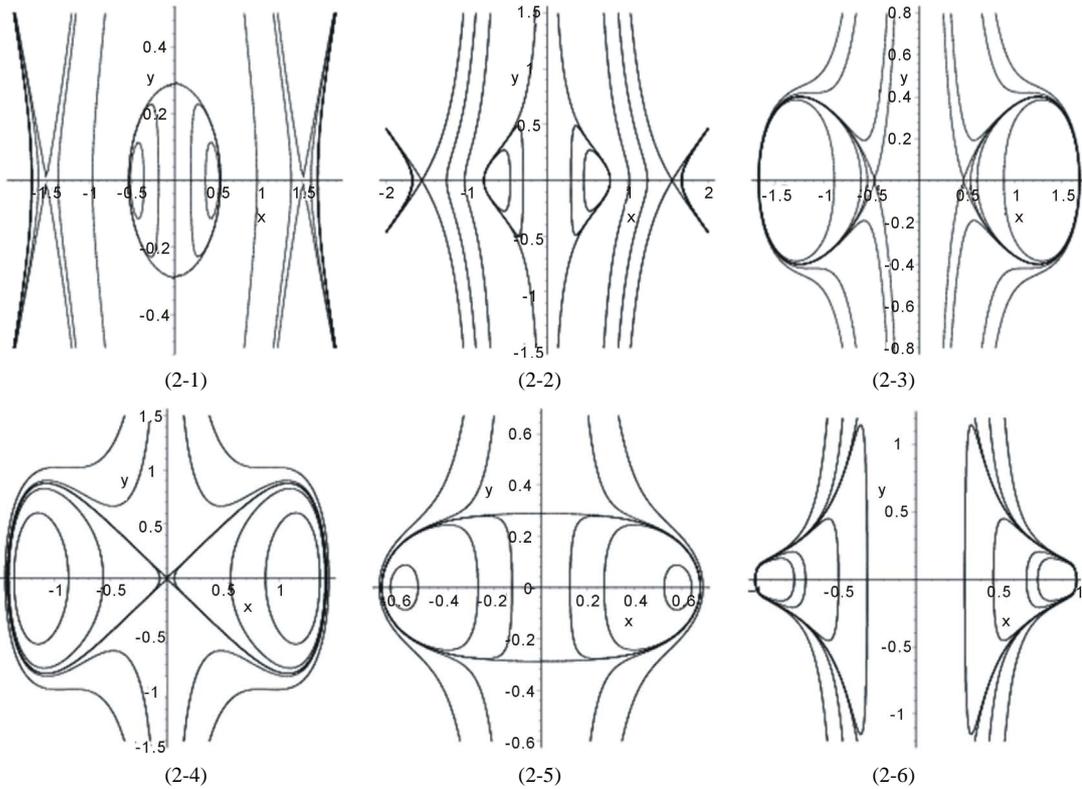


Figure 2. The phase portraits of (1.6) for $m - n = 2l, n = 2n_1, l, n_1 \in \mathbb{Z}^+$. (2-1) $r < 0, n_1 = 2, (p, q) \in (A_3)$; (2-2) $r < 0, n_1 \geq 2, (p, q) \in (A_3)$; (2-3) $r > 0, n_1 \geq 2, (p, q) \in (A_3)$; (2-4) $r > 0, n_1 = 1, (p, q) \in (A_3)$; (2-5) $r < 0, n_1 = 2, (p, q) \in (A_4)$; (2-6) $r < 0, n_1 \geq 3, (p, q) \in (A_3)$.

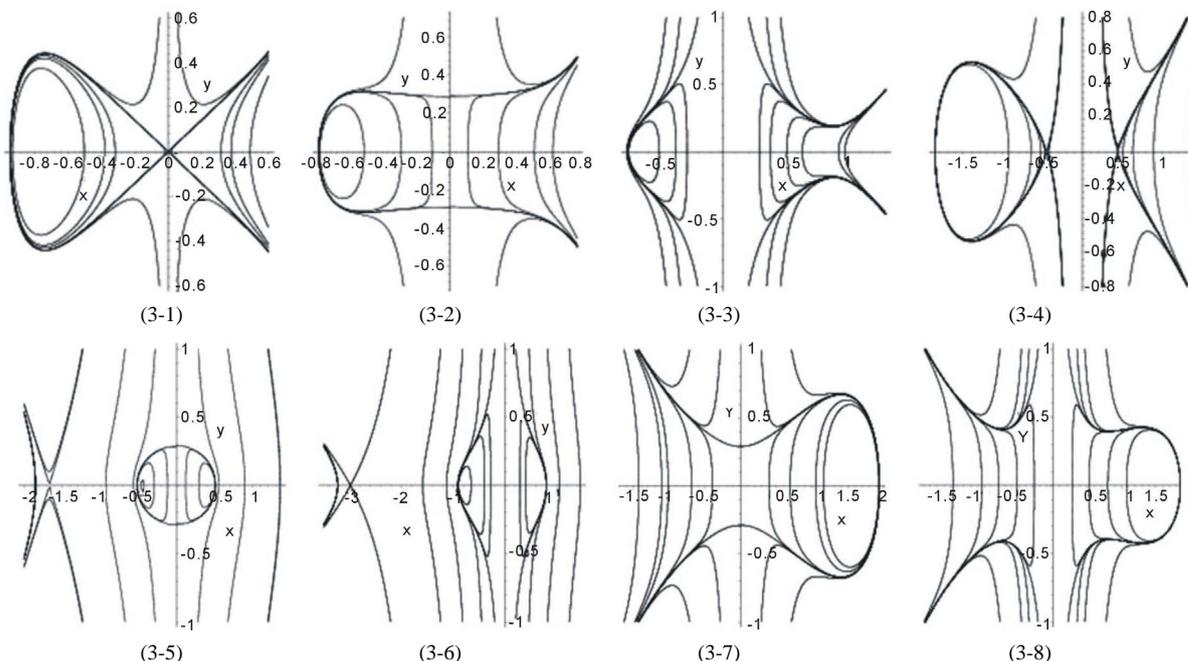


Figure 3. The phase portraits of (1.6) for $m - n = 2l - 1, n = 2n_1, 1, n_1 \in \mathbb{Z}^+$. (3-1) $r < 0, n_1 = 1, (p, q) \in (B_1)$; (3-2) $r < 0, n_1 = 2, (p, q) \in (B_1 \cup B_2)$; (3-3) $r > 0, n_1 \geq 2, (p, q) \in (B_1) \cup (B_2)$; (3-4) $r > 0, n_1 \geq 2, (p, q) \in (B_3)$; (3-5) $r < 0, n_1 = 2, (p, q) \in (B_3)$; (3-6) $r < 0, n_1 \geq 2, (p, q) \in (B_1 \cup B_2)$; (3-7) $r < 0, n_1 = 2, (p, q) \in (B_4) \cup (B_2)$; (3-8) $r > 0, n_1 \geq 2, (p, q) \in (B_4)$.

$$\phi(\xi) = \sqrt{\frac{2q}{3(p+1)}} \sin^2 \left(\sqrt{\frac{-q}{6r}} \xi \right), \tag{3.4}$$

We will show in Section 4 that (3.10) gives rise to a solitary wave solutions of peak type and valley type of (1.3).

3). Suppose that $n = 3, m = 5, r < 0, (p, q) \in C_2$. In this case, we have the phase portrait of (2.1) shown in **Figure 4** (4-5). corresponding to the orbit defined by $H(\phi, y) = 0$ to the equilibrium point $A(0, 0)$, the arch curve has the algebraic equation

$$y^2 = \frac{2}{3(-r)} (\phi - \phi_1)(\phi - \phi_2)(\phi_3 - \phi)(\phi_4 - \phi), \tag{3.5}$$

where $\phi_1 < \phi_2 < \phi_3 < \phi_4, \phi_i \left(\frac{q}{8} \phi_i^3 + \frac{p}{6} \phi_i + \frac{1}{5} \right) = 0, i = 1 - 4$.

Thus, by using the first equation of (1.6) and (3.5), we obtain the parametric representation of this arch as follows:

$$\phi(\xi) = \frac{(\phi_4 - \phi_1)\phi_2 - \phi_1(\phi_4 - \phi_2)sn^2(\Omega_2\xi; k_2)}{(\phi_4 - \phi_1) - (\phi_4 - \phi_2)sn^2(\Omega_2\xi; k_2)}, \tag{3.6}$$

where $sn(x; k)$ is the Jacobin elliptic functions with the modulo k ,

$$\Omega_2 = \sqrt{\frac{(\phi_3 - \phi_2)(\phi_3 - \phi_1)}{-6r}}, \quad k_2 = \sqrt{\frac{(\phi_3 - \phi_2)(\phi_4 - \phi_1)}{(\phi_4 - \phi_2)(\phi_3 - \phi_1)}}$$

We will show in Section 4 that (3.6) gives rise to a smooth compacton solution of (1.3).

4). Suppose that $n = 2, m = 3, r < 0, (p, q) \in B_1$. In this case, we have the phase portrait of (2.1) shown in **Figure 3** (3-1), corresponding to the orbit defined by $H(\phi, y) = 0$ to the equilibrium point $A(0, 0)$, the arch curve has the algebraic equation

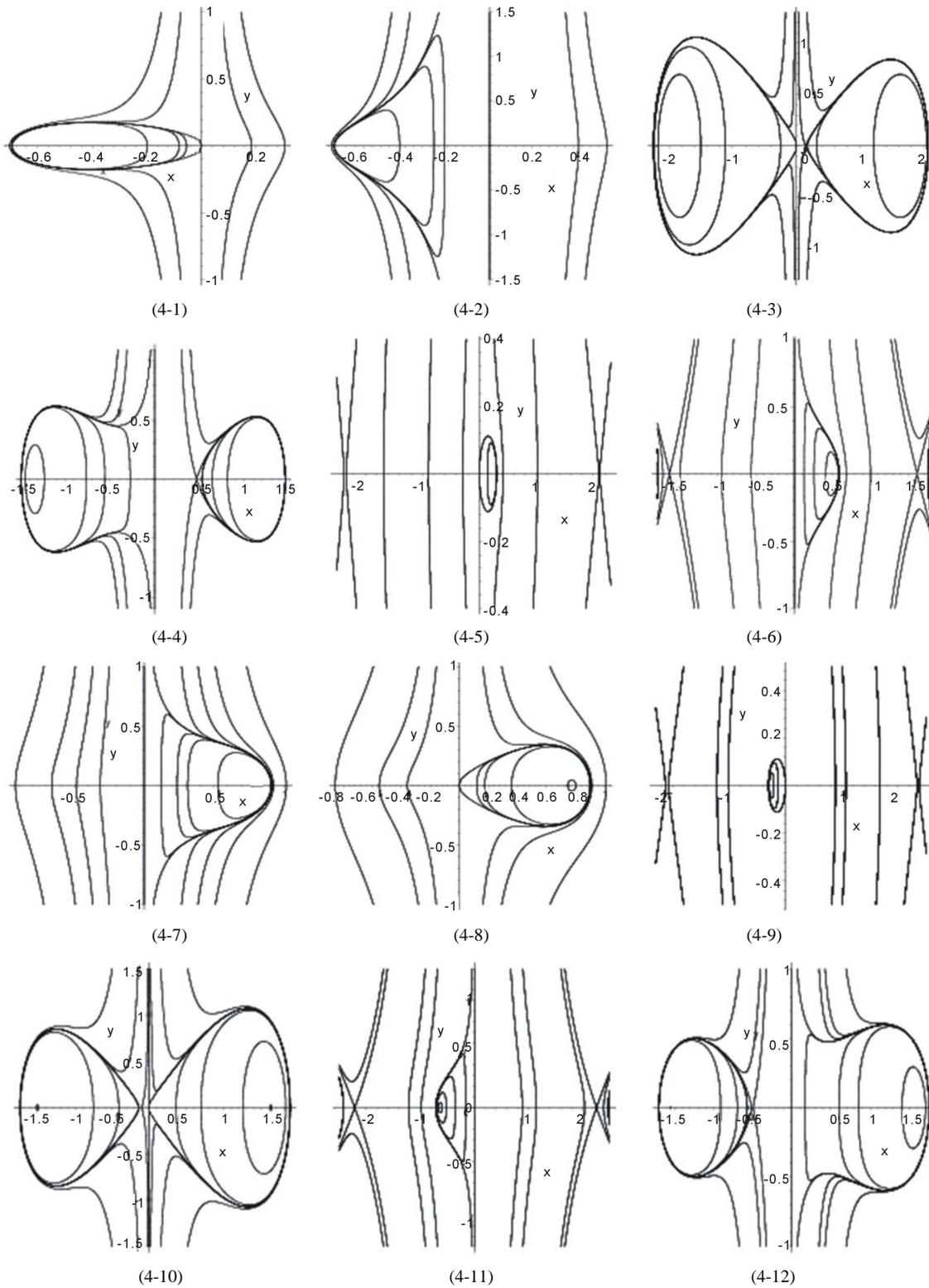


Figure 4. (4-1) $r > 0, n_1 = 1, (p, q) \in (C_1)$; (4-2) $r > 0, n_1 \geq 1, (p, q) \in (C_2)$; (4-3) $r > 0, n_1 = 1, (p, q) \in (C_2)$; (4-4) $r > 0, n_1 \geq 2, (p, q) \in (C_2)$; (4-5) $r < 0, n_1 = 1, (p, q) \in (C_2)$; (4-6) $r < 0, n_1 \geq 2, (p, q) \in (C_2)$; (4-7) $r > 0, n_1 \geq 2, (p, q) \in (C_3)$; (4-8) $r < 0, n_1 = 1, (p, q) \in (C_3)$; (4-9) $r > 0, n_1 = 1, (p, q) \in (C_4)$; (4-10) $r < 0, n_1 = 1, (p, q) \in (C_4)$; (4-11) $r > 0, n_1 \geq 2, (p, q) \in (C_4)$; (4-12) $r < 0, n_1 \geq 2, (p, q) \in (C_4)$.

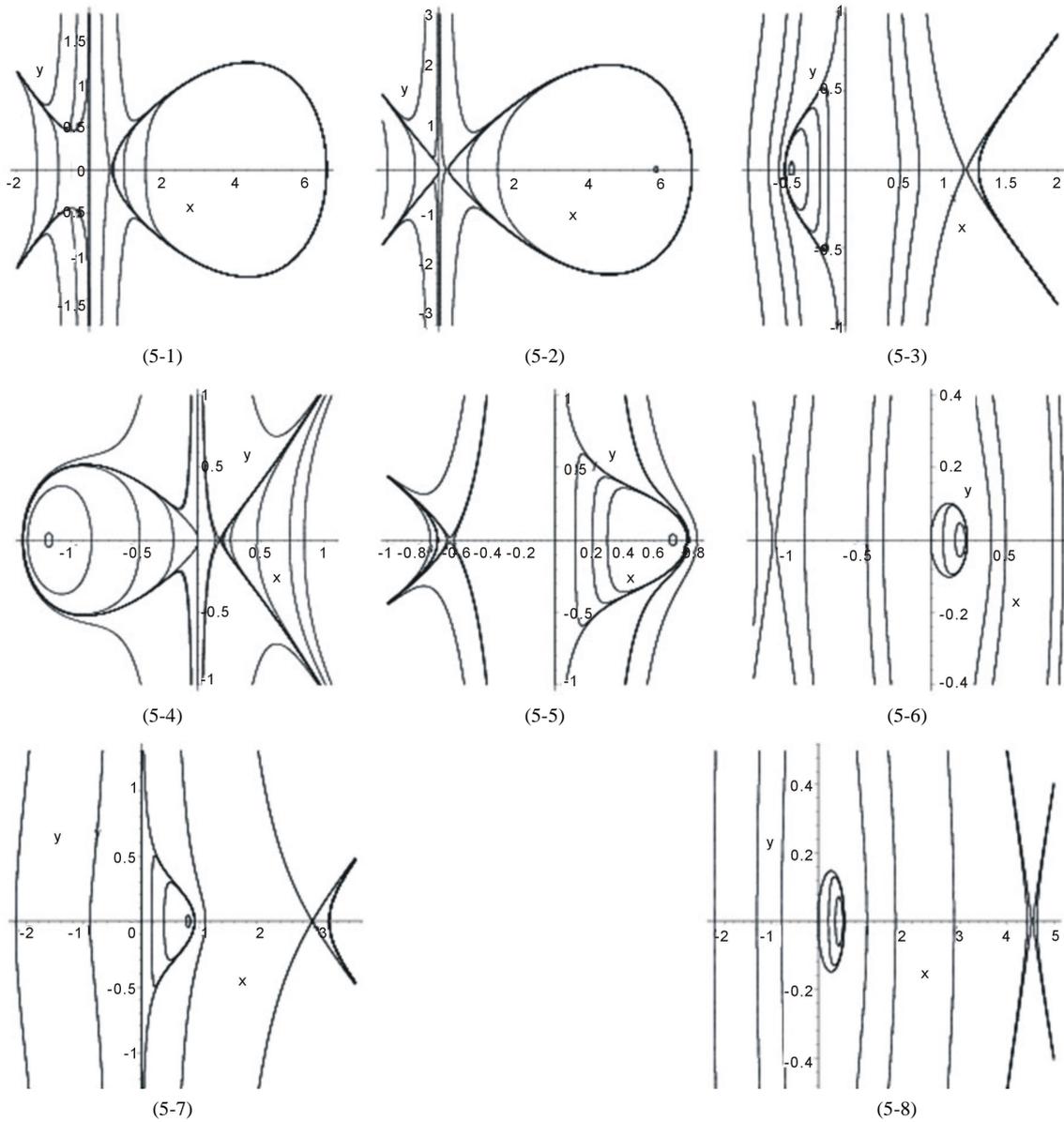


Figure 5. The phase portraits of (1.6) for $m - n = 2l - 1$, $n = 2n_1 + 1$, $l, n_1 \in \mathbb{Z}^+$ (5-1) $r > 0$, $n_1 \geq 2$, $(p, q) \in (D_2)$; (5-2) $r > 0$, $n_1 = 1$, $(p, q) \in (D_2)$; (5-3) $r > 0$, $n_1 \geq 2$, $(p, q) \in (D_3) \cup (D_4)$; (5-4) $r > 0$, $n_1 = 1$, $(p, q) \in (D_3) \cup (D_4)$; (5-5) $r < 0$, $n_1 \geq 2$, $(p, q) \in (D_3) \cup (D_4)$; (5-6) $r < 0$, $n_1 = 1$, $(p, q) \in (D_3) \cup (D_4)$; (5-7) $r < 0$, $n_1 \geq 2$, $(p, q) \in (D_2)$; (5-8) $r < 0$, $n_1 = 1$, $(p, q) \in (D_2)$.

$$y^2 = \frac{q}{-5r} \phi^2 \left(\phi + \frac{5(p+1)}{4q} \right), \tag{3.7}$$

Thus, by using the first equation of (1.6) and (3.7), we obtain the parametric representation of this arch as follows:

$$\phi(\xi) = \frac{-\frac{5(p+1)}{4q}}{1 - \tanh^2 \left(\sqrt{-\frac{4q}{5r}} \xi \right)}. \tag{3.8}$$

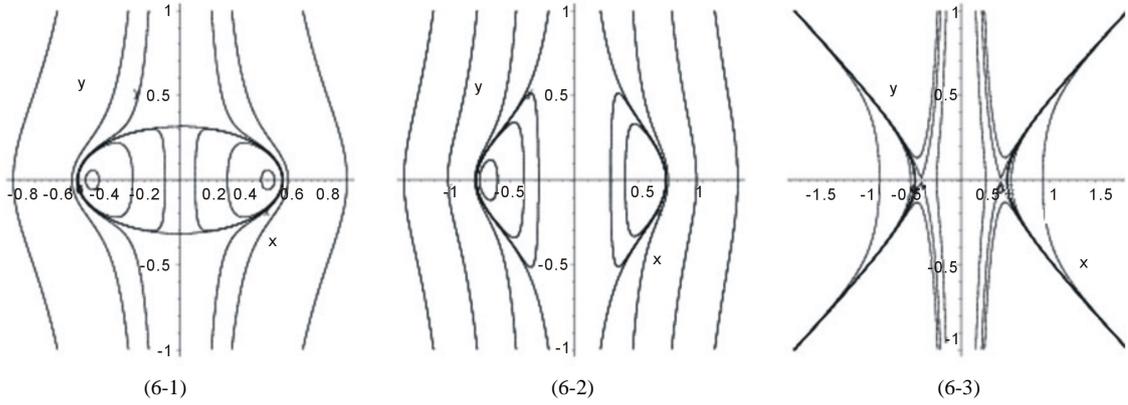


Figure 6. The phase portraits of (1.6) for $n = 2n_1, n_1 \in \mathbb{Z}^+$. (6-1) $r < 0, n_1 = 2, m_1 \geq n_1, p < 0$; (6-2) $r < 0, n_1 \geq 2, m_1 > n_1, p < 0$; (6-3) $r > 0, m_1 > n_1, p < 0$.

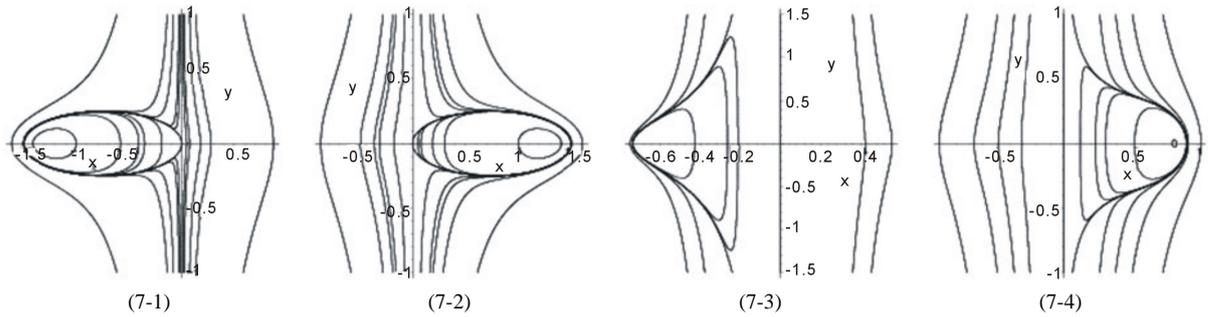


Figure 7. The phase portraits of (1.6) for $n = 2n_1 + 1, n_1 \in \mathbb{Z}^+$. (7-1) $r > 0, n_1 = 1, m_1 \geq n_1, p > 0$; (7-2) $r < 0, n_1 = 1, m_1 > n_1, p < 0$; (7-3) $r < 0, n_1 \geq 2, m_1 > n_1, p > 0$; (7-4) $r < 0, n_1 \geq 2, m_1 > n_1, p < 0$.

We will show in Section 4 that (3.8) gives rise to a solitary wave solution of peak type or valley type of (1.3).

5). Suppose that $n = 3, m = 4, r > 0, (p, q) \in D_3 \cup D_4$, in this case, we have the phase portrait of (2.1) shown in **Figure 5** (5-4). corresponding to the orbit defined by $H(\phi, y) = 0$ to the equilibrium point

$S_{\pm} \left(\pm \sqrt{\frac{-p + \sqrt{p^2 - 4q}}{2q}}, 0 \right)$, the arch curve has the algebraic equation

$$y^2 = \frac{2}{3r}(\phi - 0) \left(\phi - \left(-\frac{p}{12} + \frac{7}{2q} \sqrt{\frac{p^2 - 4q}{36 - 35}} \right) \right) \left(-\frac{p}{12} - \frac{7}{2q} \sqrt{\frac{p^2 - 4q}{36 - 35}} - \phi \right). \quad (3.9)$$

Thus, by using the first equation of (1.6) and (3.9), we obtain the parametric representation of this arch as follows:

$$\phi(\xi) = -\frac{7p}{12q} - \frac{7}{2q} \sqrt{\frac{p^2 - 4q}{36 - 35}} + \left(\frac{7p}{12q} + \frac{7}{2q} \sqrt{\frac{p^2 - 4q}{36 - 35}} \right) \text{sn}^2(\Omega_3 \xi; k_3), \quad (3.10)$$

where $\text{sn}(x; k)$ is the Jacobin elliptic functions with the modulo k and

$$\Omega_3 = \sqrt{\frac{-7\sqrt{\frac{p^2 - 4q}{36 - 35}}}{6qr}}, \quad k_3 = \sqrt{\frac{1}{12\sqrt{\frac{p^2 - 4q}{36 - 35}} + \frac{1}{2}}}$$

We will show in Section 4 that (3.10) gives rise to a smooth compacton solution of (1.3).

6). Suppose that $n = 3, m = 4, r < 0, (p, q) \in D_3 \cup D_4$, In this case, we have the phase portrait of (2.1) shown in **Figure 5** (5-6), corresponding to the orbit defined by $H(\phi, y) = 0$ to the equilibrium point

$S_{\pm} \left(\pm \sqrt{\frac{-p \pm \sqrt{p^2 - 4q}}{2q}}, 0 \right)$, the arch curve has the algebraic equation

$$y^2 = \frac{2}{-3r}(0 - \phi) \left(\phi - \left(-\frac{p}{12} + \frac{7}{2q} \sqrt{\frac{p^2 - 4q}{36 - 35}} \right) \right) \left(-\frac{p}{12} - \frac{7}{2q} \sqrt{\frac{p^2 - 4q}{36 - 35}} - \phi \right). \quad (3.11)$$

Thus, by using the first equation of (1.6) and (3.11), we obtain the parametric representation of this arch as follows:

$$\phi(\xi) = -\frac{7p}{12q} - \frac{7}{2q} \sqrt{\frac{p^2 - 4q}{36 - 35}} + \frac{\frac{7}{q} \sqrt{\frac{p^2 - 4q}{36 - 35}} \left(\frac{7p}{12q} - \frac{7}{2q} \sqrt{\frac{p^2 - 4q}{36 - 35}} \right)}{\left(\frac{7p}{12q} + \frac{7}{2q} \sqrt{\frac{p^2 - 4q}{36 - 35}} \right) \operatorname{sn}^2(\Omega_4 \xi; k_4)}, \quad (3.12)$$

where $\Omega_4 = \sqrt{\frac{-7 \sqrt{\frac{p^2 - 4q}{36 - 35}}}{6qr}}$, $k_4 = \sqrt{-\frac{1}{12 \sqrt{\frac{p^2 - 4q}{36 - 35}}} + \frac{1}{2}}$,

we will show in Section 4 that (3.20) gives rise to a smooth compacton solution of (1.3)

7). Suppose $n = 4, m = 5, r < 0, (p, q) \in B_1 \cup B_2 \cup B_4$ that. In this case, we have the phase portrait of (2.1) shown in **Figure 3** (3-2) and (3-7), corresponding to the orbit defined by $H(\phi, y) = 0$ to the equilibrium point $A(0, 0)$, the arch curve has the algebraic equation

$$y^2 = \frac{1}{-2r} \left(\frac{q}{9} \phi^3 + \frac{p}{8} \phi^2 + \frac{1}{6} \right). \quad (3.13)$$

Thus, by using the first equation of (1.6) and (3.13), we obtain the parametric representation of this arch as follows:

$$\phi(\xi) = \frac{3p}{-8q} + \wp \left(\sqrt{-\frac{q}{72r}} \xi, g_1, g_2 \right), \quad (3.14)$$

where $g_1 = \frac{27p^2}{16q^2}, g_2 = \frac{27p^3}{128q^3} - \frac{6}{q}$. We will show in Section 4 that (3.14) gives rise to a smooth compacton solution of (1.3).

8). Suppose $n = 4, m = 6, r < 0, (p, q) \in A_3$. In this case, we have the phase portrait of (2.1) shown in **Figure 2** (2-1), corresponding to the orbit defined by $H(\phi, y) = 0$ to the equilibrium point $A(0, 0)$, the arch curve has the algebraic equation

$$y^2 = \frac{1}{-4r} \left(\phi^2 + \frac{5p}{8q} + \frac{5}{2q} \sqrt{\frac{p^2 - 4q}{16 - 15}} \right) \left(\phi^2 + \frac{5p}{8q} - \frac{5}{2q} \sqrt{\frac{p^2 - 4q}{16 - 15}} \right). \quad (3.15)$$

Thus, by using the first equation of (1.6) and (3.15), we obtain the parametric representation of this arch as follows:

$$\phi(\xi) = \pm \frac{\sqrt{-\frac{5p}{8q} + \frac{5}{2q} \sqrt{\frac{p^2 - 4q}{16 - 15}}}}{\operatorname{cn}(\Omega_5 \xi; k_5)}, \quad (3.16)$$

where $\Omega_5 = \sqrt{\frac{5\sqrt{\frac{p^2-4q}{16-15}}}{-4qr}}$, $k_4 = \sqrt{\frac{p}{8\sqrt{\frac{p^2-4q}{16-15}} + \frac{1}{2}}}$,

We will show in Section 4 that (3.16) gives rise to two periodic cusp wave solutions of peak type and valley type of (1.3).

9). Suppose $n = 3, m = 5, r > 0, (p, q) \in C_4$. In this case, we have the phase portrait of (2.1) shown in **Figure 4** (4-5), corresponding to the orbit defined by $H(\phi, y) = 0$ to the equilibrium point $A(0, 0)$, the arch curve has the algebraic equation

$$y^2 = \frac{2}{3r}(\phi - \phi_1)(\phi - \phi_2)(\phi - \phi_3)(\phi_4 - \phi), \tag{3.17}$$

where $\phi_1 < \phi_2 < \phi_3 < \phi_4, \phi_i \left(-\frac{q}{8}\phi_i^3 - \frac{p}{6}\phi_i - \frac{1}{5} \right) = 0, i = 1 - 4$. Thus, by using the first equation of (1.6) and (3.17), we obtain the parametric representation of this arch as follows:

$$\phi(\xi) = \frac{(\phi_4 - \phi_3)\phi_2 - \phi_3(\phi_4 - \phi_2)sn^2(\Omega_6\xi; k_6)}{(\phi_4 - \phi_2) - (\phi_4 - \phi_3)sn^2(\Omega_6\xi; k_6)}, \tag{3.18}$$

where $sn(x; k)$ is the Jacobin elliptic functions with the modulo k and

$$\Omega_6 = \sqrt{\frac{(\phi_4 - \phi_2)(\phi_3 - \phi_1)}{6r}}, \quad k_6 = \sqrt{\frac{(\phi_4 - \phi_3)(\phi_2 - \phi_1)}{(\phi_3 - \phi_2)(\phi_3 - \phi_1)}}$$

We will show in Section 4 that (3.6) gives rise to a smooth compacton solution of (1.3).

4. The Existence of Smooth and Non-Smooth Travelling Wave Solutions of (1.6)

In this section, we use the results of Section 2 to discuss the existence of smooth and non-smooth solitary wave and periodic wave solutions. We first consider the existence of smooth solitary wave solution and periodic wave solutions.

Theorem 4.1

1). Suppose that $m - n = 2, n = 2n_1 + 1 \geq 5, l, n_1 \in \mathbb{Z}^+, r > 0, (p, q) \in C_2$: Then, corresponding to a branch of the curves $H(\phi, y) = h_2(h_3)$ defined by (1.7), Equation (1.3) has a smooth solitary wave solution of peak type, corresponding to a branch of the curves $H(\phi, y) = h, h \in (h_3, h_2)$ defined by (1.7), Equation (1.3) has a smooth family of periodic wave solutions (see **Figure 4** (4-4)).

2). Suppose that $m - n = 2, n = 3, m \in \mathbb{Z}^+, r > 0, (p, q) \in C_2$: Then, corresponding to a branch of the curves $H(\phi, y) = h_2$ defined by (1.7), equation (1.3) has a smooth solitary wave solution of peak type, corresponding to a branch of the curves $H(\phi, y) = h, h \in (h_3, h_2)$ defined by (1.7), Equation (1.3) has a smooth family of periodic wave solutions (see **Figure 4** (4-3)).

3). Suppose that $m - n = 2, n = 2n_1 + 1 \geq 5, l, n_1 \in \mathbb{Z}^+, r > 0, (p, q) \in A_2$, Then, corresponding to a branch of the curves $H(\phi, y) = 0$ defined by (1.7), equation (1.3) has a smooth solitary wave solution of peak type, corresponding to a branch of the curves $H(\phi, y) = h, h \in (h_1, h_2)$ defined by (1.7), equation (1.3) has a smooth family of periodic wave solutions (see **Figure 4** (4-12)).

4). Suppose that $m - n = 2l, n = 3, m \in \mathbb{Z}^+, r > 0, (p, q) \in C_2$, Then, corresponding to a branch of the curves $H(\phi, y) = 0$ defined by (1.7), equation (1.3) has a smooth solitary wave solution of valley type, corresponding to a branch of the curves $H(\phi, y) = h, h \in (h_1, h_3)$ defined by (1.7), equation (1.3) has a smooth family of periodic wave solutions (see **Figure 4** (4-3)).

5). Suppose that $m - n = 2l, n \geq 4, m \in \mathbb{Z}^+, r > 0, (p, q) \in A_2$, Then, corresponding to a branch of the curves $H(\phi, y) = h_2$ defined by (1.7), equation (1.3) has a smooth solitary wave solution of peak type, corresponding to a branch of the curves $H(\phi, y) = h, h \in (h_3, h_4)$ defined by (1.7), Equation (1.3) has a smooth family of periodic wave solutions (see **Figure 2** (2-3)).

6). Suppose that $m - n = 2l, n \geq 4, m \in \mathbb{Z}^+, r > 0, (p, q) \in A_2$, Then, corresponding to a branch of the curves $H(\phi, y) = h_3$ defined by (1.7), equation (1.3) has a smooth solitary wave solution of peak type, corresponding to a branch of the curves $H(\phi, y) = h, h \in (h_1, h_2)$ defined by (1.7), Equation (1.3) has a smooth family of periodic wave solutions (see **Figure 2** (2-3)).

7). Suppose that $m - n = 2l, n = 3, m \in \mathbb{Z}^+, r > 0, (p, q) \in C_3$, Then, corresponding to a branch of the curves $H(\phi, y) = 0$ defined by (1.7), Equation (1.3) has a smooth solitary wave solution of valley type, corresponding to a branch of the curves $H(\phi, y) = h, h \in (h_1, h_2)$ defined by (1.7), Equation (1.3) has a smooth family of periodic wave solutions (see **Figure 2** (2-3)).

8). Suppose that $m - n = 2l - 1, n = 4, l \in \mathbb{Z}^+, r > 0, (p, q) \in B_3$, Then, corresponding to a branch of the curves $H(\phi, y) = h, h \in (h_3, h_2)$ defined by (1.7), Equation (1.3) has a smooth family of periodic wave solutions (see **Figure 3** (3-5)).

9). Suppose that: $m - n = 2l - 1, n \geq 4, l \in \mathbb{Z}^+, r > 0, (p, q) \in B_3$, Then, corresponding to a branch of the curves $H(\phi, y) = h_2$ defined by (1.7), equation (1.3) has a smooth solitary wave solution of valley type, corresponding to a branch of the curves $H(\phi, y) = h, h \in (h_1, h_2)$ defined by (1.7), Equation (1.3) has a smooth family of periodic wave solutions (see **Figure 3** (3-4)).

10). Suppose that $m - n = 2l, n = 2, l \in \mathbb{Z}^+, r < 0, (p, q) \in B_1$, Then, corresponding to a branch of the curves $H(\phi, y) = 0$ defined by (1.7), equation (1.3) has a smooth solitary wave solution of valley type, corresponding to a branch of the curves $H(\phi, y) = h, h \in (0, h_1)$ defined by (1.7), Equation (1.3) has a smooth family of periodic wave solutions (see **Figure 2** (2-4)).

11). Suppose that $m - n = 2l - 1, n = 2, l \in \mathbb{Z}^+, r < 0, (p, q) \in B_1$: Then, corresponding to a branch of the curves $H(\phi, y) = 0$ defined by (1.7), Equation (1.3) has a smooth solitary wave solution of valley type, corresponding to a branch of the curves $H(\phi, y) = h, h \in (0, h_1)$ defined by (1.7), Equation (1.3) has a smooth family of periodic wave solutions (see **Figure 3** (3-1)).

We shall describe what types of non-smooth solitary wave and periodic wave solutions can appear for our system (1.6) which correspond to some orbits of (2.1) near the straight line $\phi = 0$. To discuss the existence of cusp waves, we need to use the following lemmarelating to the singular straight line.

Lemma 4.2 The boundary curves of a periodic annulus are the limit curves of closed orbits inside the annulus; If these boundary curves contain a segment of the singular straight line $\phi = 0$ of (1.4), then along this segment and near this segment, in very short time interval $y = \phi_\xi$ jumps rapidly.

Base on Lemma 4.2, **Figure 2**, and **Figure 3**, we have the following result.

Theorem 4.3

1). Suppose that $m - n = 2l, n = 4, l \in \mathbb{Z}^+$.

a). For $r < 0, (p, q) \in A_4$ corresponding to the arch curve $H(\phi, y) = 0$ defined by (1.7), Equation (1.3) has two periodic cusp wave solutions; corresponding to two branches of the curves $H(\phi, y) = h, h \in (h_1, 0)$ defined by (1.7), Equation (1.3) has two families of periodic wave solutions. When h varies from h_1 to 0, these periodic travelling waves will gradually lose their smoothness, and evolve from smooth periodic travelling waves to periodic cusp travelling waves, finally approach a periodic cusp wave of valley type and a periodic cusp wave of peak type defined by $H(\phi, y) = 0$ of (1.7) (see **Figure 2** (2-5)).

b). For $r > 0, (p, q) \in A_3$ corresponding to the arch curve $H(\phi, y) = h, h \in (0, h_2)$ defined by (1.7), Equation (1.3) has two periodic cusp wave solutions; corresponding to two branches of the curves $H(\phi, y) = h, h \in (h_1, 0)$ defined by (1.7), Equation (1.3) has two families of periodic wave solutions. When h varies from h_1 to 0, these periodic travelling waves will gradually lose their smoothness, and evolve from smooth periodic travelling waves to periodic cusp travelling waves, finally approach a periodic cusp wave of valley type and a periodic cusp wave of peak type defined by $H(\phi, y) = 0$ of (1.7) (see **Figure 2** (2-1)).

2). Suppose that $m - n = 2l - 1, n = 4, l \in \mathbb{Z}^+$.

c). For $r < 0, (p, q) \in B_4$ corresponding to the arch curve $H(\phi, y) = 0$ defined by (1.7), Equation (1.3) has two periodic cusp wave solutions; corresponding to two branches of the curves $H(\phi, y) = h, h \in (0, h_1)$ defined by (1.7), Equation (1.3) has two families of periodic wave solutions. When h varies from 0 to h_1 , these periodic travelling waves will gradually lose their smoothness, and evolve from smooth periodic travelling waves to periodic cusp travelling waves, finally approach a periodic cusp wave of valley type and a periodic cusp wave of peak type defined by $H(\phi, y) = 0$ of (1.7) (see **Figure 3** (3-7)).

d). For $r < 0, (p, q) \in B_3$ corresponding to the arch curve $H(\phi, y) = 0$ defined by (1.7), Equation (1.3) has two periodic cusp wave solutions; corresponding to two branches of the curves $H(\phi, y) = h, h \in (0, h_3)$ defined by (1.7), Equation (1.3) has two families of periodic wave solutions. When h varies from 0 to h_3 , these periodic travelling waves will gradually lose their smoothness, and evolve from smooth periodic travelling waves to periodic cusp travelling waves, finally approach a periodic cusp wave of valley type and a periodic cusp wave of peak type defined by $H(\phi, y) = 0$ of (1.7) (see **Figure 3** (3-5)).

d). For $r < 0, (p, q) \in B_1 \cup B_2$ corresponding to the arch curve $H(\phi, y) = 0$ defined by (1.7), Equation (1.3) has two periodic cusp wave solutions; corresponding to two branches of the curves $H(\phi, y) = h, h \in (0, h_1)$ defined by (1.7), Equation (1.3) has two families of periodic wave solutions. When h varies from 0 to h_1 , these periodic travelling waves will gradually lose their smoothness, and evolve from smooth periodic travelling waves to periodic cusp travelling waves, finally approach a periodic cusp wave of valley type and a periodic cusp wave of peak type defined by $H(\phi, y) = 0$ of (1.7) (see **Figure 3** (3-3)).

We can easily see that there exist two families of closed orbits of (1.3) in **Figure 2** (2-6), **Figure 3** (3-6) and in **Figure 6** (6-2). There is one family of closed orbits in **Figure 3** (3-3), (3-8), **Figure 4** (4-2), (4-5) - (4-7), (4-9), (4-11) and in **Figure 5** (5-3), (5-5) - (5-7) and in **Figure 7** (7-3), (7-4). In all the above cases there exists at least one family of closed orbits (1.3) for which as h from $H(\phi_e, 0)$ to 0, where ϕ_e is the abscissa of the center, the closed orbit will expand outwards to approach the straight line $\phi = 0$ and $|y| = |\phi'|$ will approach to ∞ . As a result, we have the following conclusions.

Theorem 4.4

1). Suppose that $m - n = 2l, n = 2n_1 + 1, l, n_1 \in \mathbb{Z}^+$.

a). If $r < 0, n_1 \geq 1, (p, q) \in C_2$; then when $h \in (h_1, 0)$ in (1.7), Equation (1.3) has a family of uncountably infinite many periodic traveling wave solutions; where h varies from h_1 to 0, these periodic traveling wave solutions will gradually lose their smoothness, and evolve from smooth periodic traveling waves to periodic cusp traveling waves (see **Figure 4** (4-2)).

b). If $r < 0, n_1 \geq 2, (p, q) \in C_3$; then when $h \in (0, h_1)$ in (1.7), Equation (1.3) has a family of uncountably infinite many periodic traveling wave solutions; when h varies from 0 to h_1 , these periodic traveling wave solutions will gradually lose their smoothness, and evolve from smooth periodic traveling waves to periodic cusp traveling waves (see **Figure 4** (4-7)).

c). If $r < 0, n_1 \geq 2, (p, q) \in C_2$; then when $h \in (0, h_2)$ in (1.7), Equation (1.3) has a family of uncountably infinite many periodic traveling wave solutions; when h varies from 0 to h_2 , these periodic traveling wave solutions will gradually lose their smoothness, and evolve from smooth periodic traveling waves to periodic cusp traveling waves (see **Figure 4** (4-6)).

d). If $r > 0, n_1 \geq 2, (p, q) \in C_4$; then when $h \in (h_2, 0)$ in (1.7), Equation (1.3) has a family of uncountably infinite many periodic traveling wave solutions; when h varies from h_2 to 0, these periodic traveling wave solutions will gradually lose their smoothness, and evolve from smooth periodic traveling waves to periodic cusp traveling waves (see [Figure 4](#) (4-11)).

e). If $r < 0, n_1 = 1, (p, q) \in C_2$; then when $h \in (0, h_2)$ in (1.7), Equation (1.3) has a family of uncountably infinite many periodic traveling wave solutions; when h varies from 0 to h_2 , these periodic traveling wave solutions will gradually lose their smoothness, and evolve from smooth periodic traveling waves to periodic cusp traveling waves (see [Figure 4](#) (4-5)).

f). If $r > 0, n_1 = 1, (p, q) \in C_4$; then when $h \in (0, h_2)$ in (1.7), Equation (1.3) has a family of uncountably infinite many periodic traveling wave solutions; when h varies from 0 to h_2 , these periodic traveling wave solutions will gradually lose their smoothness, and evolve from smooth periodic traveling waves to periodic cusp traveling waves (see [Figure 4](#) (4-9)).

2). Suppose that, then when $h \in (0, h_1)$ in (1.7), Equation (1.3) has two family of uncountably infinite many periodic traveling wave solutions; when h varies from 0 to h_1 , these periodic traveling wave solutions will gradually lose their smoothness, and evolve from smooth periodic traveling waves to periodic cusp traveling waves (see [Figure 2](#) (2-6)). Paralleling to [Figure 2](#) (2-6), we can see the periodic travelling wave solutions implied in [Figure 3](#) (3-6) and [Figure 6](#) (6-2) have the same characters.

3). Suppose that $m - n = 2l - 1, n = 2n_1 + 1, l, n_1 \in \mathbb{Z}^+$.

g). If $r < 0, n_1 = 1, (p, q) \in D_2 \cup D_4$; then when $h \in (0, h_2)$ in (1.7), Equation (1.3) has a family of uncountably infinite many periodic traveling wave solutions; when h varies from 0 to h_2 , these periodic traveling wave solutions will gradually lose their smoothness, and evolve from smooth periodic traveling waves to periodic cusp traveling waves (see [Figure 5](#) (5-6)).

h). If $r < 0, n_1 \geq 2, (p, q) \in D_2$; then when $h \in (0, h_1)$ in (1.7), Equation (1.3) has a family of uncountably infinite many periodic traveling wave solutions; when h varies from 0 to h_1 , these periodic traveling wave solutions will gradually lose their smoothness, and evolve from smooth periodic traveling waves to periodic cusp traveling waves (see [Figure 5](#) (5-7)).

i). If $r < 0, n_1 \geq 2, (p, q) \in D_3 \cup D_4$, then when $h \in (h_1, 0)$ in (1.7), Equation (1.3) has a family of uncountably infinite many periodic traveling wave solutions; when h varies from h_1 to 0, these periodic traveling wave solutions will gradually lose their smoothness, and evolve from smooth periodic traveling waves to periodic cusp traveling waves (see [Figure 5](#) (5-3)).

j). If $r < 0, n_1 \geq 2, (p, q) \in D_3 \cup D_4$, then when $h \in (0, h_2)$ in (1.7), Equation (1.3) has a family of uncountably infinite many periodic traveling wave solutions; when h varies from 0 to h_2 , these periodic traveling wave solutions will gradually lose their smoothness, and evolve from smooth periodic traveling waves to periodic cusp traveling waves (see [Figure 5](#) (5-5)).

4). Suppose that $n = 2n_1 + 1, l, n_1 \in \mathbb{Z}^+, q = 0$.

k). If $r < 0, n_1 \geq 2, p > 0$, then when $h \in (h_1, 0)$ in (1.7), Equation (1.3) has a family of uncountably infinite many periodic traveling wave solutions; when h varies from h_1 to 0, these periodic traveling wave solutions will gradually lose their smoothness, and evolve from smooth periodic traveling waves to periodic cusp traveling waves (see [Figure 5](#) (5-5)).

Equation (1.3) has one family of uncountably infinite many periodic traveling wave solutions; when h varies from 0 to h_1 , these periodic travelling wave solutions will gradually lose their smoothness, and evolve from smooth periodic travelling waves to periodic cusp travelling waves (see [Figure 7](#) (7-4)).

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