American Journal of Computational Mathematics, **2013, 3, 356-361**
Published Online December 2013 (http://www.scirp.org/journal/ajcm)
http://dx.doi.org/10.4236/ajcm.2013.34045

Fast Finite Difference Solutions of the Three Dimensional Poisson's Equation in Cylindrical Coordinates

Alemayehu Shiferaw, R. C. Mittal
Department of Mathematics, Indian Institute of Technology, Roorkee, India
Email: abelhaim@gmail.com, mittalrc@gmail.com

Received November 1, 2013; revised December 1, 2013; accepted December 8, 2013

ABSTRACT

In this work, the three-dimensional Poisson's equation in cylindrical coordinates system with the Dirichlet's boundary conditions in a portion of a cylinder for $r \neq 0$ is solved directly, by extending the method of Hockney. The Poisson equation is approximated by second-order finite differences and the resulting large algebraic system of linear equations is treated systematically in order to get a block tri-diagonal system. The accuracy of this method is tested for some Poisson's equations with known analytical solutions and the numerical results obtained show that the method produces accurate results.

Keywords: Poisson's Equation; Hockney's Method; Thomas Algorithm

1. Introduction

The three-dimensional Poisson's equation in cylindrical coordinates (r,θ,z) is given by

$$U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} + U_{zz} = f(r,\theta,z) \qquad (1)$$

which is often encountered in heat and mass transfer theory, fluid mechanics, elasticity, electrostatics, and other areas of mechanics and physics. In particular, the Poisson equation describes stationary temperature distribution in the presence of thermal sources or sinks in the domain under consideration.

The analytic solution for the three-dimensional Poisson's equation in cylindrical coordinate system is much more complicated and tedious because of the complexity of the nature of the problems and their geometry, and the availability of appropriate methods. To solve Poisson's equation in polar and cylindrical coordinates geometry, different approaches and numerical methods using finite difference approximation have been developed. For instance, *Chao* [1] developed a direct solver method for the electrostatic potential in a cylindrical region; *Chen* [2] developed a direct spectral collocation Poisson solver for several different domains including a part of a disk, an annulus, a unit disk, and a cylinder using the weighted interpolation technique and non-classical collocation points; *Christopher* [3] developed a solution method in an annulus using conformal mapping and Fast Fourier Transform; *Kalita and Ray* [4] have developed a high order compact scheme on a circular cylinder to solve their problem on incompressible viscous flows; *Lai and Wang* [5] developed a fast direct solvers for Poisson's equation on 2D polar and spherical coordinates based on FFT; *Swarztrauber and Sweet* [6] developed a direct solution of the discrete Poisson equation on a disk in the sense of least squares; *Mittal and Gahlaut* [7,8] developed high order finite difference schemes to solve Poisson's equation in cylindrical symmetry; *Tan* [9] developed a spectrally accurate solution for the three-dimensional Poisson's equation and Helmholtz's equation using Chebyshev series and Fourier series for a simple domain in a cylindrical coordinate system; Iyengar and Manohar [10] derived fourth-order difference schemes for the solution of the Poisson equation which occurs in problems of heat transfer; *Iyengar and Goyal* [11] developed a multigrid method in cylindrical coordinates system; *Lai and Tseng* [12] have developed a fourth-order compact scheme, and their scheme relies on the truncated Fourier series expansion, where the partial differential equations of Fourier coefficients are solved by a formally fourth-order accurate compact difference discretization; *Xu et al.* [13] developed a parallel three-dimensional Poisson solver in cylindrical coordinate system for the electrostatic potential of a charged particle beam in a circular, which used Fourier expansions in the longitudinal and azimuthal directions, and spectral element discretization in the radial direction, and some other developments had also been observed. The need to obtain the best solution for the Poisson's equation is still in progress.

In this paper, we develop a second-order finite difference approximation scheme and solve the resulting large algebraic system of linear equations systematically using block tridiagonal system [14] and extend the Hockney's method [15] to solve the three dimensional Poisson's equation on Cylindrical coordinates system.

2. Finite Difference Approximation

Consider the three dimensional Poisson's equation in cylindrical coordinates (r,θ,z) given by

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}+\frac{\partial^2 U}{\partial z^2}=f(r,\theta,z) \text{ on } D$$

and the boundary condition

$$U(r,\theta,z)=g(r,\theta,z) \text{ on } C \tag{2}$$

where C is the boundary of D and D is

$$D_1=\{(r,\theta,z): R_0<r<R_1, a<z<b, \theta_0<\theta<\theta_1, \theta_0<\theta_1<2\pi\}$$

and

$$D_2=\{(r,\theta,z): R_0<r<R_1, a<z<b, 0\le\theta<2\pi\}$$

Consider **Figure 1** as the geometry of the problem. Let $u(r,\theta,z)$ be discretized at the point (r_i,θ_j,z_k) and for simplicity write a point (r_i,θ_j,z_k) as (i,j,k) and $u(r_i,\theta_j,z_k)$ as $u_{i,j,k}$.

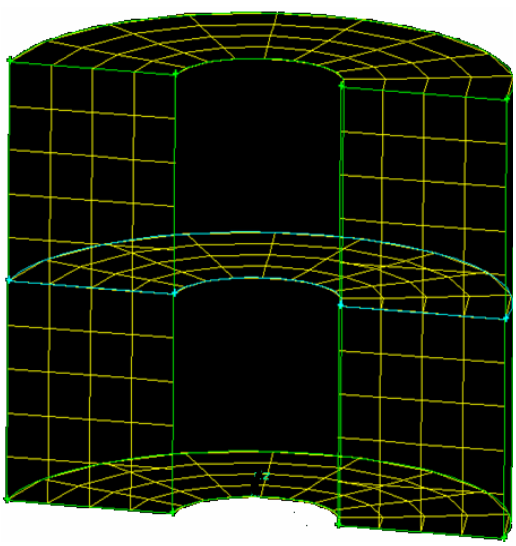

Figure 1. Portion of a Cylinder.

Assume that there are M points along r, N points along θ and P points along the z directions to form the mesh, and let the step size along the direction of r be Δr, of θ be $\Delta\theta$ and z be Δz.

Here $r_i=R_0+i\Delta r$, $\theta_j=\theta_0+j\Delta\theta$ and $z_k=a+k\Delta z$ where $i=1,2,\cdots,M$, $j=1,2,\cdots,N$ and $k=1,2,\cdots,P$ For $r\neq 0$, using the central difference approximation scheme that

$$\begin{aligned}
\frac{\partial u}{\partial r}&=\frac{u_{i+1,j,k}-u_{i-1,j,k}}{2\Delta r}+O(\Delta r)\\
\frac{\partial^2 u}{\partial r^2}&=\frac{u_{i+1,j,k}-2u_{i,j,k}+u_{i-1,j,k}}{(\Delta r)^2}+O\left((\Delta r)^2\right)\\
\frac{\partial^2 u}{\partial \theta^2}&=\frac{u_{i,j+1,k}-2u_{i,j,k}+u_{i,j-1,k}}{(\Delta\theta)^2}+O\left((\Delta\theta)^2\right) \text{ and}\\
\frac{\partial^2 u}{\partial z^2}&=\frac{u_{i,j,k+1}-2u_{i,j,k}+u_{i,j,k-1}}{(\Delta z)^2}+O\left((\Delta z)^2\right)
\end{aligned} \tag{3}$$

Truncating higher order differences of (3) and substituting (3) in (2), we have

$$f_{i,j,k}=\frac{u_{i+1,j,k}-2u_{i,j,k}+u_{i-1,j,k}}{(\Delta r)^2}+\frac{u_{i+1,j,k}-u_{i-1,j,k}}{(2\Delta r)r_i}+\frac{1}{r_i^2}\left(\frac{u_{i,j+1,k}-2u_{i,j,k}+u_{i,j-1,k}}{(\Delta\theta)^2}\right)+\frac{u_{i,j,k+1}-2u_{i,j,k}+u_{i,j,k-1}}{(\Delta z)^2} \tag{4}$$

Let $\omega_i=\frac{\Delta r}{2r_i}$, $\alpha_i=\frac{(\Delta r)^2}{r_i^2(\Delta\theta)^2}$, $\rho=\frac{(\Delta r)^2}{(\Delta z)^2}$ and $y_i=-2(1+\alpha_i+\rho)$.

Multiplying both sides of (4) by $(\Delta r)^2$, rearranging and simplifying further, we get

$$\begin{aligned}
(\Delta r)^2 f_{i,j,k}&=(1+\omega_i)u_{i+1,j,k}+(1-\omega_i)u_{i-1,j,k}\\
&+\alpha_i\left(u_{i,j+1,k}+u_{i,j-1,k}\right)+\rho\left(u_{i,j,k+1}+u_{i,j,k-1}\right)+y_i u_{i,j,k}
\end{aligned} \tag{5}$$

When there are two or more space dimensions the band width is larger and the number of operations goes up and thus the computation for the solution is not such an easy task.

The system of Equations in (5) is a linear sparse system, and thereby saving on both work and storage compared with a general system of equations. Such savings are basically true of finite difference methods: they yield sparse systems because each equation involves only a few variables. Now we use these advantages.

Consider Equation (5) first in the θ direction, next in the z direction and lastly in the r direction, and hence Equation (5) can be put in matrix form as

$$AU=\mathcal{B} \tag{6}$$

where $A = \begin{pmatrix} R_1 & S_1 & & & & \\ S_2^* & R_2 & S_2 & & & \\ & S_3^* & R_3 & S_3 & & \\ & & & \ddots & & \\ & & & S_{M-1}^* & R_{M-1} & S_{M-1} \\ & & & & S_M^* & R_M \end{pmatrix}$

it has *M* blocks and each block is of order NP.

$$R_i = \begin{pmatrix} L_i & \mathrm{T} & & & & \\ \mathrm{T} & L_i & \mathrm{T} & & & \\ & \mathrm{T} & L_i & \mathrm{T} & & \\ & & & \ddots & & \\ & & & \mathrm{T} & L_i & \mathrm{T} \\ & & & & \mathrm{T} & L_i \end{pmatrix} \quad (7)$$

For D_1,

$$L_i = \begin{pmatrix} y_i & \alpha_i & & & & \\ \alpha_i & y_i & \alpha_i & & & \\ & \alpha_i & y_i & \alpha_i & & \\ & & & \ddots & & \\ & & & \alpha_i & y_i & \alpha_i \\ & & & & \alpha_i & y_i \end{pmatrix} \quad (8)$$

For D_2,

$$L_i = \begin{pmatrix} y_i & \alpha_i & & & & \alpha_i \\ \alpha_i & y_i & \alpha_i & & & \\ & \alpha_i & y_i & \alpha_i & & \\ & & & \ddots & & \\ & & & \alpha_i & y_i & \alpha_i \\ \alpha_i & & & & \alpha_i & y_i \end{pmatrix} \quad (9)$$

is a circulant matrix;

Both matrices (8) and (9) are of order N; and $\mathrm{T} = \mathrm{diag}(\rho, \rho, \rho, \cdots, \rho)$ is of order N.

$S_i = \mathrm{diag}(\boldsymbol{\omega}_i, \boldsymbol{\omega}_i, \boldsymbol{\omega}_i, \cdots, \boldsymbol{\omega}_i)$ has P blocks and

$\boldsymbol{\omega}_i = \mathrm{diag}(1+\omega_i, 1+\omega_i, \cdots, 1+\omega_i)$ is of order N

$S_i^* = \mathrm{diag}(\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_i, \boldsymbol{\varphi}_i, \cdots, \boldsymbol{\varphi}_i)$ has P blocks and

$\boldsymbol{\varphi}_i = \mathrm{diag}(1-\omega_i, 1-\omega_i, \cdots, 1-\omega_i)$ is of order N.

$\mathcal{B} = [\boldsymbol{B}_1 \quad \boldsymbol{B}_2 \quad \boldsymbol{B}_3 \quad \cdots \quad \boldsymbol{B}_M]^{\mathrm{T}}$,

$\boldsymbol{B}_k = [\boldsymbol{d}_{1k} \quad \boldsymbol{d}_{2k?} \quad \cdots \quad \boldsymbol{d}_{Nk}]^{\mathrm{T}}$ and

$\boldsymbol{d}_{jk} = [d_{1jk} \quad d_{2jk} \quad \cdots \quad d_{Mjk}]^{\mathrm{T}}$ such that each d_{ijk} represents a known boundary values of U and values of f, and $\boldsymbol{U} = [\boldsymbol{U}_1 \quad \boldsymbol{U}_2 \quad \boldsymbol{U}_3 \quad \cdots \quad \boldsymbol{U}_M]^{\mathrm{T}}$,

$\boldsymbol{U}_i = (U_{i1} \quad U_{i2} \quad U_{i3} \quad \cdots \quad U_{iP})^{\mathrm{T}}$ and

$U_{ik} = (U_{ij1} \quad U_{ij2} \quad U_{ij3} \quad \cdots \quad U_{ijP})^{\mathrm{T}}$

We write (6) a

$$\begin{aligned} R_1\boldsymbol{U}_1 + S_1\boldsymbol{U}_2 &= \boldsymbol{B}_1 \\ S_2^*\boldsymbol{U}_1 + R_2\boldsymbol{U}_2 + S_2\boldsymbol{U}_3 &= \boldsymbol{B}_2 \\ S_3^*\boldsymbol{U}_2 + R_3\boldsymbol{U}_3 + S_3\boldsymbol{U}_4 &= \boldsymbol{B}_3 \\ &\vdots \\ S_M^*\boldsymbol{U}_{M-1} + R_M\boldsymbol{U}_M &= \boldsymbol{B}_M \end{aligned} \quad (10)$$

3. Extended Hockney's Method

Observe that the matrix L_i is a real symmetric matrix and hence its eigenvalues and eigenvectors can easily be obtained.

for D_1 $\lambda_{ij} = y_i + 2\alpha_i \cos\left(\frac{j\pi}{N+1}\right)$ and

for D_2 $\lambda_{ij} = y_i + 2\alpha_i \cos\left(\frac{2\pi j}{N}\right)$ $i = 1, 2, 3, \cdots, M$,

$j = 1, 2, 3, \cdots, N$.

Let $\boldsymbol{q}_j$ be an eigenvector of L_i corresponding to the eigenvalue λ_{ij} and the matrix

$Q = [\boldsymbol{q}_1 \quad \boldsymbol{q}_2 \quad \boldsymbol{q}_3 \quad \cdots \quad \boldsymbol{q}_n]^{\mathrm{T}}$ be a modal matrix of L_i, $\forall i$ such that $QQ^{\mathrm{T}} = I$ and

$Q^{\mathrm{T}} L_i Q = \mathrm{diag}(\eta_{i1}, \eta_{i2}, \eta_{i3}, \cdots, \eta_{iN})$

The $N \times N$ modal matrix Q is defined by

$q_{i,j} = \sqrt{\frac{2}{N+1}} \sin\left(\frac{ij\pi}{N+1}\right)$ for D_1 and

$q_{i,j} = \frac{(\cos\theta + \sin\theta)}{\sqrt{N}}$, where $\theta = \left(\frac{2\pi}{N}\right)(i-1)(j-1)$ for D_2 and $i = 1, 2, 3, \cdots, M$, $j = 1, 2, 3, \cdots, N$

Let $\mathbb{Q} = \mathrm{diag}(Q, Q, \cdots, Q)$ be a matrix of order NP.

Thus $\mathbb{Q}$ satisfy the property that $\mathbb{Q}^{\mathrm{T}}\mathbb{Q} = I$ since $Q^{\mathrm{T}}Q = I$ and due to the matrix R_i is symmetric, we have $\mathbb{Q}^{\mathrm{T}} R_i \mathbb{Q} = \mathrm{diag}(\eta_{j1}^i, \eta_{j2}^i, \cdots, \eta_{jP}^i) = \Upsilon_i$ (say);

$\mathbb{Q}^{\mathrm{T}} S_i \mathbb{Q} = S_i$ and $\mathbb{Q}^{\mathrm{T}} S_i^* \mathbb{Q} = S_i^*$ since both S_i and S_i^* are diagonal matrices.

Let $\mathbb{Q}^{\mathrm{T}} \boldsymbol{U}_i = \boldsymbol{V}_i \Rightarrow \boldsymbol{U}_i = \mathbb{Q}\boldsymbol{V}_i$

$$\mathbb{Q}^{\mathrm{T}} \boldsymbol{B}_i = \bar{\boldsymbol{b}}_i \Rightarrow \boldsymbol{B}_i = \mathbb{Q}\bar{\boldsymbol{b}}_i \quad (11)$$

where $\boldsymbol{V}_i = [V_{i1} \quad V_{i2} \quad V_{i3} \quad \cdots \quad V_{iP}]^{\mathrm{T}}$,

$V_{ik} = [v_{i1k} \quad v_{i2k} \quad v_{i3k} \quad \cdots \quad v_{iNk}]^{\mathrm{T}}$;

$\bar{\boldsymbol{b}}_i = [\boldsymbol{b}_{i1} \quad \boldsymbol{b}_{i2?} \quad \cdots \quad \boldsymbol{b}_{ik}]^{\mathrm{T}}$ and

$\boldsymbol{b}_{ik} = [b_{i1k} \quad b_{i2k} \quad \cdots \quad b_{iNk}]^{\mathrm{T}}$

Pre-multiplying Equation (10) by $\mathbb{Q}^{\mathrm{T}}$ and applying (11), we get

$$\begin{aligned} \Upsilon_1\boldsymbol{V}_1 + S_1\boldsymbol{V}_2 &= \bar{\boldsymbol{b}}_1 \\ S_2^*\boldsymbol{V}_1 + \Upsilon_2\boldsymbol{V}_2 + S_2\boldsymbol{V}_3 &= \bar{\boldsymbol{b}}_2 \\ S_3^*\boldsymbol{V}_2 + \Upsilon_3\boldsymbol{V}_3 + S_3\boldsymbol{V}_4 &= \bar{\boldsymbol{b}}_3 \\ &\vdots \\ S_M^*\boldsymbol{V}_{M-1} + \Upsilon_M\boldsymbol{V}_M &= \bar{\boldsymbol{b}}_M \end{aligned} \quad (12)$$

Now from each Equation of (14) we collect the first equations and put them as one group of equation

$$\begin{aligned}
&\eta_{jk}^1 V_k^1 + (1+\omega_1) V_k^2 = \bar{\boldsymbol{b}}_1 \\
&(1-\omega_2) V_k^1 + \eta_{jk}^2 V_k^2 + (1+\omega_2) V_k^3 = \bar{\boldsymbol{b}}_2 \\
&(1-\omega_2) V_k^2 + \eta_{jk}^3 V_k^3 + (1+\omega_2) V_k^4 = \bar{\boldsymbol{b}}_3 \\
&\vdots \\
&(1-\omega_{M-1}) V_k^{M-2} + \eta_{jk}^M V_k^{M-1} + (1+\omega_{M-1}) V_k^M = \bar{\boldsymbol{b}}_{M-1} \\
&(1-\omega_M) V_k^{M-1} + \eta_{jk}^M V_k^M = \bar{\boldsymbol{b}}_M
\end{aligned} \quad (13)$$

Collect as the first set of equations by putting $k=1$ in Equation (15), for $i=1,2,3,\cdots,M$ and $j=1,2,3,\cdots,N$ $(1-\omega_i) v_{j1}^{i-1} + \eta_{j1}^i v_{j1}^i + (1+\omega_i) v_{j1}^{i+1} = b_{j1}^i$ and

$$v_{j1}^0 = 0 = v_{j1}^m \quad (14a)$$

Again consider the second equations by putting $k=2$, and get

$(1-\omega_i) v_{j2}^{i-1} + \eta_{j2}^i v_{j2}^i + (1+\omega_i) v_{j2}^{i+1} = b_{j2}^i$ and

$$v_{j2}^0 = 0 = v_{j2}^m \quad (14b)$$

Continuing in this manner and finally considering the last equations for $k=P$, we obtain

$(1-\omega_i) v_{jP}^{i-1} + \eta_{j2}^i v_{jP}^i + (1+\omega_i) v_{jP}^{i+1} = b_{jP}^i$ and

$$v_{jP}^0 = 0 = v_{jP}^m \quad (14c)$$

All these set of Equations (14a)-(14c) are tri-diagonal ones and hence we solve for v_{jk}^i by using Thomas algorithm. With the help of (11) again we get all u_{jk}^i and this solves (5) as desired. By doing this we generally reduce the number of computations and computational time.

4. Numerical Results

In order to test the efficiency and adaptability of the proposed method, computational experiments are done on some selected problems that may arise in practice, for which the analytical solutions of U are known to us. The computed solutions are found for any values of M, N, and P. Here results are reported at some randomly taken points from **Tables 1** to **7**.

Example 1. Consider $\nabla^2 U = -3\cos\theta$ with the boundary conditions

$$U(0,\theta,z) = U(1,\theta,z) = -2z$$

$$U(r,0,z) = r(1-r) - 2z, \quad U\left(r,\frac{\pi}{2},z\right) = -2z$$

$$(r,\theta,0) = r(1-r)\cos\theta, \quad U(r,\theta,1) = r(1-r)\cos\theta - 2$$

The analytical solution is $U(r,\theta,z) = r(1-r)\cos\theta - 2z$ and the results of this example are shown in **Table 1**.

Example 2. Consider $\nabla^2 U = -\pi^2 r\cos\theta\sin(\pi z)$ with the boundary conditions

$$U(1,\theta,z) = \cos\theta\sin(\pi z), \; U(2,\theta,z) = 2\cos\theta\sin(\pi z)$$

$$U(r,0,z) = r\sin(\pi z), \; U(r,\pi,z) = -r\sin(\pi z)$$

$$U(r,\theta,0) = U(r,\theta,1) = 0$$

The analytical solution is $U(r,\theta,z) = r\cos\theta\sin(\pi z)$ and results of this example are shown in **Table 2**

Example 3. Consider

$$\nabla^2 U = -\pi^2\left(r^2 - \frac{1}{r^2}\right)\sin(2\theta)\sin(\pi z)$$

with the boundary conditions

$$U(1,\theta,z) = 0, \quad U(2,\theta,z) = \frac{15}{4}\sin(2\theta)\sin(\pi z)$$

$U(r,0,z) = 0 = U\left(r,\frac{\pi}{2},z\right)$ and

$U(r,\theta,0) = 0 = U(r,\theta,1)$

The analytical solution is

Table 1. Maximum absolute error of example 1.

(N,P,M)	Max. absolute error	(N,P,M)	Max. absolute error
(9,9,9)	6.83901e−005	(29,9,39)	7.63105e−006
(9,9,29)	6.85359e−005	(29,19,9)	7.64055e−006
(9,19,9)	6.85882e−005	(29,29,19)	7.63684e−006
(9,19,19)	6.85881e−005	(29,29,29)	7.65726e−006
(9,29,39)	6.87927e−005	(29,39,19)	7.63846e−006
(9,39,29)	6.88027e−005	(29,39,29)	7.65884e−006
(19,9,9)	1.71132e−005	(39,9,19)	4.28351e−006
(19,9,19)	1.71127e−005	(39,9,39)	4.29603e−006
(19,19,19)	1.71675e−005	(39,19,29)	4.30845e−006
(19,29,39)	1.72164e−005	(39,29,19)	4.29998e−006
(19,39,9)	1.71752e−005	(39,39,9)	4.30369e−006
(19,39,39)	1.72198e−005	(39,39,39)	4.31261e−006

Table 2. Maximum absolute error of example 2.

(N,P,M)	Max. absolute error	(N,P,M)	Max. absolute error
(9,9,9)	5.97565e−003	(29,9,39)	6.25865e−003
(9,9,29)	6.04232e−003	(29,19,9)	1.56933e−003
(9,19,9)	1.68198e−003	(29,29,19)	7.21066e−004
(9,19,19)	1.69672e−003	(29,29,29)	7.21498e−004
(9,29,39)	8.9486e−004	(29,39,19)	4.18664e−004
(9,39,29)	6.14273e−004	(29,39,29)	4.19068e−004
(19,9,9)	6.18349e−003	(39,9,19)	6.23836e−003
(19,9,19)	6.24022e−003	(39,9,39)	6.24401e−003
(19,19,19)	1.6095e−003	(39,19,29)	1.5717e−003
(19,29,39)	7.53504e−004	(39,29,19)	7.07837e−004
(19,39,9)	4.50526e−004	(39,39,9)	4.01493e−004
(19,39,39)	4.53562e−004	(39,39,39)	4.05867e−004

$U=\left(r^2-\frac{1}{r^2}\right)\sin(2\theta)\sin(\pi z)$ and results of this example are shown in **Table 3**

Example 4. Consider

$$\nabla^2 U=(\cos\theta_1+\sin\theta_1)(\cos z_1+\sin z_1)$$
$$\times\left(\left(\frac{\pi}{2r}-\frac{\pi^2}{2}-4\frac{\pi^2}{r^2}\right)\cos r^*-\left(\frac{\pi}{2r}+\frac{\pi^2}{2}+4\frac{\pi^2}{r^2}\right)\sin r^*\right)$$

where $r^*=\frac{\pi}{2}(r-4)$, $\theta_1=\pi(2\theta-1)$, $z_1=\frac{\pi}{2}(z-1)$, $2\le r\le 4$, $0\le\theta\le 0.5$ and $-1\le z\le 1$

The analytical solution is

$$U(r,\theta,z)=(\cos r^*+\sin r^*)(\cos\theta_1+\sin\theta_1)\times(\cos z_1+\sin z_1)$$

This problem was considered as one test problem by Iyengar and Goyal [11] and their result and our are found to be the same for h = 1/8, but their method cannot be applied for non-uniform values h_1, h_2 and h_3. We have shown the results of this example in **Table 4**.

Table 3. Maximum absolute error of example 3.

(N,P,M)	Max. absolute error	(N,P,M)	Max. absolute error
(9,9,9)	1.0041e−002	(29,9,39)	9.14135e−003
(9,9,29)	1.03659e−002	(29,19,9)	2.09239e−003
(9,19,9)	3.33415e−003	(29,29,19)	1.08113e−003
(9,19,19)	3.5776e−003	(29,29,29)	1.12241e−003
(9,29,39)	2.39279e−003	(29,39,19)	6.45366e−004
(9,39,29)	1.94206e−003	(29,39,29)	6.85093e−004
(19,9,9)	8.97807e−003	(39,9,19)	8.98428e−003
(19,9,19)	9.24935e−003	(39,9,39)	9.07348e−003
(19,19,19)	2.51841e−003	(39,19,29)	2.30437e−003
(19,29,39)	1.33191e−003	(39,29,19)	1.01451e−003
(19,39,9)	6.40825e−004	(39,39,9)	3.98681e−004
(19,39,39)	8.95344e−004	(39,39,39)	6.31752e−004

Table 4. Maximum absolute error of example 4.

(N,P,M)	Max. absolute error	(N,P,M)	Max. absolute error
(9,9,9)	1.52153e−002	(29,9,39)	4.59667e−003
(9,9,29)	1.18196e−002	(29,19,9)	5.9446e−003
(9,19,9)	1.26939e−002	(29,29,19)	2.29605e−003
(9,19,19)	9.82575e−003	(29,29,29)	1.72475e−003
(9,29,39)	8.67411e−003	(29,39,19)	2.13081e−003
(9,39,29)	8.68314e−003	(29,39,29)	1.558e−003
(19,9,9)	9.48954e−003	(39,9,19)	4.98915e−003
(19,9,19)	6.4572e−003	(39,9,39)	4.21468e−003
(19,19,19)	3.87647e−003	(39,19,29)	1.82246e−003
(19,29,39)	2.66641e−003	(39,29,19)	1.91856e−003
(19,39,9)	6.34865e−003	(39,39,9)	4.97913e−003
(19,39,39)	2.49938e−003	(39,39,39)	9.7357e−004

Example 5. Consider $\nabla^2 U=-3\cos\theta$ with the boundary conditions

$$U(0,\theta,z)=z=U(1,\theta,z),$$
$$U(r,\theta,0)=r(1-r)\cos\theta,\ U(r,\theta,1)=1+r(1-r)\cos\theta$$

The analytical solution is $U=r(1-r)\cos\theta+z$ and results of this example are shown in **Table 5**.

Example 6. Consider $\nabla^2 U=6rz\cos\theta$ with the boundary conditions

$$U(0,\theta,z)=0, U(1,\theta,z)=z\cos^3\theta,$$
$$U(r,\theta,0)=0 \text{ and } (r,\theta,1)=r^3\cos^3\theta$$

The analytical solution is $U=r^3 z\cos^3\theta$ and results of this example are shown in **Table 6**.

Example 7. Consider

$$\nabla^2 U=-\pi^2\left(r^2-1/r^2\right)\sin(2\theta)\sin(\pi z)$$

with the boundary conditions

$$U(1,\theta,z)=0,\ U(2,\theta,z)=\frac{15}{4}\sin(2\theta)\sin(\pi z)$$
$$U(r,\theta,0)=0=U(r,\theta,1)$$

The analytical solution is

$U=\left(r^2-\frac{1}{r^2}\right)\sin(2\theta)\sin(\pi z)$ and results of this example are shown in **Table 7**. Here, we have displayed only for some points which are taken randomly, but we can show the results at any point inside the cylinder.

5. Conclusion

In this work, first we apply Hockney's method in order to reduce (5) as a tri-diagonal matrix and after that all the computations directly rely on the Thomas Algorithm. By

Table 5. Maximum absolute error of example 5.

(N,P,M)	Max. absolute error	(N,P,M)	Max. absolute error
(10,9,9)	2.91054e−003	(29,9,39)	3.27118e−004
(10,9,29)	2.94732e−003	(29,19,9)	3.2355e−004
(10,19,9)	2.91942e−003	(29,29,19)	3.26362e−004
(10,19,19)	2.94503e−003	(29,29,29)	3.27939e−004
(10,29,39)	2.96297e−003	(29,39,19)	3.26426e−004
(10,39,29)	2.95912e−003	(29,39,29)	3.28008e−004
(20,9,9)	7.26075e−004	(39,9,19)	1.82891e−004
(20,9,19)	7.32065e−004	(39,9,39)	1.83977e−004
(20,19,19)	7.34274e−004	(39,19,29)	1.84332e−004
(20,29,39)	7.39078e−004	(39,29,19)	1.83546e−004
(20,39,9)	7.28843e−004	(39,39,9)	1.8211e−004
(20,39,39)	7.3923e−004	(39,39,39)	1.84702e−004

Table 6. Maximum absolute error of example 6.

(N,P,M)	Max. absolute error	(N,P,M)	Max. absolute error
(10,9,9)	8.08303e−003	(29,9,39)	8.96012e−004
(10,9,29)	7.78161e−003	(29,19,9)	1.26723e−003
(10,19,9)	8.13051e−003	(29,29,19)	9.73704e−004
(10,19,19)	7.87559e−003	(29,29,29)	9.19304e−004
(10,29,39)	7.82057e−003	(29,39,19)	9.74094e−004
(10,39,29)	7.84024e−003	(29,39,29)	9.19684e−004
(20,9,9)	2.33516e−003	(39,9,19)	5.8771e−004
(20,9,19)	2.0482e−003	(39,9,39)	5.1461e−004
(20,19,19)	2.06074e−003	(39,19,29)	5.35984e−004
(20,29,39)	1.99772e−003	(39,29,19)	5.9185e−004
(20,39,9)	2.35229e−003	(39,39,9)	9.27532e−004
(20,39,39)	1.99854e−003	(39,39,39)	5.18641e−004

Table 7. Maximum absolute error of example 7.

(N,P,M)	Max. absolute error	(N,P,M)	Max. absolute error
(10,9,9)	2.8935e−002	(29,9,39)	1.14219e−002
(10,9,29)	2.9406e−002	(29,19,9)	4.40139e−003
(10,19,9)	2.2444e−002	(29,29,19)	3.40452e−003
(10,19,19)	2.29089e−002	(29,29,29)	3.4587e−003
(10,29,39)	2.18002e−002	(29,39,19)	2.97592e−003
(10,39,29)	2.13515e−002	(29,39,29)	3.02879e−003
(20,9,9)	1.35531e−002	(39,9,19)	8.42863e−002
(20,9,19)	1.38225e−002	(39,9,39)	7.43613e−002
(20,19,19)	7.39905e−003	(39,19,29)	3.30954e−002
(20,29,39)	6.2994e−003	(39,29,19)	6.92702e−002
(20,39,9)	5.55157e−003	(39,39,9)	7.68463e−003
(20,39,39)	5.89069e−003	(39,39,39)	5.36082e−002

doing this, we saved the number of computation and computational time. The method is shown to produce good results.

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