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High Accurate Fourth-Order Finite Difference Solutions of the Three Dimensional Poisson's Equation in Cylindrical Coordinate

Alemayehu Shiferaw, Ramesh Chand Mittal

Department of Mathematics, Indian Institute of Technology, Roorkee, India
Email: abelhaim@gmail.com, mittalrc@gmail.com

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Abstract

In this work, by extending the method of Hockney into three dimensions, the Poisson's equation in cylindrical coordinates system with the Dirichlet's boundary conditions in a portion of a cylinder for $r \neq 0$ is solved directly. The Poisson equation is approximated by fourth-order finite differences and the resulting large algebraic system of linear equations is treated systematically in order to get a block tri-diagonal system. The accuracy of this method is tested for some Poisson's equations with known analytical solutions and the numerical results obtained show that the method produces accurate results.

Keywords

Poisson's Equation; Tri-Diagonal Matrix; Fourth-Order Finite Difference Approximation; Hockney's Method; Thomas Algorithm

1. Introduction

The three-dimensional Poisson's equation in cylindrical coordinates (r,θ,z) is given by

$$U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} + U_{zz} = f(r,\theta,z) \tag{1}$$

has a wide range of application in engineering and science fields (especially in physics).

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In physical problems that involve a cylindrical surface (for example, the problem of evaluating the temperature in a cylindrical rod), it will be convenient to make use of cylindrical coordinates. For the numerical solution of the three dimensional Poisson's equation in cylindrical coordinates system, several attempts have been made in particular for physical problems that are related directly or indirectly to this equation. For instance, *Lai* [1] developed a simple compact fourth-order Poisson solver on polar geometry based on the truncated Fourier series expansion, where the differential equations of the Fourier coefficients are solved by the compact fourth-order finite difference scheme; *Mittal and Gahlaut* [2] have developed high order finite difference schemes of second- and fourth- order in polar coordinates using a direct method similar to Hockney's method; Mittal and Gahlaut [3] developed a second- and fourth-order finite difference scheme to solve Poisson's equation in the case of cylindrical symmetry; *Alemayehu and Mittal* [4] have derived a second-order finite difference approximation scheme to solve the three dimensional Poisson's equation in cylindrical coordinates by extending Hockney's method; *Tan* [5] developed a spectrally accurate solution for the three dimensional Poisson's equation and Helmholtz's equation using Chebyshev series and Fourier series for a simple domain in a cylindrical coordinate system; *Iyengar and Manohar* [6] derived fourth-order difference schemes for the solution of the Poisson equation which occurs in problems of heat transfer; *Iyengar and Goyal* [7] developed a multigrid method in cylindrical coordinates system; *Lai and Tseng* [8] have developed a fourth-order compact scheme, and their scheme relies on the truncated Fourier series expansion, where the partial differential equations of Fourier coefficients are solved by a formally fourth-order accurate compact difference discretization. The need to obtain the best solution for the three dimensional Poisson's equation in cylindrical coordinates system is still in progress.

In this paper, we develop a fourth-order finite difference approximation scheme and solve the resulting large algebraic system of linear equations systematically using block tridiagonal system [9] [10] and extend the Hockney's method [9] [11] to solve the three dimensional Poisson's equation on Cylindrical coordinates system.

2. Finite Difference Approximation

Consider the three dimensional Poisson's equation in cylindrical coordinates (r,θ,z) given by

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}+\frac{\partial^2 U}{\partial z^2}=f(r,\theta,z) \text{ on } D$$

and the boundary condition

$$U(r,\theta,z)=g(r,\theta,z) \text{ on } C \tag{2}$$

where C is the boundary of D and D is

$D_1 = \{(r,\theta,z): R_0<r<R_1, a<z<b, \theta_0<\theta<\theta_1, \theta_0<\theta_1<2\pi\}$ and $D_2=\{(r,\theta,z): R_0<r<R_1, a<z<b, 0\le\theta<2\pi\}$

Consider **Figure 1** as the geometry of the problem. Let $u(r,\theta,z)$ be discretized at the point (r_i,θ_j,z_k) and for simplicity write a point (r_i,θ_j,z_k) as (i,j,k) and $u(r_i,\theta_j,z_k)$ as $u_{i,j,k}$.

Assume that there are *M* points in the direction of r, *N* points in θ and *P* points in the z directions to form the mesh, and let the step size along the direction of r be Δr, of θ be $\Delta\theta$ and z be Δz.

Here $r_i=R_0+i\Delta r, \theta_j=\theta_0+j\Delta\theta$ and $z_k=a+k\Delta z$

Where $i=1,2,\cdots,M$, $j=1,2,\cdots,N$ and $k=1,2,\cdots,P$.

When $r=0$ is an interior or a boundary point of (2), then the Poisson's equation becomes singular and to take care of the singularity a different approach will be taken. Thus in this paper we consider only for the case $r\ne 0$.

Using the approximations that

$$\left(\frac{\partial^2 U}{\partial r^2}\right)_{i,j,k}=\frac{1}{(\Delta r)^2}\left(1+\frac{1}{12}\delta_r^2\right)^{-1}\delta_r^2 U_{i,j,k}+O\left((\Delta r)^4\right) \tag{3}$$

$$\left(\frac{\partial^2 U}{\partial \theta^2}\right)_{i,j,k}=\frac{1}{(\Delta \theta)^2}\left(1+\frac{1}{12}\delta_\theta^2\right)^{-1}\delta_\theta^2 U_{i,j,k}+O\left((\Delta \theta)^4\right) \tag{4}$$

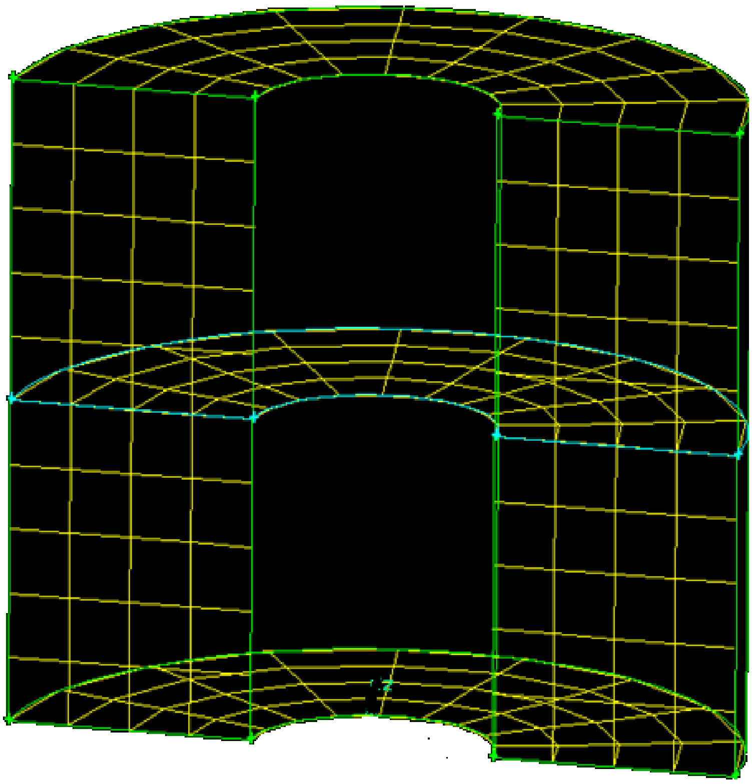

Figure 1. Portion of a cylinder.

$$\left(\frac{\partial^2 U}{\partial z^2}\right)_{i,j,k} = \frac{1}{(\Delta z)^2}\left(1+\frac{1}{12}\delta_z^2\right)^{-1}\delta_z^2 U_{i,j,k} + O\left((\Delta z)^4\right) \quad (5)$$

Now using (3), (4) and (5), we get (Refer the work of Mittal and Ghalaut in [2])

From (1) consider only the approximation of the sum of the first and the third terms, that is, the sum of $\frac{\partial^2 U}{\partial r^2}$ and $\frac{1}{r_i^2}\frac{\partial^2 U}{\partial \theta^2}$

$$\begin{aligned}&\left(\frac{\partial^2 U}{\partial r^2}+\frac{1}{r_i^2}\frac{\partial^2 U}{\partial \theta^2}\right)_{i,j,k}\\ &= \frac{1}{12(\Delta r)^2}\Bigg[-20\left(1+\frac{\omega}{r_i^2}\right)U_{i,j,k}+2\left(5-\frac{\omega}{r_i^2}\right)\left(U_{i+1,j,k}+U_{i-1,j,k}\right)+2\left(\frac{5\omega}{r_i^2}-1\right)\left(U_{i,j+1,k}+U_{i,j-1,k}\right)\\ &+\left(1+\frac{\omega}{r_i^2}\right)\left(U_{i+1,j+1,k}+U_{i+1,j-1,k}+U_{i-1,j+1,k}+U_{i-1,j-1,k}\right)\Bigg]-\frac{1}{12}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r_i^2}\frac{\partial^2}{\partial \theta^2}\right)\\ &\left((\Delta r)^2\frac{\partial^2}{\partial r^2}+(\Delta\theta)^2\frac{\partial^2}{\partial \theta^2}\right)U_{i,j,k}+O\left((\Delta r)^4+(\Delta\theta)^4\right)\end{aligned} \quad (6)$$

where $\omega = \frac{(\Delta r)^2}{(\Delta\theta)^2}$

Again from (1) consider only the approximation of the sum of the first and the fourth terms, that is, the sum of $\frac{\partial^2 U}{\partial r^2}$ and $\frac{\partial^2 U}{\partial z^2}$, and we get

$$\left(\frac{\partial^2 U}{\partial r^2}+\frac{\partial^2 U}{\partial z^2}\right)_{i,j,k}$$
$$=\frac{1}{12(\Delta r)^2}\left[\left(1+\frac{(\Delta r)^2}{(\Delta z)^2}\right)\left(U_{i+1,j,k+1}+U_{i+1,j,k-1}+U_{i-1,j,k+1}+U_{i-1,j,k-1}\right)\right.$$
$$\left.+2\left(5-\frac{(\Delta r)^2}{(\Delta z)^2}\right)\left(U_{i+1,j,k}+U_{i-1,j,k}\right)+2\left(5\frac{(\Delta r)^2}{(\Delta z)^2}-1\right)\left(U_{i,j+1,k}+U_{i,j-1,k}\right)-20\left(1+\frac{(\Delta r)^2}{(\Delta z)^2}\right)U_{i,j,k}\right] \tag{7}$$
$$-\frac{1}{12}\left(\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z^2}\right)\left((\Delta r)^2\frac{\partial^2}{\partial r^2}+(\Delta z)^2\frac{\partial^2}{\partial z^2}\right)U_{i,j,k}+O\left((\Delta r)^4+(\Delta z)^4\right)$$

Once again from (1) consider only the approximation of the sum of the second and the fourth terms, that is, the sum of $\frac{1}{r_i^2}\frac{\partial^2 U}{\partial \theta^2}$ and $\frac{\partial^2 U}{\partial z^2}$; to get

$$\left(\frac{1}{r_i^2}\frac{\partial^2 U}{\partial \theta^2}+\frac{\partial^2 U}{\partial z^2}\right)_{i,j,k}$$
$$=\frac{1}{12}\left[\left(\frac{1}{(r_i\Delta\theta)^2}+\frac{1}{(\Delta z)^2}\right)\left(U_{i,j+1,k+1}+U_{i,j+1,k-1}+U_{i,j-1,k+1}+U_{i,j-1,k-1}\right)+2\left(\frac{5}{(r_i\Delta\theta)^2}-\frac{1}{(\Delta z)^2}\right)\left(U_{i,j+1,k}+U_{i,j-1,k}\right)\right.$$
$$\left.+2\left(\frac{5}{(\Delta z)^2}-\frac{1}{(r_i\Delta\theta)^2}\right)\left(U_{i,j,k+1}+U_{i,j,k-1}\right)-20\left(\frac{1}{(r_i\Delta\theta)^2}+\frac{1}{(\Delta z)^2}\right)U_{i,j,k}\right] \tag{8}$$
$$-\frac{1}{12}\left(\frac{1}{r_i^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}\right)\left((\Delta\theta)^2\frac{\partial^2}{\partial r^2}+(\Delta z)^2\frac{\partial^2}{\partial z^2}\right)U_{i,j,k}+O\left((\Delta\theta)^4+(\Delta z)^4\right)$$

Again taking the approximation of the term $\frac{\partial U}{\partial r}$ by

$$\left(\frac{\partial U}{\partial r}\right)_{i,j,k}=\frac{\phi\delta_{2r}\left(U_{i,j+1,k}+U_{i,j-1,k}+U_{i,j,k+1}+U_{i,j,k-1}\right)+(1-4\phi)\delta_{2r}U_{i,j,k}}{2\Delta r}$$
$$-\frac{1}{3}(\Delta r)^2\frac{\partial^3 U_{i,j,k}}{\partial r^3}-\phi(\Delta\theta)^2\frac{\partial^3 U_{i,j,k}}{\partial r\partial\theta^2}-\phi(\Delta z)^2\frac{\partial^3 U_{i,j,k}}{\partial r\partial z^2} \tag{9}$$
$$+O\left((\Delta r)^4+(\Delta\theta)^4+(\Delta z)^4\right),\ 0\le\phi\le 1$$

Equation (9) implying that

$$\frac{1}{r_i}\left(\frac{\partial U}{\partial r}\right)_{i,j,k}=\frac{\phi\delta_{2r}\left(U_{i,j+1,k}+U_{i,j-1,k}+U_{i,j,k+1}+U_{i,j,k-1}\right)+(1-4\phi)\delta_{2r}U_{i,j,k}}{2r_i\Delta r}$$
$$-\frac{1}{3r_i}(\Delta r)^2\frac{\partial^3 U_{i,j,k}}{\partial r^3}-\phi(\Delta\theta)^2\frac{1}{r_i}\frac{\partial^3 U_{i,j,k}}{\partial r\partial\theta^2} \tag{10}$$
$$-\phi(\Delta z)^2\frac{1}{r_i}\frac{\partial^3 U_{i,j,k}}{\partial r\partial z^2}+O\left((\Delta r)^4+(\Delta\theta)^4+(\Delta z)^4\right)$$

Now letting $\alpha=\frac{(\Delta r)^2}{(\Delta z)^2}$ and adding (6), (7), (8) and twice of (10), we get

$$
\begin{aligned}
&2\left(\frac{\partial^2 U}{\partial r^2}+\frac{1}{r_i}\frac{\partial U}{\partial r}+\frac{1}{r_i^2}\frac{\partial^2 U}{\partial \theta^2}+\frac{\partial^2 U}{\partial z^2}\right)_{i,j,k} \\
&=\frac{1}{12(\Delta r)^2}\Bigg[\left(1+\frac{\omega}{r_i^2}\right)\left(U_{i+1,j+1,k}+U_{i+1,j-1,k}+U_{i-1,j+1,k}+U_{i-1,j-1,k}\right)+2\left(5-\frac{\omega}{r_i^2}\right)\left(U_{i+1,j,k}+U_{i-1,j,k}\right) \\
&\quad+2\left(\frac{5\omega}{r_i^2}-1\right)\left(U_{i,j+1,k}+U_{i,j-1,k}\right)+(1+\alpha)\left(U_{i+1,j,k+1}+U_{i+1,j,k-1}+U_{i-1,j,k+1}+U_{i-1,j,k-1}\right) \\
&\quad+2(5-\alpha)\left(U_{i+1,j,k}+U_{i-1,j,k}\right)+2(5\alpha-1)\left(U_{i,j+1,k}+U_{i,j-1,k}\right)-20\left(2+\alpha+\frac{\omega}{r_i^2}\right)U_{i,j,k}\Bigg] \\
&\quad+\frac{1}{12}\Bigg[\left(\frac{1}{(r_i\Delta\theta)^2}+\frac{1}{(\Delta z)^2}\right)\left(U_{i,j+1,k+1}+U_{i,j+1,k-1}+U_{i,j-1,k+1}+U_{i,j-1,k-1}\right)-20\left(\frac{1}{(r_i\Delta\theta)^2}+\frac{1}{(\Delta z)^2}\right)U_{i,j,k} \\
&\quad+2\left(\frac{5}{(r_i\Delta\theta)^2}-\frac{1}{(\Delta z)^2}\right)\left(U_{i,j+1,k}+U_{i,j-1,k}\right)+2\left(\frac{5}{(\Delta z)^2}-\frac{1}{(r_i\Delta\theta)^2}\right)\left(U_{i,j,k+1}+U_{i,j,k-1}\right) \\
&\quad-\frac{1}{12}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r_i^2}\frac{\partial^2}{\partial\theta^2}\right)\left((\Delta r)^2\frac{\partial^2}{\partial r^2}+(\Delta\theta)^2\frac{\partial^2}{\partial\theta^2}\right)U_{i,j,k}-\frac{1}{12}\left(\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z^2}\right)\left((\Delta r)^2\frac{\partial^2}{\partial r^2}+(\Delta z)^2\frac{\partial^2}{\partial z^2}\right)U_{i,j,k} \\
&\quad-\frac{1}{12}\left(\frac{1}{r_i^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right)\left((\Delta\theta)^2\frac{\partial^2}{\partial r^2}+(\Delta z)^2\frac{\partial^2}{\partial z^2}\right)U_{i,j,k}+\frac{\phi\delta_{2r}\left(U_{i,j+1,k}+U_{i,j-1,k}+U_{i,j,k+1}+U_{i,j,k-1}\right)}{\Delta r} \\
&\quad+\frac{(1-4\phi)\delta_{2r}U_{i,j,k}}{\Delta r}-\frac{1}{3}(\Delta r)^2\frac{\partial^3 U_{i,j,k}}{\partial r^3}-\phi(\Delta\theta)^2\frac{\partial^3 U_{i,j,k}}{\partial r\partial\theta^2}-\phi(\Delta z)^2\frac{\partial^3 U_{i,j,k}}{\partial r\partial z^2}+O\left((\Delta r)^4+(\Delta\theta)^4+(\Delta z)^4\right)
\end{aligned}
\tag{11}
$$

Now choose $\phi=\frac{1}{12}$ and consider the following terms in (11)

$$
\begin{aligned}
&-\frac{1}{12}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r_i^2}\frac{\partial^2}{\partial\theta^2}\right)\left((\Delta r)^2\frac{\partial^2}{\partial r^2}+(\Delta\theta)^2\frac{\partial^2}{\partial\theta^2}\right)U_{i,j,k} \\
&-\frac{1}{12}\left(\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z^2}\right)\left((\Delta r)^2\frac{\partial^2}{\partial r^2}+(\Delta z)^2\frac{\partial^2}{\partial z^2}\right)U_{i,j,k}-\frac{1}{3r_i}(\Delta r)^2\frac{\partial^3 U_{i,j,k}}{\partial r^3} \\
&-\frac{1}{12}\left(\frac{1}{r_i^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right)\left((\Delta\theta)^2\frac{\partial^2}{\partial r^2}+(\Delta z)^2\frac{\partial^2}{\partial z^2}\right)U_{i,j,k}-\frac{1}{12r_i}\left((\Delta\theta)^2\frac{\partial^3 U_{i,j,k}}{\partial r\partial\theta^2}+(\Delta z)^2\frac{\partial^3 U_{i,j,k}}{\partial r\partial z^2}\right) \\
&=-\frac{1}{3r_i}(\Delta r)^2\frac{\partial^3 U}{\partial r^3}-\frac{1}{12}\left((\Delta r)^2\frac{\partial^2}{\partial r^2}+(\Delta\theta)^2\frac{\partial^2}{\partial\theta^2}+(\Delta z)^2\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r_i^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right)U_{i,j,k} \\
&\quad-\frac{1}{12}\left((\Delta r)^2\frac{\partial^2}{\partial r^2}\left(\frac{\partial^2}{\partial r^2}\right)+(\Delta\theta)^2\frac{\partial^2}{\partial\theta^2}\left(\frac{\partial^2}{\partial\theta^2}\right)+(\Delta z)^2\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2}{\partial z^2}\right)\right)U_{i,j,k} \\
&\quad-\frac{1}{12r_i}\left((\Delta\theta)^2\frac{\partial^3 U_{i,j,k}}{\partial r\partial\theta^2}+(\Delta z)^2\frac{\partial^3 U_{i,j,k}}{\partial r\partial z^2}\right) \\
&=-\frac{1}{12}\left((\Delta r)^2\frac{\partial^2}{\partial r^2}+(\Delta\theta)^2\frac{\partial^2}{\partial\theta^2}+(\Delta z)^2\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r_i^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right)U_{i,j,k} \\
&\quad-\frac{1}{12}\left(\frac{1}{r_i}\frac{\partial}{\partial r}\left((\Delta r)^2\frac{\partial^2}{\partial r^2}+(\Delta\theta)^2\frac{\partial^2}{\partial\theta^2}+(\Delta z)^2\frac{\partial^2}{\partial z^2}\right)U_{i,j,k}\right)-\frac{1}{4r_i}(\Delta r)^2\frac{\partial^3 U_{i,j,k}}{\partial r^3} \\
&=-\frac{1}{12}\left((\Delta r)^2\frac{\partial^2}{\partial r^2}+(\Delta\theta)^2\frac{\partial^2}{\partial\theta^2}+(\Delta z)^2\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r_i}\frac{\partial}{\partial r}+\frac{1}{r_i^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right)U_{i,j,k}-\frac{1}{4r_i}(\Delta r)^2\frac{\partial^3 U_{i,j,k}}{\partial r^3}
\end{aligned}
\tag{12}
$$

Again we can write the term $-\frac{1}{4r_i}(\Delta r)^2 \frac{\partial^3 U_{i,j,k}}{\partial r^3}$ in (12) as

$$\begin{aligned}
&-\frac{(\Delta r)^2}{4r_i}\frac{\partial^3 U_{i,j,k}}{\partial r^3} \\
&= -\frac{(\Delta r)^2}{4r_i}\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r_i}\frac{\partial}{\partial r}+\frac{1}{r_i^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}\right)U_{i,j,k} \\
&\quad +\frac{(\Delta r)^2}{4r_i}\frac{\partial}{\partial r}\left(\frac{1}{r_i}\frac{\partial U_{i,j,k}}{\partial r}\right)+\frac{(\Delta r)^2}{4r_i}\frac{\partial}{\partial r}\left(\frac{1}{r_i^2}\frac{\partial^2 U_{i,j,k}}{\partial \theta^2}\right)+\frac{(\Delta r)^2}{4r_i}\frac{\partial}{\partial r}\left(\frac{\partial^2 U_{i,j,k}}{\partial z^2}\right) \\
&= -\frac{(\Delta r)^2}{4r_i}\frac{\partial f}{\partial r}-\frac{(\Delta r)^2}{4r_i^3}\frac{\partial U_{i,j,k}}{\partial r}+\frac{(\Delta r)^2}{4r_i^2}\frac{\partial^2 U_{i,j,k}}{\partial r^2}-\frac{1}{2}\frac{(\Delta r)^2}{r_i^4}\frac{\partial^2 U_{i,j,k}}{\partial \theta^2}+\frac{(\Delta r)^2}{4r_i^3}\frac{\partial}{\partial r}\left(\frac{\partial^2 U_{i,j,k}}{\partial \theta^2}\right) \\
&\quad +\frac{(\Delta r)^2}{4r_i}\frac{\partial}{\partial r}\left(\frac{\partial^2 U_{i,j,k}}{\partial z^2}\right)
\end{aligned} \tag{13}$$

Using (12), (13), and multiplying both sides of (11) by $12(\Delta r)^2$ and rearranging and simplifying further, we get

$$\begin{aligned}
&(\Delta r)^2\left(24+\delta_r^2+\delta_\theta^2+\delta_z^2+\frac{3\Delta r}{2r_i}\delta_{2r}\right)f_{i,j,k} \\
&= a_0(i)U_{i,j,k}+a_1(i)U_{i+1,j,k}+a_2(i)U_{i-1,j,k} \\
&\quad +a_3(i)\left(U_{i,j+1,k}+U_{i,j-1,k}\right)+a_4(i)\left(U_{i,j,k+1}+U_{i,j,k-1}\right)+a_5(i)\left(U_{i+1,j+1,k}+U_{i+1,j-1,k}\right) \\
&\quad +a_6(i)\left(U_{i-1,j+1,k}+U_{i-1,j-1,k}\right)+a_7(i)\left(U_{i+1,j,k+1}+U_{i+1,j,k-1}\right)+a_8(i)\left(U_{i-1,j,k+1}+U_{i-1,j,k-1}\right) \\
&\quad +a_9(i)\left(U_{i,j+1,k+1}+U_{i,j-1,k+1}+U_{i,j+1,k-1}+U_{i,j-1,k-1}\right)
\end{aligned} \tag{14}$$

where

$$a_0(i) = -40\left(1+\alpha+\frac{\omega}{r_i^2}\right)-6\frac{(\Delta r)^2}{r_i^2}+12\frac{\omega}{r_i^2}\frac{(\Delta r)^2}{r_i^2}$$

$$a_1(i) = 20-2\alpha-\frac{2\omega}{r_i^2}+8\frac{\Delta r}{r_i}-\frac{3}{2}\left(\frac{\Delta r}{r_i}\right)^3+3\left(\frac{\Delta r}{r_i}\right)^2-3\frac{\omega}{r_i^2}\frac{\Delta r}{r_i}-3\alpha\frac{\Delta r}{r_i}$$

$$a_2(i) = 20-2\alpha-\frac{2\omega}{r_i^2}-8\frac{\Delta r}{r_i}+\frac{3}{2}\left(\frac{\Delta r}{r_i}\right)^3+3\left(\frac{\Delta r}{r_i}\right)^2+3\frac{\omega}{r_i^2}\frac{\Delta r}{r_i}+3\alpha\frac{\Delta r}{r_i}$$

$$a_3(i) = -2\alpha+12\frac{\omega}{r_i^2}-2 \qquad a_4(i) = 20\alpha-\frac{2\omega}{r_i^2}-2$$

$$a_5(i) = 1+\frac{\omega}{r_i^2}+\frac{\Delta r}{r_i}+\frac{3}{2}\frac{\omega}{r_i^2}\frac{\Delta r}{r_i} \qquad a_6(i) = 1+\frac{\omega}{r_i^2}-\frac{\Delta r}{r_i}-\frac{3}{2}\frac{\omega}{r_i^2}\frac{\Delta r}{r_i}$$

$$a_7(i) = 1+\alpha+\frac{\Delta r}{r_i}+\frac{3}{2}\alpha\frac{\Delta r}{r_i} \qquad a_8(i) = 1+\alpha-\frac{\Delta r}{r_i}-\frac{3}{2}\alpha\frac{\Delta r}{r_i} \qquad a_9(i) = \alpha+\frac{\omega}{r_i^2}$$

The system of equations in (14) is a linear sparse system, and thereby when solving we save both work and storage compared with a general system of equations. Such savings are basically true of finite difference methods: they yield sparse systems because each equation involves only few variables.

To solve equation (14), consider first in the θ direction, next in the z direction and lastly in the r direction, and thus (14) can be written in matrix form as

$$AU = \mathcal{B} \tag{15}$$

where

$$A = \begin{pmatrix} R_1 & S_1 & & & & & \\ T_2 & R_2 & S_2 & & & & \\ & T_3 & R_3 & S_3 & & & \\ & & & \ddots & & & \\ & & & & T_{M-1} & R_{M-1} & S_{M-1} \\ & & & & & T_M & R_M \end{pmatrix}$$

and it has M blocks and each is of order NP.

$$R_i = \begin{pmatrix} R_i' & R_i'' & & & & \\ R_i'' & R_i' & R_i'' & & & \\ & R_i'' & R_i' & R_i'' & & \\ & & & \ddots & & \\ & & & R_i'' & R_i' & R_i'' \\ & & & & R_i'' & R_i' \end{pmatrix}, \quad S_i = \begin{pmatrix} S_i' & S_i'' & & & & \\ S_i'' & S_i' & S_i'' & & & \\ & S_i'' & S_i' & S_i'' & & \\ & & & \ddots & & \\ & & & S_i'' & S_i' & S_i'' \\ & & & & S_i'' & S_i' \end{pmatrix}$$

$$T_i = \begin{pmatrix} T_i' & T_i'' & & & & \\ T_i'' & T_i' & T_i'' & & & \\ & T_i'' & T_i' & T_i'' & & \\ & & & \ddots & & \\ & & & T_i'' & T_i' & T_i'' \\ & & & & T_i'' & T_i' \end{pmatrix}$$

R_i, S_i, and T_i are of order NP.

For the domain D_1

$$R_i' = \begin{pmatrix} a_0(i) & a_3(i) & & & & \\ a_3(i) & a_0(i) & a_3(i) & & & \\ & a_3(i) & a_0(i) & a_3(i) & & \\ & & & \ddots & & \\ & & & a_3(i) & a_0(i) & a_3(i) \\ & & & & a_3(i) & a_0(i) \end{pmatrix}$$

$$R_i'' = \begin{pmatrix} a_4(i) & a_9(i) & & & & \\ a_9(i) & a_4(i) & a_9(i) & & & \\ & a_9(i) & a_4(i) & a_9(i) & & \\ & & & \ddots & & \\ & & & a_9(i) & a_4(i) & a_9(i) \\ & & & & a_9(i) & a_4(i) \end{pmatrix}$$

$$S_i' = \begin{pmatrix} a_1(i) & a_5(i) & & & & \\ a_5(i) & a_1(i) & a_5(i) & & & \\ & a_5(i) & a_1(i) & a_5(i) & & \\ & & & \ddots & & \\ & & & a_5(i) & a_1(i) & a_5(i) \\ & & & & a_5(i) & a_1(i) \end{pmatrix}$$

$$S_i'' = \begin{pmatrix} a_7(i) & & & & \\ & a_7(i) & & & \\ & & a_7(i) & & \\ & & & \ddots & \\ & & & & a_7(i) \end{pmatrix}, \quad T_i' = \begin{pmatrix} a_2(i) & a_6(i) & & & & \\ a_6(i) & a_2(i) & a_6(i) & & & \\ & a_6(i) & a_2(i) & a_6(i) & & \\ & & & \ddots & & \\ & & & a_6(i) & a_2(i) & a_6(i) \\ & & & & a_6(i) & a_2(i) \end{pmatrix}$$

$$T_i'' = \begin{pmatrix} a_8(i) & & & & \\ & a_8(i) & & & \\ & & a_8(i) & & \\ & & & \ddots & \\ & & & & a_8(i) \end{pmatrix}$$

For the domain D_2,

$$R_i' = \begin{pmatrix} a_0(i) & a_3(i) & & & & & a_3(i) \\ a_3(i) & a_0(i) & a_3(i) & & & & \\ & a_3(i) & a_0(i) & a_3(i) & & & \\ & & & \ddots & & & \\ & & & & a_3(i) & a_0(i) & a_3(i) \\ a_3(i) & & & & & a_3(i) & a_0(i) \end{pmatrix}$$

$$R_i'' = \begin{pmatrix} a_4(i) & a_9(i) & & & & & a_9(i) \\ a_9(i) & a_4(i) & a_9(i) & & & & \\ & a_9(i) & a_4(i) & a_9(i) & & & \\ & & & \ddots & & & \\ & & & & a_9(i) & a_4(i) & a_9(i) \\ a_9(i) & & & & & a_9(i) & a_4(i) \end{pmatrix}$$

$$S_i' = \begin{pmatrix} a_1(i) & a_5(i) & & & & & a_5(i) \\ a_5(i) & a_1(i) & a_5(i) & & & & \\ & a_5(i) & a_1(i) & a_5(i) & & & \\ & & & \ddots & & & \\ & & & & a_5(i) & a_1(i) & a_5(i) \\ a_5(i) & & & & & a_5(i) & a_1(i) \end{pmatrix}$$

$$T_i' = \begin{pmatrix} a_2(i) & a_6(i) & & & & & a_6(i) \\ a_6(i) & a_2(i) & a_6(i) & & & & \\ & a_6(i) & a_2(i) & a_6(i) & & & \\ & & & \ddots & & & \\ & & & & a_6(i) & a_2(i) & a_6(i) \\ a_6(i) & & & & & a_6(i) & a_2(i) \end{pmatrix}$$

S_i'' and T_i'' are the same as in the domain D_1.

Here in D_2, the matrices $R_i', R_i'', S_i', S_i'', T_i'$, and T_i'' are circulant matrices of order N; and

$$\mathcal{B} = \begin{bmatrix} \boldsymbol{B}_0 & \boldsymbol{B}_1 & \boldsymbol{B}_2 & \cdots & \boldsymbol{B}_M \end{bmatrix}^{\mathrm{T}}, \ \boldsymbol{B}_i = \begin{bmatrix} \boldsymbol{d}_{i1} & \boldsymbol{d}_{i2} & \boldsymbol{d}_{i3} & \cdots & \boldsymbol{d}_{iP} \end{bmatrix}^{\mathrm{T}} \text{ and } \boldsymbol{d}_{ik} = \begin{bmatrix} d_{ij1} & d_{ij2} & \cdots & d_{ijP} \end{bmatrix}^{\mathrm{T}}$$

such that each d_{ijk} represents a known boundary values of U and values of f, and

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{U}_1 & \boldsymbol{U}_2 & \boldsymbol{U}_3 & \cdots & \boldsymbol{U}_M \end{bmatrix}^{\mathrm{T}}, \ \boldsymbol{U}_i = \begin{pmatrix} U_{i1} & U_{i2} & U_{i3} & \cdots & U_{iP} \end{pmatrix}^{\mathrm{T}} \text{ and } U_{ij} = \begin{pmatrix} U_{ij1} & U_{ij2} & U_{ij3} & \cdots & U_{ijP} \end{pmatrix}^{\mathrm{T}}$$

Thus, we write (15) as

$$\begin{aligned}&R_1\boldsymbol{U}_1+S_1\boldsymbol{U}_2=\boldsymbol{B}_1\\&T_2\boldsymbol{U}_1+R_2\boldsymbol{U}_2+S_2\boldsymbol{U}_3=\boldsymbol{B}_2\\&T_3\boldsymbol{U}_2+R_3\boldsymbol{U}_3+S_3\boldsymbol{U}_4=\boldsymbol{B}_3\\&\cdots\\&T_M\boldsymbol{U}_{M-1}+R_M\boldsymbol{U}_M=\boldsymbol{B}_M\end{aligned}\tag{16}$$

3. Extended Hockney's Method

Observe that matrices R_i', R_i'', S_i' and T_i' are real symmetric matrices and hence their eigenvalues and eigenvectors can easily be obtained as

For D_1

$$\lambda_{ij}=a_0(i)+2a_3(i)\cos\left(\frac{j\pi}{N+1}\right),\quad \beta_{ij}=a_4(i)+2a_9(i)\cos\left(\frac{j\pi}{N+1}\right),\quad \eta_{ij}=a_1(i)+2a_5(i)\cos\left(\frac{j\pi}{N+1}\right)$$

$$\zeta_{ij}=a_2(i)+2a_6(i)\cos\left(\frac{j\pi}{N+1}\right),\quad i=1(1)M \text{ and } j=1(1)N$$

and **for** D_2

$$\lambda_{ij}=a_0(i)+2a_3(i)\cos\left(\frac{2\pi j}{N}\right),\quad \beta_{ij}=a_4(i)+2a_9(i)\cos\left(\frac{2\pi j}{N}\right),\quad \eta_{ij}=a_1(i)+2a_5(i)\cos\left(\frac{2\pi j}{N}\right)$$

$$\zeta_{ij}=a_2(i)+2a_6(i)\cos\left(\frac{2\pi j}{N}\right),\quad i=1(1)M \text{ and } j=1(1)N$$

Let $\boldsymbol{q}_j$ be an eigenvector of R_i', R_i'', S_i' and T_i' corresponding to the eigenvalue $\lambda_{ij}, \beta_{ij}, \eta_{ij}$ and ξ_{ij}; and matrix $Q=\begin{bmatrix}\boldsymbol{q}_1 & \boldsymbol{q}_2 & \boldsymbol{q}_3 & \cdots & \boldsymbol{q}_N\end{bmatrix}^{\mathrm{T}}$ be a modal matrix of R_i', R_i'', S_i' and T_i', $\forall i$ such that $Q^TQ=I$

The $N\times N$ modal matrix Q is defined by

$$q_{ij}=\sqrt{\frac{2}{N+1}}\sin\left(\frac{ij\pi}{N+1}\right),\ i,j=1(1)N \text{ for } D_1;\quad q_{ij}=\left(\frac{\cos\theta+\sin\theta}{\sqrt{N}}\right) \text{ where } \theta=\frac{2\pi}{N}(i-1)(j-1)$$

$i,j=1(1)N$ for D_2

Let $\mathbb{Q}=\operatorname{diag}(Q,Q,Q,\cdots,Q)$ be a matrix of order NP; thus $\mathbb{Q}$ satisfy $\mathbb{Q}^T\mathbb{Q}=I$ since $Q^TQ=I$.

Since R_i, S_i and T_i are symmetric matrices, we have

$$\mathbb{Q}^TR_i\mathbb{Q}=\operatorname{diag}\left(\mu_{j1}^i,\mu_{j2}^i,\cdots,\mu_{jP}^i\right)=\Upsilon_i \quad\text{where}\quad \mu_{jP}^i=\lambda_{ij}+2\beta_{ij}\cos\left(\frac{k\pi}{P+1}\right)$$

$$\mathbb{Q}^TS_i\mathbb{Q}=\operatorname{diag}\left(\xi_{j1}^i,\xi_{j2}^i,\cdots,\xi_{jP}^i\right)=\Phi_i \quad\text{where}\quad \mu_{jP}^i=\eta_{ij}+2a_7(i)\cos\left(\frac{k\pi}{P+1}\right)$$

$$\mathbb{Q}^TT_i\mathbb{Q}=\operatorname{diag}\left(\tau_{j1}^i,\tau_{j2}^i,\cdots,\tau_{jP}^i\right)=\Psi_i \quad\text{where}\quad \tau_{jP}^i=\zeta_{ij}+2a_8(i)\cos\left(\frac{k\pi}{P+1}\right)$$

Let

$$\mathbb{Q}^T\boldsymbol{U}_i=\boldsymbol{V}_i\Rightarrow\boldsymbol{U}_i=\mathbb{Q}\boldsymbol{V}_i,\quad \mathbb{Q}^T\boldsymbol{B}_i=\bar{\boldsymbol{b}}_i\Rightarrow\boldsymbol{B}_i=\mathbb{Q}\bar{\boldsymbol{b}}_i\tag{17}$$

where

$$\boldsymbol{V}_i=\begin{bmatrix}V_{i1} & V_{i2} & V_{i3} & \cdots & V_{iP}\end{bmatrix}^{\mathrm{T}},\ V_{ik}=\begin{bmatrix}v_{i1k} & v_{i2k} & v_{i3k} & \cdots & v_{iNk}\end{bmatrix}^{\mathrm{T}};$$

$$\bar{\boldsymbol{b}}_i=\begin{bmatrix}\boldsymbol{b}_{i1} & \boldsymbol{b}_{i2?} & \cdots & \boldsymbol{b}_{ik}\end{bmatrix}^{\mathrm{T}} \text{ and } \boldsymbol{b}_{ik}=\begin{bmatrix}b_{i1k} & b_{i2k} & \cdots & b_{iNk}\end{bmatrix}^{\mathrm{T}}$$

Pre-multiplying Equation (16) by $\mathbb{Q}^T$ and applying (17), we get

$$\begin{aligned} &\Upsilon_1 \boldsymbol{V}_1 + \Phi_1 \boldsymbol{V}_2 = \bar{\boldsymbol{b}}_1 \\ &\Psi_2 \boldsymbol{V}_1 + \Upsilon_2 \boldsymbol{V}_2 + \Phi_2 \boldsymbol{V}_3 = \bar{\boldsymbol{b}}_2 \\ &\Psi_3 \boldsymbol{V}_2 + \Upsilon_3 \boldsymbol{V}_3 + \Phi_3 \boldsymbol{V}_4 = \bar{\boldsymbol{b}}_3 \\ &\cdots \\ &\Psi_M \boldsymbol{V}_{M-1} + \Upsilon_M \boldsymbol{V}_M = \bar{\boldsymbol{b}}_M \end{aligned} \tag{18}$$

Now from each Equation of (18) we collect the first equations and put them as one group of equation

$$\begin{aligned} &\mu_{jk}^1 V_k^1 + \xi_{jk}^i V_k^2 = \bar{\boldsymbol{b}}_1 \\ &\tau_{jk}^2 V_k^1 + \mu_{jk}^2 V_k^2 + \xi_{jk}^2 V_k^3 = \bar{\boldsymbol{b}}_2 \\ &\tau_{jk}^3 V_k^2 + \mu_{jk}^3 V_k^3 + \xi_{jk}^3 V_k^4 = \bar{\boldsymbol{b}}_3 \\ &\tau_{jk}^{M-1} V_k^{M-2} + \mu_{jk}^{M-1} V_k^{M-1} + \xi_{jk}^{M-1} V_k^M = \bar{\boldsymbol{b}}_{M-1} \\ &\tau_{jk}^M V_k^{M-1} + \mu_{jk}^M V_k^M = \bar{\boldsymbol{b}}_M \end{aligned} \tag{19}$$

Now put $k=1$ in Equation (19) and collect the entire first set of equations, for $i=1,2,3,\cdots,M$ and $j=1,2,3,\cdots,N$ to get

$$\tau_{j1}^i v_{j1}^{i-1} + \mu_{j1}^i v_{j1}^i + \xi_{j1}^i v_{j1}^{i+1} = \bar{\boldsymbol{b}}_i \quad \text{and} \quad v_{j1}^0 = 0 = v_{j1}^M \tag{20a}$$

Again consider the second equations by putting $k=2$, and get

$$\tau_{j2}^i v_{j2}^{i-1} + \mu_{j2}^i v_{j2}^i + \xi_{j2}^i v_{j2}^{i+1} = \bar{\boldsymbol{b}}_i \quad \text{and} \quad v_{j2}^0 = 0 = v_{j2}^M \tag{20b}$$

Continuing in this manner and finally considering the last equations for $k=P$, we obtain

$$\tau_{jP}^i v_{jP}^{i-1} + \mu_{jP}^i v_{jP}^i + \xi_{jP}^i v_{jP}^{i+1} = \bar{\boldsymbol{b}}_i \quad \text{and} \quad v_{jP}^0 = 0 = v_{jP}^M \tag{20c}$$

All these set of Equations (20a)-(20c) are tri-diagonal ones and hence we solve for v_{jk}^i by using Thomas algorithm. With the help of (17) again we get all u_{jk}^i and this solves (14) as desired. By doing this we generally reduce the number of computations and computational time.

4. Numerical Results

In order to test the efficiency and adaptability of the proposed method, computational experiments are done on some selected problems that may arise in practice, for which the analytical solutions of U are known to us. The computed solutions are found for all grid points for any values of M, N and P. Here results are reported at some randomly taken mesh points in terms of the absolute maximum error from **Table 1** to **7**.

Example 1. Consider $\nabla^2 U = 0$ with the boundary conditions $U(0,\theta,z) = 0$, $U(1,\theta,z) = z\sin\theta$

$$U(r,0,z) = 0 = U(r,\pi,z), \text{ and } U(r,\theta,0) = 0, U(r,\theta,1) = r\sin\theta$$

The analytical solution is $U(r,\theta,z) = rz\sin\theta$ and the computed results of this example are shown in **Table 1**.

Example 2. Consider $\nabla^2 U = -\pi^2 r\cos\theta\sin\pi z$ with the boundary conditions

$$U(1,\theta,z) = \cos\theta\sin\pi z, \quad U(2,\theta,z) = 2\cos\theta\sin\pi z$$

$$U(r,0,z) = r\sin\pi z, U\left(r,\frac{\pi}{2},z\right) = 0, \text{ and } U(r,\theta,0) = 0 = U(r,\theta,1)$$

The analytical solution is $U(r,\theta,z) = r\cos\theta\sin\pi z$ and the computed results of this example are shown in **Table 2**.

Example 3. Consider $\nabla^2 U = -3\cos\theta$ with the boundary conditions

$$U(0,\theta,z) = U(1,\theta,z) = -2z, \quad U(r,0,z) = r(1-r) - 2z, \quad U\left(r,\frac{\pi}{2},z\right) = -2z$$

Table 1. Maximum absolute error of example 1.

(N,P,M)	Max. absolute error	(N,P,M)	Max. absolute error
(9,9,9)	3.51670e-005	(29,9,39)	1.37257e-006
(9,9,29)	1.46565e-005	(29,19,9)	4.15180e-006
(9,19,9)	3.53325e-005	(29,29,19)	2.45633e-006
(9,19,19)	2.06578e-005	(29,29,29)	1.74383e-006
(9,29,39)	1.13280e-005	(29,39,19)	2.45924e-006
(9,39,29)	1.46438e-005	(29,39,29)	1.74829e-006
(19,9,9)	9.21838e-006	(39,9,19)	1.35171e-006
(19,9,19)	5.32850e-006	(39,9,39)	7.75143e-007
(19,19,19)	5.46733e-006	(39,19,29)	9.82456e-007
(19,29,39)	3.02425e-006	(39,29,19)	1.38647e-006
(19,39,9)	9.27536e-006	(39,39,9)	2.34568e-006
(19,39,39)	3.02636e-006	(39,39,39)	7.68613e-007

Table 2. Maximum absolute error of example 2.

(N,P,M)	Max. absolute error	(N,P,M)	Max. absolute error
(9,9,9)	2.93159e-003	(29,9,39)	2.98714e-003
(9,9,29)	2.95649e-003	(29,19,9)	7.39877e-004
(9,19,9)	7.32025e-004	(29,29,19)	3.31950e-004
(9,19,19)	7.38648e-004	(29,29,29)	3.31771e-004
(9,29,39)	3.27574e-004	(29,39,19)	1.86718e-004
(9,39,29)	1.83450e-004	(29,39,29)	1.86618e-004
(19,9,9)	2.95328e-003	(39,9,19)	2.98618e-003
(19,9,19)	2.97861e-003	(39,9,39)	2.98710e-003
(19,19,19)	7.44907e-004	(39,19,29)	7.46353e-004
(19,29,39)	3.31145e-004	(39,29,19)	3.31953e-004
(19,39,9)	1.84585e-004	(39,39,9)	1.84916e-004
(19,39,39)	1.86232e-004	(39,39,39)	1.86784e-004

$$U(r,\theta,0)=r(1-r)\cos\theta,\ U(r,\theta,1)=r(1-r)\cos\theta-2$$

The analytical solution is $U(r,\theta,z)=r(1-r)\cos\theta-2z$ and the computed results of this example are shown in **Table 3**.

Example 4. Consider $\nabla^2 U=-\pi^2\left(r^2-\frac{1}{r^2}\right)\sin(2\theta)\sin(\pi z)$ with the boundary conditions

$$U(1,\theta,z)=0,\ U(2,\theta,z)=\frac{15}{4}\sin(2\theta)\sin(\pi z),\ U(r,0,z)=0=U\left(r,\frac{\pi}{2},z\right)\ \text{and}\ U(r,\theta,0)=0=U(r,\theta,1)$$

The analytical solution is $U(r,\theta,z)=\left(r^2-\frac{1}{r^2}\right)\sin(2\theta)\sin(\pi z)$ and the computed results of this example are shown in **Table 4**.

Example 5 Consider $\nabla^2 U=\left(8rz(1-z)-2r^3\right)(\sin\theta+\cos\theta)$, where $0\le\theta<2\pi$ with the boundary conditions

$$U(0,\theta,z)=0,\ U(1,\theta,z)=z(1-z)(\sin\theta+\cos\theta)\ \ U(r,\theta,0)=0=U(r,\theta,1)$$

The analytical solution is $U(r,\theta,z)=r^3 z(1-z)(\sin\theta+\cos\theta)$ and the computed results of this example are shown in **Table 5**.

Table 3. Maximum absolute error of example 3.

(N,P,M)	Max. absolute error	(N,P,M)	Max. absolute error
(9,9,9)	1.81124e-004	(29,9,39)	1.16544e-005
(9,9,29)	4.45263e-005	(29,19,9)	1.82484e-004
(9,19,9)	1.81185e-004	(29,29,19)	4.61297e-005
(9,19,19)	6.02480e-005	(29,29,29)	2.04978e-005
(9,29,39)	3.97430e-005	(29,39,19)	4.61300e-005
(9,39,29)	4.46327e-005	(29,39,29)	2.04979e-005
(19,9,9)	1.81939e-004	(39,9,19)	4.61828e-005
(19,9,19)	4.59426e-005	(39,9,39)	1.17058e-005
(19,19,19)	4.59583e-005	(39,19,29)	2.05467e-005
(19,29,39)	1.50833e-005	(39,29,19)	4.61879e-005
(19,39,9)	1.82013e-004	(39,39,9)	1.82652e-004
(19,39,39)	1.50852e-005	(39,39,39)	1.15493e-005

Table 4. Maximum absolute error of example 4.

(N,P,M)	Max. absolute error	(N,P,M)	Max. absolute error
(9,9,9)	3.68396e-003	(29,9,39)	3.98135e-003
(9,9,29)	4.07400e-003	(29,19,9)	6.33780e-004
(9,19,9)	7.68229e-004	(29,29,19)	3.64070e-004
(9,19,19)	1.04366e-003	(29,29,29)	4.17368e-004
(9,29,39)	5.73867e-004	(29,39,19)	1.75928e-004
(9,39,29)	3.62888e-004	(29,39,29)	2.24720e-004
(19,9,9)	3.58663e-003	(39,9,19)	3.89251e-003
(19,9,19)	3.92179e-003	(39,9,39)	3.97355e-003
(19,19,19)	9.34774e-004	(39,19,29)	9.60868e-004
(19,29,39)	4.61633e-004	(39,29,19)	3.55183e-004
(19,39,9)	7.29565e-004	(39,39,9)	7.23913e-004
(19,39,39)	2.68695e-004	(39,39,39)	2.34933e-004

Table 5. Maximum absolute error of example 5.

(N,P,M)	Max. absolute error	(N,P,M)	Max. absolute error
(9,9,9)	5.97062e-004	(29,9,39)	1.65910e-004
(9,9,29)	4.42157e-004	(29,19,9)	4.11093e-004
(9,19,9)	5.09956e-004	(29,29,19)	1.01380e-004
(9,19,19)	3.72361e-004	(29,29,29)	6.92680e-005
(9,29,39)	3.26827e-004	(29,39,19)	1.03392e-004
(9,39,29)	3.27891e-004	(29,39,29)	6.38312e-005
(19,9,9)	3.72181e-004	(39,9,19)	1.80739e-004
(19,9,19)	2.39220e-004	(39,9,39)	1.49613e-004
(19,19,19)	1.52973e-004	(39,19,29)	6.95506e-005
(19,29,39)	1.04227e-004	(39,29,19)	1.06985e-004
(19,39,9)	3.96923e-004	(39,39,9)	4.28673e-004
(19,39,39)	9.84850e-005	(39,39,39)	3.91182e-005

This example was considered by M.C. Lai [1] as a test problem and our results are better than their results in terms of accuracy. For instance, for (8,16,16) the maximum absolute error in their result is 9.1438e-004 and while ours is 3.28689e-004.

Example 6 Consider $\nabla^2 U = 6rz\cos\theta$, where $0 \le \theta < 2\pi$ with the boundary conditions

$$U(0,\theta,z) = 0,\ U(1,\theta,z) = z\cos^3\theta;\ U(r,\theta,0) = 0 \ \text{ and } \ U(r,\theta,1) = r^3\cos^3\theta$$

The analytical solution is $U(r,\theta,z) = r^3 z\cos^3\theta$ and the computed results are shown in **Table 6**.

Example 5.7 Consider $\nabla^2 U = -\pi^2\left(r^2 - \frac{1}{r^2}\right)\sin(2\theta)\sin(\pi z)$ where $0 \le \theta < 2\pi$ with the boundary conditions

$$U(1,\theta,z) = 0,\ U(2,\theta,z) = \frac{15}{4}\sin(2\theta)\sin(\pi z);\ U(r,\theta,0) = 0 = U(r,\theta,1)$$

The analytical solution is $U(r,\theta,z) = \left(r^2 - \frac{1}{r^2}\right)\sin(2\theta)\sin(\pi z)$ and the computed results of this example are shown in **Table 7**.

Table 6. Maximum absolute error of example 6.

(N,P,M)	Max. absolute error	(N,P,M)	Max. absolute error
(9,9,9)	3.04648e-003	(29,9,39)	3.06543e-004
(9,9,29)	3.18297e-003	(29,19,9)	2.00777e-004
(9,19,9)	3.05549e-003	(29,29,19)	2.85659e-004
(9,19,19)	3.16606e-003	(29,29,29)	3.02059e-004
(9,29,39)	3.19893e-003	(29,39,19)	2.85718e-004
(9,39,29)	3.19459e-003	(29,39,29)	3.02122e-004
(19,9,9)	6.03143e-004	(39,9,19)	1.42033e-004
(19,9,19)	6.87721e-004	(39,9,39)	1.62951e-004
(19,19,19)	6.89766e-004	(39,19,29)	1.58004e-004
(19,29,39)	7.13568e-004	(39,29,19)	1.42553e-004
(19,39,9)	6.05428e-004	(39,39,9)	1.58596e-004
(19,39,39)	7.13712e-004	(39,39,39)	1.63583e-004

Table 7. Maximum absolute error of example 7.

(N,P,M)	Max. absolute error	(N,P,M)	Max. absolute error
(9,9,9)	3.00418e-003	(29,9,39)	4.13706e-003
(9,9,29)	2.36262e-003	(29,19,9)	8.41886e-004
(9,19,9)	4.54924e-003	(29,29,19)	5.78646e-004
(9,19,19)	4.13088e-003	(29,29,29)	6.30456e-004
(9,29,39)	4.49099e-003	(29,39,19)	3.91287e-004
(9,39,29)	4.67590e-003	(29,39,29)	4.41870e-004
(19,9,9)	3.54731e-003	(39,9,19)	4.01710e-003
(19,9,19)	3.84938e-003	(39,9,39)	4.09806e-003
(19,19,19)	1.08325e-003	(39,19,29)	1.10354e-003
(19,29,39)	6.43373e-004	(39,29,19)	5.01257e-004
(19,39,9)	9.83307e-004	(39,39,9)	7.61254e-004
(19,39,39)	4.66590e-004	(39,39,39)	3.82100e-004

5. Conclusions

In this work, we have transformed the three dimensional Poisson's equation in cylindrical coordinates system into a system of algebraic linear equations using its equivalent fourth-order finite difference approximation scheme. The resulting large number of algebraic equation is, then, systematically arranged in order to get a block matrix. By extending Hockney's method to three dimensions, we reduced the obtained matrix into a block tridiagonal matrix, and each block is solved by the help of Thomas algorithm. We have successfully implemented this method to find the solution of the three dimensional Poisson's equation in cylindrical coordinates system and it is found that the method can easily be applied and adapted to find a solution of some related applied problems. The method produced accurate results considering double precision. This method is direct and allows considerable savings in computer storage as well as execution speed.

Therefore, the method is suitable to apply to some three dimensional Poisson's equations.

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