

Differential Games of Pursuing in the Systems with Distributed Parameters and Geometrical Restrictions

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ABSTRACT

A problem of pursuit in the controlled systems of elliptic type without mixed derivatives with variable coefficients was considered. The model of the considered system is described by partial differential equations. The players (opponents) control parameters occur on the right-hand side of the equation and are subjected to various constraints. The first player's goal is to bring the system from one state into another desired state; the second player's goal is to prevent this from happening. We represent new sufficient conditions for bringing the system from one state into another. The finite-difference method is used to solve this problem.

Keywords: Pursuit; Pursuer; Evader; Terminal Set; Pursuit Control; Evasion control

1. Introduction

Some problem formulations in the theory of differential games may be illustrated by motion of two controlled objects, pursuer and evader. Let in the course of motion the objects continuously observe each other and at each time instant correct their motions depending on the information about the adversary. Depending on the pursuer's aim, the problem of pursuit is then formulated as follows: using the information about the evader, at each time instant t select a control such that coincidence of the objects' spatial coordinates is reached as soon as possible.

The majority of studies consider the case where behavior of the lumped-parameter model described by a system of ordinary differential equations. This scheme encompasses many problems of differential games arising in diverse fields of the natural sciences. The mathematical issues of the differential games describing the lumped-parameter systems were developed in detail.

In many applications, however, the lumped-parameter models describe phenomena inadequately. It often turns out that a system which is optimal in the sense of a simplified model does not use the additional designed-in potentialities of control. The distributed-parameter models obeying the differential equations with partial derivatives offer a better, more adequate description. Use of these equations also gives rise to various game problems of which one is the subject matter of the present paper. It focuses only on the problem of pursuit. Therefore, we make an assumption about the nature of information for

this problem.

2. Formulation of the Problem

The operated distributed system described by the elliptic equations (see, for example, [1,2]) is considered

$$a(x,y)\partial^2 z/\partial x^2 + b(x,y)\partial^2 z/\partial y^2 = f(u(x,y),v(x,y)), \quad (1)$$

$$\partial z/\partial \xi + \alpha(x,y)z = \varphi(x,y), \quad (x,y) \in \Gamma$$

where $z = z(x,y)$ – unknown function, $a(x,y)$, $b(x,y)$ – continuous functions in $\Omega = \{(x,y): 0 < x < 1, 0 < y < 1\}$ with border Γ , $\varphi(x,y)$ – smooth function on Γ , ξ – external normal. It is supposed that there is a positive constant ν such that for any $(x,y) \in \Omega$ the inequality, $b(x,y) \geq \nu$, $u = u(x,y)$, $v = v(x,y)$ – operating functions is executed from a class $L_2(\Omega)$. The first (pursuing) player, (pursued or escaping) the player, $u \in \bar{P}$, $v \in \bar{Q}$, \bar{P} and \bar{Q} – nonempty compacts in R^1 disposes of function $v(x,y)$ second of function $u(x,y)$. The terminal set $\bar{M}_1 \subset R^1$ is allocated.

Definition 1. In a task (1) it is possible ε – completion of ($\varepsilon > 0$) prosecutions from “boundary” situation $\phi(\cdot, \cdot)$, if exist function $u(v, x, y) \in \bar{P}$, $v \in \bar{Q}$, $(x, y) \in \Omega$, such that for any function $v_0(x, y) \in \bar{Q}$, $(x, y) \in \Omega$ the solution $z_0(x, t)$ of a task (1) where $u = u(v_0(x, y), x, y)$, $v = v_0(x, y)$, gets on a set $\varepsilon I + \bar{M}_1$, at some (\tilde{x}, \tilde{y}) , $(\tilde{x}, \tilde{y}) \in \Omega: z_0(\tilde{x}, \tilde{y}) \in \varepsilon I + \bar{M}_1$ where $I = (-1, 1)$.

Decompose the Euclidean space R^2 of variables (x, y) by the planes $x_i = ih$, $h = 1/r$, $i = 0, 1, \dots$, and $y_j = jl$, $l = 1/\theta$, $j = 0, 1, 2, \dots$, into parallelepipeds

$\Omega_{(i,j)} = \{(x, y) : ih < x_i < (i+1)h, jl < y < (j+1)l\}$, r and θ being some natural numbers. The points (x_i, y_j) belonging to a set Ω are the nodes of the grid Ω_{hl} . Each node has its neighbors. If all these neighbor nodes also belong to the grid Ω_{hl} , then the node $(x_i, y_j) = (ih, jl)$ is referred to as “internal”, otherwise, (x_i, y_j) is called the “boundary” node. The set of all boundary nodes is called as border of net area and is designated through Γ_h .

Replace the internal nodes of the derivatives (1) differential second-order accuracy of approximation ratios with formulas

$$\begin{aligned} \partial z / \partial x(x_i, y_j) &= (z(x_{i+1}, y_j) - z(x_{i-1}, y_j)) / 2h + O(h)^2, \\ \partial z / \partial y(x_i, y_j) &= (z(x_i, y_{j+1}) - z(x_i, y_{j-1})) / 2l + O(l)^2, \\ \partial^2 z / \partial x^2(x_i, y_j) &= (z(x_{i+1}, y_j) - 2z(x_i, y_j) + z(x_{i-1}, y_j)) / h^2 + O(h)^2, \\ \partial^2 z / \partial y^2(x_i, y_j) &= (z(x_i, y_{j+1}) - 2z(x_i, y_j) + z(x_i, y_{j-1})) / l^2 + O(l)^2, \end{aligned}$$

Substituting these ratios in (1), having rejected an error of approximation of derivatives, we will receive the differential equations for unknown $z_{i,j}$

$$a_{i,j}(z_{i+1,j} - 2z_{i,j} + z_{i-1,j}) / h^2 + b_{i,j}(z_{i,j+1} - 2z_{i,j} + z_{i,j-1}) / l^2 = f_{i,j}, \quad (2)$$

$$i=0, 1, \dots, r-1; \quad j=0, 1, \dots, \theta-1.$$

where the following designations of values of coefficients and the right part in a hub (x_i, y_j) : $a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j}, g_{i,j}, f_{i,j}$, for example are entered

$$f_{i,j} = f(u(x_i, y_j), v(x_i, y_j)), \quad (x_i, y_j) \in \Omega_{hl}$$

Ratios (2) contains except unknown $z_{i,j}$ in internal nodes also unknown $z_{i,j}$ on border of net area. For boundary nodes we will write down a ratio

$$\begin{aligned} (z_{1,j} - z_{0,j}) / h + \alpha_{0,j}(z_{0,j} + z_{1,j}) / 2 &= \varphi_{0,j}, \quad (j=0, 1, 2, \dots, \theta-1) \\ (z_{r,j} - z_{r-1,j}) / h + \alpha_{r,j}(z_{r-1,j} + z_{r,j}) / 2 &= \varphi_{r,j}, \\ (z_{i,1} - z_{i,0}) / l + \alpha_{i,0}(z_{i,1} + z_{i,0}) / 2 &= \varphi_{i,0}, \quad (i=0, 1, 2, \dots, r-1) \\ (z_{i,\theta} - z_{i,\theta-1}) / l + \alpha_{i,\theta}(z_{i,\theta} + z_{i,\theta-1}) / 2 &= \varphi_{i,\theta} \end{aligned} \quad (3)$$

Thus, we will receive system of $r\theta + 2(r + \theta)$ equations with the same number of unknown $z_{i,j}$.

Using boundary conditions (3), we will express $z_{i,1}, z_{i,\theta}$ through $z_{i,0}, z_{i,\theta-1}$. Let's have

$$\begin{aligned} z_{i,1} &= (2 - l\alpha_{i,0}) / (2 + l\alpha_{i,0}) z_{i,0} + 2l\varphi_{i,0} / (2 + l\alpha_{i,0}) \\ z_{i,\theta} &= (2 - l\alpha_{i,\theta}) / (2 + l\alpha_{i,\theta}) z_{i,\theta-1} + 2l\varphi_{i,\theta} / (2 + l\alpha_{i,\theta}). \end{aligned} \quad (4)$$

Using these ratios, we will exclude in system (3) unknown $z_{i,1}, z_{i,\theta}$. If to enter designation $\gamma = h^2 / l^2$, we will receive system

$$\left. \begin{aligned} z_{i+1,0} - (2 + 2\gamma - k_{0,0}\gamma)z_{i,0} + \gamma z_{i,1} + z_{i-1,0} &= F_{i,0}, \\ z_{i+1,j} + \gamma z_{i,j-1} - 2(1 + \gamma)z_{i,j} + \gamma z_{i,j+1} + z_{i-1,j} &= F_{i,j} \\ &(j=1, 2, \dots, \theta-2), \\ z_{i+1,\theta-1} + \gamma z_{i,\theta-2} - (2 + 2\gamma - k_{i,\theta}\gamma)z_{i,\theta-1} + z_{i-1,\theta-1} &= F_{i,\theta-1} \\ &(i=0, 1, 2, \dots, r-1), \end{aligned} \right\} \quad (5)$$

where

$$F_{i,0} = h^2 f_{i,0} - \gamma 2l\varphi_{i,0} / (2 + l\alpha_{i,0}); \quad F_{ij} = h^2 f_{i,j}, \quad (j=1, 2, \dots, \theta-2); \quad (6)$$

$$F_{i,\theta-1} = h^2 f_{i,\theta-1} - \gamma 2l\varphi_{i,\theta} / (2 + l\alpha_{i,\theta}).$$

This system can shortly be written down in a look

$$z_{i+1} + A_i z_i + z_{i-1} = F_i, \quad (i=0, 1, 2, \dots, r-1). \quad (7)$$

where

$$z_i = (z_{i,0}, z_{i,1}, \dots, z_{i,\theta-1}); \quad F_i = (F_{i,0}, F_{i,1}, \dots, F_{i,\theta-1}),$$

$$A_i = \begin{pmatrix} -2(1 + \gamma) + k_{i,0}\gamma & \gamma & 0 & \dots & 0 & 0 \\ \gamma & -2(1 + \gamma) & \gamma & \dots & 0 & 0 \\ 0 & \gamma & -2(1 + \gamma) & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \gamma & -2(1 + \gamma) + k_{i,\theta}\gamma \end{pmatrix} \quad (8)$$

Boundary conditions (3) and (5) can be copied in a look

$$\begin{aligned} z_{1,j} &= (2 - h\alpha_{0,j}) / (2 + h\alpha_{0,j}) z_{0,j} + (2h\varphi_{0,j}) / (2 + h\alpha_{0,j}) \\ &= k_{0,j} z_{0,j} + y_{0,j}, \quad (j=0, 1, 2, \dots, \theta-1) \\ z_{r-1,j} &= (2 + h\alpha_{r,j}) / (2 - h\alpha_{r,j}) z_{r,j} - (2h\varphi_{r,j}) / (2 - h\alpha_{r,j}) \\ &= k_{r,j} z_{r,j} + y_{r,j}, \quad (j=0, 1, 2, \dots, \theta-1) \end{aligned} \quad (9)$$

where

$$\begin{aligned} k_{0,j} &= (2 - h\alpha_{0,j}) / (2 + h\alpha_{0,j}); \quad y_{0,j} = (2h\varphi_{0,j}) / (2 + h\alpha_{0,j}); \\ k_{r,j} &= (2 + h\alpha_{r,j}) / (2 - h\alpha_{r,j}); \quad y_{r,j} = -(2h\varphi_{r,j}) / (2 - h\alpha_{r,j}). \end{aligned} \quad (10)$$

Having put

$$y_0 = (y_{0,0}, y_{0,1}, \dots, y_{0,\theta-1}); \quad y_r = (y_{r,0}, y_{r,1}, \dots, y_{r,\theta-1}).$$

$$X_0 = \begin{pmatrix} k_{0,0} & 0 & 0 & \dots & 0 & 0 \\ 0 & k_{0,1} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & k_{0,\theta-1} \end{pmatrix}; \quad (11)$$

$$X_1 = \begin{pmatrix} k_{r,0} & 0 & 0 & \dots & 0 & 0 \\ 0 & k_{r,1} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & k_{r,\theta-1} \end{pmatrix},$$

it is possible to write down systems (9) in such look:

$$\begin{cases} z_1 = X_0 z_0 + y_0, \\ z_{r-1} = X_1 z_r + y_r \end{cases} \quad (12)$$

Finally we have the following system of the equations:

$$\begin{aligned} z_1 &= X_0 z_0 + y_0, \\ z_{i+1} + A_i z_i - z_{i-1} &= F_i \quad (i=1, 2, \dots, r-1) \\ z_{r-1} &= X_1 z_r + y_r \end{aligned} \quad (13)$$

Instead of game (13) we will consider more the general game described by system of the equations

$$\begin{aligned} C_0 z_0 - B_0 z_1 &= f_0, \\ -A_n z_{n-1} + C_n z_n - B_n z_{n+1} &= f_n(u_n, v_n), \quad 1 \leq n \leq N-1, \\ -A_N z_{N-1} + C_N z_N &= f_N, \end{aligned} \quad (14)$$

where $z_n \in R^m$, $n = \overline{0, N}$, A_n, C_n, B_n - $m \times m$ constant square matrixes, u_n, v_n - operating parameters, u_n -

prosecution parameter, v_n – beanie parameter, $u_n \in P_n \subset R^p$, $v_n \in Q_n \subset R^q$, P_n and Q_n – nonempty sets; f_n – the set function displaying $R^p \times R^q$ in R^m . Besides, in R^m the terminal set is M allocated.

Definition 2. We shall say that from “boundary” situation (f_0, f_N) it is possible to complete pursuit for N steps if from any sequence $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_{N-1}$ of the values of evasion controls it is possible to construct a sequence $\bar{u}_1, \bar{u}_2, \dots, \bar{u}_{N-1}$ of values of the pursuit control values such that the solution $\{z_0, \bar{z}_1, \bar{z}_2, \dots, \bar{z}_{N-1}, z_N\}$ of the equation

$$\begin{aligned} C_0 z_0 - B_0 z_1 &= f_0, \\ -A_n z_{n-1} + C_n z_n - B_n z_{n+1} &= f_n(\bar{u}_n, \bar{v}_n), \quad 1 \leq n \leq N-1, \\ -A_N z_{N-1} + C_N z_N &= f_N. \end{aligned} \quad (15)$$

Gets on $M: \bar{z}_i \in M$ for some i . Thus for finding of value \bar{u}_n it is allowed to use values \bar{v}_n and \bar{z}_n .

Note that the type of systems (14) is difference schemes for elliptic equations of second order with variable coefficients in any field of any number of dimensions [3-14].

Solution of problem (14) will be sought in the form

$$z_n = \alpha_{n+1} z_{n+1} + \beta_{n+1}, \quad n = N-1, N-2, \dots, 0, \quad (16)$$

where α_{n+1} – uncertain while a square matrix of the sizes $m \times m$, and β_{n+1} – a vector of dimension m . From a formula (16) and the equations of system (14) for $1 \leq n \leq N-1$ there are recurrent ratios for calculation of matrixes α_n and vectors β_n . Really from a formula (16) $z_{n-1} = \alpha_n z_n + \beta_n$ substituting it in (14) we will receive

$$\begin{aligned} -A_n(\alpha_n z_n + \beta_n) + C_n z_n - B_n z_{n+1} &= f_n(u_n, v_n), \\ 1 \leq n \leq N-1; \\ (C_n - A_n \alpha_n) z_n &= B_n z_{n+1} + f_n(u_n, v_n) + A_n \beta_n; \\ z_n &= (C_n - A_n \alpha_n)^{-1} B_n z_{n+1} + (C_n - A_n \alpha_n)^{-1} [f_n(u_n, v_n) + A_n \beta_n]. \end{aligned}$$

Equating now the right parts of the last and (16) equalities we will receive

$$\begin{aligned} \alpha_{n+1} &= (C_n - A_n \alpha_n)^{-1} B_n, \quad n = 1, 2, \dots, N-1; \\ \beta_{n+1} &= (C_n - A_n \alpha_n)^{-1} [f_n(u_n, v_n) + A_n \beta_n], \\ n &= 1, 2, \dots, N. \end{aligned}$$

Further from (16) and the equations (14) for $n=0, N$, there are the initial values α_1, β_1 and z_N , allowing beginning the account on recurrent ratios. From (14) and (16) for $n=0$ we will have

$$z_0 = C_0^{-1} B_0 z_1 + C_0^{-1} f_0, \quad z_0 = \alpha_1 z_1 + \beta_1,$$

And, therefore

$$\alpha_1 = C_0^{-1} B_0, \quad \beta_1 = C_0^{-1} f_0.$$

In the same way for $n=N$ we have

$$-A_N(\alpha_N z_N + \beta_N) + C_N z_N = f_N$$

or

$$z_N = (C_N - A_N \alpha_N)^{-1} (f_N + A_N \beta_N).$$

Uniting, we will write out final formulas

$$\alpha_{n+1} = (C_n - A_n \alpha_n)^{-1} B_n, \quad n = 1, 2, \dots, N, \quad \alpha_1 = C_0^{-1} B_0 \quad (17)$$

$$\begin{aligned} \beta_{n+1} &= (C_n - A_n \alpha_n)^{-1} (f_n(u_n, v_n) + A_n \beta_n), \\ n &= 1, 2, \dots, N-1. \quad \beta_1 = C_0^{-1} f_0, \end{aligned} \quad (18)$$

$$\begin{aligned} z_n &= \alpha_{n+1} z_{n+1} + \beta_{n+1}(u_n, v_n), \\ n &= N-1, N-2, \dots, 0, \quad z_N = \beta_{N+1} \end{aligned} \quad (19)$$

It is clear, that if in game (17), (18), (19) $z_n \in M$ that in game (14) too game comes to the end. Therefore further instead of game (14) we will consider discrete game described by system of the equations (17), (18), (19).

Before giving determination of stability of algorithm (17), (18), (19), we will provide some data from linear algebra.

Let A – any square matrix $m \times m$ and $\|x\|_m$ be norm of a vector in R^m , then the norm A is defined by equality

$$\|A\| = \sup_{x \neq 0} \|Ax\|_m / \|x\|_m.$$

For a case of Euclidean norms in R^m we have $\|A\| = \sqrt{\rho}$, where ρ – maximum on the module own value of a matrix $A \cdot A$.

Without the proof we will give the following known lemma (see [15]).

Lemma 1. Let for some matrix norm the square matrix meet a condition $\|A\| \leq q < 1$. Then there is a matrix $(E+A)^{-1}$ and $\|(E+A)^{-1}\| \leq 1/(1-q)$.

Let's say that the algorithm is steady if the assessment $\|\alpha_j\| \leq 1$ for $1 \leq j \leq N$ is carried out.

Lemma 2. If C_j for $0 \leq j \leq N$ – no degenerate matrixes and A_j and B_j – nonzero matrixes for $1 \leq j \leq N-1$ also are satisfied conditions

$$\begin{aligned} \|C_0^{-1} B_0\| \leq 1, \quad \|C_N^{-1} A_N\| \leq 1, \\ \|C_j^{-1} A_j\| + \|C_j^{-1} B_j\| \leq 1, \quad 1 \leq j \leq N-1. \end{aligned}$$

And at least in one of inequalities the strict inequality takes place, there are return to the $C_j - A_j \alpha_j$ matrix and $\|\alpha_j\| \leq 1$, here $\alpha_1 = C_0^{-1} B_0$,

$$\alpha_{j+1} = (C_j - A_j \alpha_j)^{-1} B_j, \quad 1 \leq j \leq N-1.$$

Proof. $\|\alpha_1\| = \|C_0^{-1} B_0\| \leq 1$, suppose, that $\|\alpha_j\| \leq 1$ also we will show $\|\alpha_{j+1}\| \leq 1$. After a course the proof of this fact we will receive existence of a matrix $(C_j - A_j \alpha_j)^{-1}$. Really from conditions of a lemma we will have

$$\|C_j^{-1} A_j \alpha_j\| \leq \|C_j^{-1} A_j\| \|\alpha_j\| \leq \|C_j^{-1} A_j\| \leq 1 - \|C_j^{-1} B_j\| < 1.$$

As $C_j^{-1} A_j \alpha_j$ square matrix that owing to a lemma 1 there are return to $E - C_j^{-1} A_j \alpha_j$ and $C_j - A_j \alpha_j$ matrixes and $\|(E - C_j^{-1} A_j \alpha_j)\| \leq 1 / \|C_j^{-1} B_j\|$. From here and from (17) we will receive

$$\begin{aligned} \|\alpha_{j+1}\| &\leq \|(E - C_j^{-1} A_j \alpha_j)^{-1} C_j^{-1} B_j\| \\ &\leq \|(E - C_j^{-1} A_j \alpha_j)^{-1}\| \|C_j^{-1} B_j\| \leq 1. \end{aligned}$$

The proof of the lemma is complete.

3. Main Results

Everywhere further it is supposed that $M=M_0+M_1$, where M_0 – linear subspace R^m , M_1 – a subset a subspace, L – orthogonal complement of M_0 in R^n . Denote Π we will designate a matrix of orthogonal design from R^m on L .

Let $W(0)=\{0\}$,

$$W(k)=\sum_{i=0}^{k-1} \bigcap_{\nu_{N-k+i} \in Q_{N-k+i}} \prod \alpha_{N-k} \dots \alpha_{N-k+i} \beta_{N-k+i+1}(P_{N-k+i}, \nu_{N-k+i})$$

$$W_1(k)=-M_1+W(k), \quad 1 \leq k \leq N. \tag{20}$$

Theorem 1. Let \bar{N} be the smallest of the numbers k , such that

$$-\prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_N z_N \in W_1(k). \tag{21}$$

Then from “boundary” situation (f_0, f_N) it is possible to complete pursuit for \bar{N} steps.

Let now $W_2(0)=-M_1$,

$$W_2(1)=\bigcap_{\nu_{N-1} \in Q_{N-1}} [W_2(0)+\prod \beta_{N-k}(P_{N-k-1}, \nu_{N-k-1})],$$

$$W_2(k)=\bigcap_{\nu_{N-1} \in Q_{N-1}} [W_2(k-1)+\prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_{N-1} \beta_N(P_{N-1}, \nu_{N-1})]$$

$$\tag{22}$$

Theorem 2. If \bar{N} be smallest of those numbers k , for each of which takes place inclusion

$$-\prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_N z_N \in W_2(k) \tag{23}$$

that of “boundary” situation (f_0, f_N) it is possible to complete pursuit for \bar{N} steps.

Let

$$\gamma_k(\cdot)=\left\{ \gamma_0, \gamma_1, \dots, \gamma_{k-1}; \gamma_i \geq 0, \sum_{i=0}^{k-1} \gamma_i = 1 \right\}$$

and

$$W(\gamma_k(\cdot)) = \sum_{i=0}^{k-1} \bigcap_{\nu_{N-k+i} \in Q_{N-k+i}} [-\gamma_i M_1 + \prod \alpha_{N-k} \dots \alpha_{N-k+i} \beta_{N-k+i+1}(P_{N-k+i}, \nu_{N-k+i})],$$

$$0 \leq k \leq N,$$

$$W_3(0)=-M_1, \quad W_3(k)=\bigcup_{\gamma_k(\cdot)} W(\gamma_k(\cdot)), \quad 0 \leq k \leq N. \tag{24}$$

Theorem 3. If M_1 – a convex set and \bar{N} be smallest of those numbers k . For each of which inclusion takes place

$$-\prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_N z_N \in W_3(k). \tag{25}$$

That of “boundary” situation (f_0, f_N) it is possible to complete pursuit for \bar{N} steps.

It is easy to be convinced [15] that the solution of $z_{i,j}$ differential task (2) meets to the solution z of an initial task (1), the following assessment of speed of convergence takes place

$$\|(z)_{hl} - z_{i,j}\|_{\Phi_{hl}} \leq K_1 h^2 + K_2 l^2, \tag{26}$$

where $(z)_{hl}$ – values of the exact decision a task (1) in grid functions, Φ_{hl} – spaces of net functions, $\|\cdot\|_{\Phi_{hl}}$ – is its norm and, K_1 and K_2 constants.

Theorem 4. Let in an inequality (26) $K_1 h^2 + K_2 l^2 < \varepsilon$, and in game (13) from a “boundary” situation $(f_0, f_N) = (-\gamma_0, -\gamma_N)$ completion of prosecution that is definitions 2 be possible. Then in game (1) from “boundary” situation $\partial z / \partial \xi + \alpha(x, y) z = \varphi(x, y)$, $(x, y) \in \Gamma$ it is possible to complete pursuit that are definitions 1.

4. Proof of Theorem

Proof of Theorem 1. Let $\bar{\nu}_1, \bar{\nu}_2, \dots, \bar{\nu}_{N-1}$, $\bar{\nu}_i \in Q_i$, $1 \leq i \leq N-1$ – any sequence. Instead of inclusion (21) we will consider other inclusion equivalent to it

$$-\prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_N z_N \in W_1(k-1) + \bigcap_{\nu_{N-1} \in Q_{N-1}} \prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_{N-1} \beta_N(P_{N-1}, \nu_{N-1})$$

Means, exists a_{N-1}

$$a_{N-1} \in \bigcap_{\nu_{N-1} \in Q_{N-1}} \prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_{N-1} \beta_N(P_{N-1}, \nu_{N-1}).$$

Such that

$$-\prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_N z_N \in W_1(k-1) + a_{N-1}. \tag{27}$$

Now control of the pursuing player \bar{u}_{N-1} , the relevant control of the escaping player $\bar{\nu}_{N-1}$, we will construct as the solution of the following control

$$\prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_{N-1} \beta_N(u_{N-1}, \bar{\nu}_{N-1}) = a_{N-1}.$$

It is clear, that the equation has the decision. From here owing to (27) we have

$$-\prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_N z_N \in W_1(k-1) + \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_{N-1} \beta_N(\bar{u}_{N-1}, \bar{\nu}_{N-1})$$

We write down this inclusion in other look.

$$-\prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_{N-1} [\alpha_N z_N + \beta_N(\bar{u}_{N-1}, \bar{\nu}_{N-1})] \in W_1(k-1). \tag{28}$$

As a result from equalities (18) and (28) we will receive

$$-\prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_{N-1} z_{N-1} \in W_1(k-1) \tag{29}$$

Done above a reasoning allow us to construct on the set control $\bar{\nu}_{N-1}$ providing inclusion (29). If now the control $\bar{\nu}_{N-2}$ becomes known that, we above can receive in the stated way control \bar{u}_{N-1} providing inclusion

$$-\prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_{N-2} z_{N-2} \in W_1(k-2).$$

Repeating this process, further we can construct step by step control \bar{u}_i , proceeding from becoming known controls $\bar{\nu}_i$, therefore, that in any step inclusion takes place

$$-\prod z_{N-k-1} \in W_1(0) = -M_1.$$

It means that

$$\prod z_{N-k-1} \in M$$

As we set out to prove.

Proof of Theorem 2. Let $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_{N-1}$, $\bar{v}_i \in Q_i$, $1 \leq i \leq N-1$ – any sequence. For concrete \bar{v}_{N-1} owing to (22) and (23) we will receive inclusion

$$\begin{aligned} & -\prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_N z_N \in W_2(k-1) \\ & + \prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_{N-1} \beta_N(P_{N-1}, \bar{v}_{N-1}) \end{aligned} \quad (30)$$

Now as \bar{u}_{N-1} we take that element from P_{N-1} for which inclusion (30) remained. Then we will receive

$$\begin{aligned} & -\prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_N z_N \in W_2(k-1) \\ & + \prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_{N-1} \beta_N(\bar{u}_{N-1}, \bar{v}_{N-1}) \end{aligned}$$

From this it follows that

$$-\prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_{N-1} [\alpha_N z_N + \beta_N(\bar{u}_{N-1}, \bar{v}_{N-1})] \in W_2(k-1).$$

And therefore, owing to (19) we have

$$-\prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_{N-1} z_{N-1} \in W_2(k-1).$$

If now the control \bar{v}_{N-1} becomes the stated way known that we above us can construct control \bar{u}_{N-1} providing inclusion

$$-\prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_{N-2} z_{N-2} \in W_2(k-2).$$

Further arguing similarly in any step we will receive

$$-\prod z_{N-k-1} \in W_2(0) = -M_1,$$

that is

$$z_{N-k-1} \in M.$$

The theorem is proved completely.

Proof of Theorem 3. Instead of inclusion (25) meaning (24) we will consider inclusion equivalent to it

$$-\prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_N z_N \in W(\bar{\gamma}(\cdot)).$$

Existence $\bar{\gamma}(\cdot) = \left\{ \bar{\gamma}_0, \bar{\gamma}_1, \dots, \bar{\gamma}_{k-1}, \bar{\gamma}_i \geq 0, \sum_{i=0}^{k-1} \bar{\gamma}_i = 1 \right\}$ follows from (24). From here follows

$$\begin{aligned} & -\prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_N z_N \in \sum_{i=0}^{k-2} \prod_{v_{N-k+i} \in Q_{N-k+i}} \\ & [-\bar{\gamma}_i M_1 + \prod \alpha_{N-k} \dots \alpha_{N-k+i} \beta_{N-k+i+1}(P_{N-k+i}, v_{N-k+i})] \\ & + \prod_{v_{N-1} \in Q_{N-1}} [-\bar{\gamma}_{k-1} M_1 + \prod \alpha_{N-k} \dots \alpha_{N-1} \beta_N(P_{N-1}, v_{N-1})] \end{aligned} \quad (31)$$

Let now $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_{N-1}$, $\bar{v}_i \in Q_i$, $1 \leq i \leq N-1$ – any sequence. Owing to (31) exists such a_{N-1} that

$$\begin{aligned} & v_{N-1} \in \prod_{v_{N-1} \in Q_{N-1}} [-\bar{\gamma}_{k-1} M_1 + \prod \alpha_{N-k} \dots \alpha_{N-1} \beta_N(P_{N-1}, v_{N-1})], \\ & -\prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_N z_N \in \sum_{i=0}^{k-2} \prod_{v_{N-k+i} \in Q_{N-k+i}} \\ & [-\bar{\gamma}_i M_1 + \prod \alpha_{N-k} \dots \alpha_{N-k+i} \beta_{N-k+i+1}(P_{N-k+i}, v_{N-k+i})] + a_{N-1} \end{aligned} \quad (32)$$

Therefore, controls \bar{u}_{N-1} we will construct as the solution of the following equation

$$-\bar{\gamma}_{k-1} m_1 + \prod \alpha_{N-k} \dots \alpha_{N-1} \beta_N(u_{N-1}, \bar{v}_{N-1}) = a_{N-1}, \quad m_{1,k-1} \in M_1.$$

Further owing to (32) we have

$$\begin{aligned} & -\prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_N z_N \in \sum_{i=0}^{k-2} \prod_{v_{N-k+i} \in Q_{N-k+i}} \\ & [-\bar{\gamma}_i M_1 + \prod \alpha_{N-k} \dots \alpha_{N-k+i} \beta_{N-k+i+1}(P_{N-k+i}, v_{N-k+i})] \\ & + \prod \alpha_{N-k} \dots \alpha_{N-1} \beta_N(\bar{u}_{N-1}, \bar{v}_{N-1}) - \bar{\gamma}_{k-1} m_{1,k-1}. \end{aligned}$$

It is equivalent to the following

$$\begin{aligned} & -\prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_{N-1} [\alpha_N z_N + \beta_N(\bar{u}_{N-1}, \bar{v}_{N-1})] \in \bar{\gamma}_{k-1} m_{1,k-1} \\ & + \sum_{i=0}^{k-2} \prod_{v_{N-k+i} \in Q_{N-k+i}} [-\bar{\gamma}_i M_1 + \prod \alpha_{N-k} \dots \alpha_{N-k+i} \beta_{N-k+i+1}(P_{N-k+i}, v_{N-k+i})] \end{aligned}$$

Therefore owing to (32) we have

$$\begin{aligned} & -\prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_{N-1} z_{N-1} \in \bar{\gamma}_{k-1} m_{1,k-1} \\ & + \sum_{i=0}^{k-2} \prod_{v_{N-k+i} \in Q_{N-k+i}} [-\bar{\gamma}_i M_1 + \prod \alpha_{N-k} \dots \alpha_{N-k+i} \beta_{N-k+i+1}(P_{N-k+i}, v_{N-k+i})]. \end{aligned}$$

In the same way, if the control \bar{v}_{N-2} becomes the stated way known that we above us can construct controls \bar{u}_{N-2} providing inclusion

$$\begin{aligned} & -\prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_{N-2} z_{N-2} \in \bar{\gamma}_{k-1} m_{k-1} + \bar{\gamma}_{k-2} m_{k-2} \sum_{i=0}^{k-3} \prod_{v_{N-k+i} \in Q_{N-k+i}} \\ & [-\bar{\gamma}_i M_1 + \prod \alpha_{N-k} \dots \alpha_{N-k+i} \beta_{N-k+i+1}(P_{N-k+i}, v_{N-k+i})] \end{aligned}$$

etc. Thus, we will receive

$$\begin{aligned} & -\prod \alpha_{N-k} \alpha_{N-k+1} \dots \alpha_{N-2} z_{N-2} \in \bar{\gamma}_{k-1} m_{k-1} + \bar{\gamma}_{k-2} m_{k-2} \sum_{i=0}^{k-3} \prod_{v_{N-k+i} \in Q_{N-k+i}} \\ & [-\bar{\gamma}_i M_1 + \prod \alpha_{N-k} \dots \alpha_{N-k+i} \beta_{N-k+i+1}(P_{N-k+i}, v_{N-k+i})] \end{aligned}$$

from here we receive

$$z_{N-k-1} \in M.$$

The theorem is proved completely.

Proof of Theorem 4. Let in game (13) one be able to complete the pursuit from “boundary” situation $(f_0, f_N) = (-y_0, -y_N)$ in N steps. Then, it follows from Definition 2 that from any sequence $\bar{v}_0, \bar{v}_1, \dots, \bar{v}_{N-1}, \bar{v}_k \in Q$, $0 \leq k \leq N-1$, of the evasion control it is possible to construct a sequence $\bar{u}_0, \bar{u}_1, \dots, \bar{u}_{N-1}, \bar{u}_k \in P$, $0 \leq k \leq N-1$, of pursuit control such that the solution $(z_0, \bar{z}_1, \dots, z_{N-1}, z_N)$ of the equation $z_i = X_0 z_0 + y_0$, $z_{n+1} + A_n z_n - z_{n-1} = F_n$, $1 \leq n \leq N-1$, $z_{N-1} = X_1 z_N + y_N$, for some $d \leq N$ hits $M: \bar{z}_d \in M$. Let now in game (2) $v = \bar{v}(x, y) \in \bar{Q}$, $(x, y) \in \Omega$, be an arbitrary control of an evader from the class $L_2(\Omega)$. With the knowledge of the evader control $v = \bar{v}(x, y)$, it is possible to determine $\bar{v}_{i,k}$ as the values of this function at the node points of the grid Ω_{hl} , that is,

$$v_k = \bar{v}_k = (\bar{v}_{1,k}, \bar{v}_{2,k}, \dots, \bar{v}_{r-1,k}).$$

Whence it follows that in virtue of Theorem 4 we can construct the pursuer control in game (13) providing completion of pursuit

$$u_k = \bar{u}_k = (\bar{u}_{1,k}, \bar{u}_{2,k}, \dots, \bar{u}_{r-1,k}).$$

Now in game (2) we construct the pursuer control $u = \bar{u}(x, y)$ as follows: $\bar{u}(x, y) = \{\bar{u}_{i,k} = \bar{u}_{i,k} : ih \leq x_i < (i+1)h, i=0, 1, \dots, r-1, kl \leq y < (k+1)l, k=0, 1, \dots, \theta-1\}$.

Obviously, $u \in P$ and $\bar{u}(x, y) \in L_2(\Omega)$. By substituting $v = \bar{v}(x, y)$ and $u = \bar{u}(x, y)$ in (2), we obtain a differen-

tial equation. Similarly, by substituting $\bar{v}_{i,k}$ and $\bar{u}_{i,k}$ in (3), we obtain a grid equation approximating equation (2).

Let $(z)_{hl}$ be the value of the exact solution corresponding to the controls $v=\bar{v}(x,y)$ and $u=\bar{u}(x,y)$ of problem (2) at the nodes of the grid Ω_{hl} , $\bar{z}_{i,k}$ be the solution corresponding to the controls $\bar{v}_{i,k}$ and $\bar{u}_{i,k}$ of the difference problem (3). Then, we obtain from (13) and the condition of Theorem 4 that

$$\|(z)_{hl}-\bar{z}_{i,k}\|_{\Phi_{hl}} \leq K_1 l + K_2 h^2 < \varepsilon.$$

From this fact and $\bar{z}_{i,k} \in \bar{M}_1$, we obtain $(z)_{hl}-\bar{z}_{i,k} \in \varepsilon S$, $(z)_{hl} \in \varepsilon S + \bar{z}_{i,k}$, $(z)_{hl} \in \varepsilon S + \bar{M}_1$, which proves the theorem.

5. Conclusions

Thus, to solve the game problem of pursuit in the form (1) we pass to the discrete game (13) or (14), and Theorems 1-3 establish the sufficient condition for such problems. Theorem 4 establishes the sufficient conditions for solving the problem of pursuit (1). Here, the difference $(z)_{hl}-z_{i,j}$ (see Section 3) plays the main part in the solution of problem and implies that the solutions of the grid equation (2) are stable.

The problem of stability of the grid equation (2) lies in determining the conditions under which the numerical error $p_{i,j} \equiv (z)_{hl}-z_{i,j}$ tends to zero with growing j uniformly in all i , $0 \leq i \leq n$, or at least remains bounded.

Equation (2) is called stable if the round off errors generated in the course of calculations have tendency to decrease or at least not to increase. Otherwise, the accumulated errors may reach a value such that the numerical solution $(z)_{hl}$ has nothing in common with the exact solution of the grid problem (2). It goes without saying that such unstable grid equations cannot be used for numerical solution of the differential games.

Theorems 1-4 are easily generalized to a wider class of differential games, for example, when

$$\sum_{\alpha=1}^n a_{\alpha}(x_1, x_2, \dots, x_n) \frac{\partial^2 z}{\partial x_{\alpha}^2} = f(u(x_1, x_2, \dots, x_n), v(x_1, x_2, \dots, x_n))$$

with discontinuous coefficients.

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