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The Planar Ramsey Numbers PR (K_4-e, K_l)*

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ABSTRACT

The planar Ramsey number PR (H_1, H_2) is the smallest integer n such that any planar graph on n vertices contains a copy of H_1 or its complement contains a copy of H_2. It is known that the Ramsey number $R(K_4 - e, K_6) = 21$, and the planar Ramsey numbers $PR(K_4 - e, K_l)$ for $l \leq 5$ are known. In this paper, we give the lower bounds on PR ($K_4 - e$, K_l) and determine the exact value of PR ($K_4 - e$, K_6).

Keywords: Planar Graph; Ramsey Number; Forbidden Subgraph

1. Introduction

We consider only finite undirected graphs without loops or multiple edges. For a graph G with vertex set $V(G)$ and edge set $E(G)$, we denote the order and size of G by $p(G) = |V(G)|$ and $q(G) = |E(G)|$, respectively. We refer to the regions defined by a plane graph as its *faces*. A face is said to be *incident* with the vertices and edges in its boundary. The *length* of a face is the number of edges with which it is incident. If a face has length α, we say it is an *α-face*. For a plane graph G, let f denote the number of faces, and f_α the number of α-faces. Let $d(v)$ denote the degree of a vertex $v \in V(G)$, $\delta(G)$ the minimum degree of G. The neighborhood and the closed neighborhood of a vertex $v \in V(G)$ are denoted by $N(v) = \{u \in V(G) | uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$, respectively. Let $G \cup H$ denote a disjoint sum of G and H, and nG is a disjoint sum of n copies of G. $\overline{G}$ denotes the complement of G. For $W \subseteq V(G)$, let $G\langle W\rangle$ denote the subgraph of G induced by W, and $W \backslash v$ the subset of W obtained by removing the vertex v.

A graph G of order n will be called an (H_1, H_2; n)*-graph* if $H_1 \nsubseteq G$ and $H_2 \nsubseteq \overline{G}$. If a ($H_1$, H_2; n)-graph is planar, we call it an (H_1, H_2; n)*-P-graph*. The planar Ramsey number $PR(H_1, H_2)$ is the smallest integer n such that there is no (H_1, H_2; n)-P-graph. The definition of planar Ramsey number was firstly introduced by Walker [7]. Steinberg and Tovey [3] studied the case when both H_1 and H_2 are complete. For a connected graph H_1 of order at least 5, Gorgol proved that PR (H_1, K_l)=$4l$−3 [2]. Bielak and Gorgol [1] determined that PR ($K_4 - e$, K_3)= 7 and $PR(K_4 - e, K_4) = 10$. It was shown that $PR(K_4 - e, K_5) = 14$ and $PR(K_4 - e, K_6 - e) = 16$ [5, 6].

For the Ramsey number $R(K_4 - e, K_6)$, McNamara proved that its exact value is 21 (cf. [4]). In this paper, we prove that $PR(K_4 - e, K_6)$=17 and $PR(K_4 - e, K_l) \geq 3l + \lfloor (l-1)/4 \rfloor - 2$. So, the values of PR ($K_4 - e$, K_l) for $l \geq 7$ are still open.

2. Preliminary Results

Lemma 2.1. If G is a ($K_4 - e$, K_l; n)-P-graph, then $\delta(G) \geq n - PR(K_4 - e, K_{l-1})$.

Proof. Assume that $\delta(G) < n - PR(K_4 - e, K_{l-1})$. Let v be a vertex of degree $\delta(G)$ and $H = G\langle V(G) - N[v]\rangle$, then $p(H) = n - \delta(G) - 1 \geq PR(K_4 - e, K_{l-1})$. Since $K_4 - e \nsubseteq H$, we have $K_{l-1} \subseteq \overline{H}$. The appropriate l−1 vertices of H together with v would yield a K_l in $\overline{G}$, a contradiction. So, $\delta(G) \geq n - PR(K_4 - e, K_{l-1})$.

Lemma 2.2 and 2.3 were proved in [5].

Lemma 2.2. If G is a planar graph such that $K_4 - e \nsubseteq G$, then $q(G) \leq \lfloor 12(p(G) - 2)/5 \rfloor$.

Lemma 2.3. If G is a ($K_4 - e$, K_4; 9)-P-graph, then $3K_3 \subseteq G$ or $G_{9\text{-}0} \subseteq G$, where $G_{9\text{-}0}$ is shown in **Figure 2.1**.

By an argument similar to the one in the proof of paper [5], we can prove Lemma 2.4,

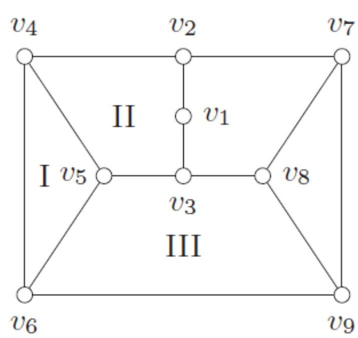

Figure 2.1. The graphs $G_{9\text{-}0}$.

Lemma 2.4. Let G be a $(K_4 - \mathrm{e}, K_5; n)$-P-graph,

a) If $n = 12$, then $4K_3 \subseteq G$, $(G_{9\text{-}0} \cup K_3) \subseteq G$ or $G_{12\text{-}i} \subseteq G$ for $1 \le i \le 8$, where $G_{12\text{-}i}$ are shown in **Figure 2.2**, and

If $n = 13$, then G is isomorphic to $G_{13\text{-}0}$ or $G_{13\text{-}0} + v_3v_4$, where $G_{13\text{-}0}$ is shown in **Figure 2.3**.

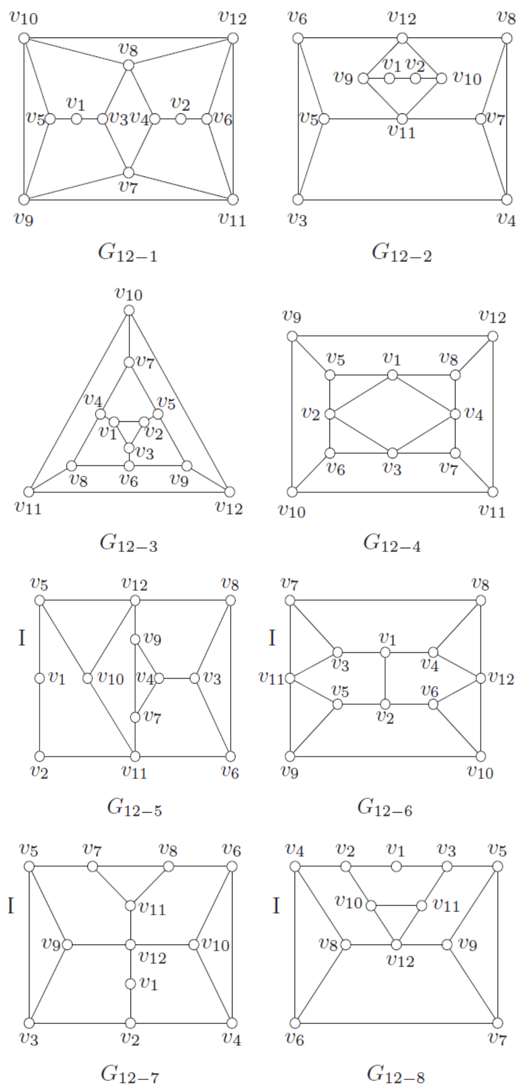

Figure 2.2. The graphs $G_{12\text{-}i}$ for $1 \le i \le 8$.

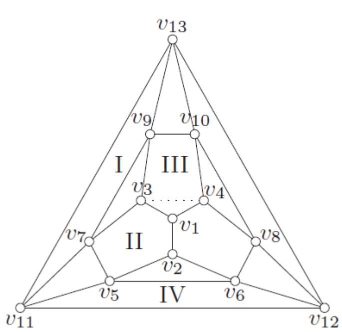

Figure 2.3. The graphs $G_{13\text{-}0}$.

Lemma 2.5. If G is a $(K_4 - e, K_6; 17)$-graph with $\delta(G) = 4$, then it is not a planar graph.

Proof. By contradiction, we assume that G is a $(K_4 - e, K_6; 17)$-P-graph with $\delta(G) = 4$. Let v be a vertex of degree $\delta(G)$ and $H = G\langle V(G) - N[v]\rangle$, then $|V(H)| = 12$. Let $N(v) = \{u_1, u_2, u_3, u_4\}$. Since $K_4 - e \nsubseteq G$ and $\delta(G) = 4$, we have

Claim 1. $G\langle N[v]\rangle$ can not lie in any α-face of H for $\alpha \le 5$ alone.

Since H is a $(K_4 - e, K_5; 12)$-P-graph, by Lemma 2.4(a), we have $(G_{9\text{-}0} \cup K_3) \subseteq H$, $G_{12\text{-}i} \subseteq H$ $(1 \le i \le 8)$ or $4K_3 \subseteq H$.

Case 1. Suppose that $(G_{9\text{-}0} \cup K_3) \subseteq H$. Let $V(G_{9\text{-}0}) = \{v_i \mid 1 \le i \le 9\}$ shown in **Figure 2.1**. By Claim 1, both $N[v]$ and K_3 have to lie in same region of $G_{9\text{-}0}$. By symmetry it is sufficient to consider that they lie in region I, II or III. If they lie in region II, since $K_4 - e \nsubseteq G$, v_6 is nonadjacent to any vertex of $\{v_2, v_3, v_7, v_8\}$. It is forced that $d(v_6) = 3$, a contradiction. If they lie in region I or III, since $K_4 - e \nsubseteq G$, v_1 has to be adjacent to both a_4 and a_8 (or a_5 and a_7). Without loss of generality, let $v_1v_4, v_1v_8 \in E(G)$. Then v_2 is nonadjacent to any vertex of $\{v_3, v_5, v_6, v_8, v_9\}$. Hence $d(v_2) \le 3$, a contradiction.

Case 2. Suppose that H contains one subgraph of $G_{12\text{-}i}$ for $1 \le i \le 6$. By Claim 1, H does not contain any subgraph of $\{G_{12\text{-}1}, G_{12\text{-}2}, G_{12\text{-}3}, G_{12\text{-}4}\}$. Hence there remaining two subcases.

Case 2.1. $G_{12\text{-}5} \subseteq H$. By Claim 1, $G\langle N[v]\rangle$ have to lie in region I. Since $K_4 - \mathrm{e} \nsubseteq G$, v_3 is nonadjacent to any vertex of $\{v_7, v_9, v_{11}, v_{12}\}$. Hence $d(v_3) = 3$, a contradiction.

Case 2.2. $G_{12\text{-}6} \subseteq H$. By Claim 1, $G\langle N[v]\rangle$ have to lie in region I. Since $d(v_1) \ge 4$ and $K_4 - e \nsubseteq G$, v_1 has to be adjacent to just one vertex of $\{v_5, v_6\}$, say v_5. And v_2 is nonadjacent to any vertex of $\{v_4, v_9, v_{10}, v_{12}\}$. Hence $d(v_2) = 3$, a contradiction.

Case 3. $G_{12\text{-}7} \subseteq H$. By Claim 1, $G\langle N[v]\rangle$ have to lie in region I. Since $d(v_{11}) \ge 4$ and $K_4 - e \nsubseteq G$, v_{11} has to be adjacent to just one vertex of $\{v_9, v_{10}\}$, say v_{10}. Since $d(v_1) \ge 4$ and $K_4 - e \nsubseteq G$, v_1 has to be adjacent to both v_4 and v_9. Let $W_7 = \{v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$. By Claim 1 and $K_4 - e \nsubseteq G$, we have $G\langle W_7\rangle \cong C_7$.

Since $K_4 - e \nsubseteq G$, $G\langle N(v)\rangle$ is isomorphic to one graph of $\{4K_1, 2K_1 \cup K_2, 2K_2\}$. If $G\langle N(v)\rangle$ is isomorphic to $4K_1$, then u_1, u_2, u_3, u_4, v_1 and v_{10} would yield a K_6 in $\overline{G}$, a contradiction. Hence $G\langle N(v)\rangle$ is isomorphic to $2K_1 \cup K_2$ or $2K_2$. Without loss of generality, let $u_3u_4 \in E(G)$. If there is one vertex of $\{u_1, u_2, u_3, u_4\}$, say u_1 which is adjacent to v_4, then since $K_4 - e \nsubseteq G$, u_1 is adjacent neither to v_2 nor to v_6. Therefore since $d(u_1) \ge 4$, u_1 is adjacent to at least one vertex of $\{v_3, v_5, v_7, v_8\}$. In any case, there exists one vertex of $\{v_2, v_6\}$ whose degree is at most 3, a contradiction. Hence v_4 is nonadjacent to any vertex of $\{u_1, u_2, u_3, u_4\}$. If $u_1u_2 \notin E(G)$, then u_1, u_2, u_3, v_4, v_9 and

v_{11} would yield a K_6 in $\overline{G}$, a contradiction. So, we have $u_1u_2 \in E(G)$, that is, $G\langle N(v)\rangle \cong 2K_2$.

Since $\delta(G) = 4$, there are at least 8 edges between the vertices of $\{u_1, u_2, u_3, u_4\}$ and $W_7\backslash v_4$. Since $K_4 - e \not\subseteq G$, each vertex of $W_7\backslash v_4$ is adjacent to at most two vertices of $\{u_1, u_2, u_3, u_4\}$. Hence there are just two vertices of $W_7\backslash v_4$ which are adjacent to two vertices of $\{u_1, u_2, u_3, u_4\}$. By symmetry there are three cases.

Case 3.1. Suppose that there is one vertex of $\{v_3, v_8\}$ which is adjacent to u_1 and u_3 (or u_2 and u_4). Without loss of generality, let $v_3u_1, v_3u_3 \in E(G)$. Then since $d(v_5) \geq 4$ and $K_4 - e \not\subseteq G$, v_5 has to be adjacent to one vertex of $\{u_2, u_4\}$, say u_2. Thus there is at least one vertex of $\{u_1, u_3, u_4\}$ whose degree is at most 3, a contradiction (see $G_{17.1}$ in **Figure 2.4**).

Case 3.2. Suppose that there is one vertex of $\{v_5, v_7\}$ which is adjacent to u_1 and u_3 (or u_2 and u_4). Without loss of generality, let $v_5u_1, v_5u_3 \in E(G)$. Then since $d(v_3) \geq 4$ and $K_4 - e \not\subseteq G$, v_3 has to be adjacent to one vertex of $\{u_2, u_4\}$, say u_2. Thus there is at least one vertex of $\{u_1, u_3, u_4\}$ whose degree is at most 3, a contradiction(see $G_{17..2}$ in Fig. 2.4).

Case 3.3. Suppose that both v_2 and v_6 are adjacent to two vertices of $\{u_1, u_2, u_3, u_4\}$ respectively, say $v_2u_1, v_2u_3 \in E(G)$, v_6 is adjacent to one vertex of $\{u_1, u_2\}$ and one vertex of $\{u_3, u_4\}$. Then since $K_4 - e \not\subseteq G$, there is at least one vertex of $\{u_1, u_2, u_3, u_4\}$ whose degree is at most 3, a contradiction (see $G_{17.3}$ in **Figure 2.4**).

Case 4. $G_{12-8} \subseteq H$. By Claim 1, $G\langle N[v]\rangle$ have to lie in region I. Since $d(v_8) \geq 4$ and $K_4 - e \not\subseteq G$, v_8 has to be adjacent to v_9. Since $d(v_{10}) \geq 4$ and $K_4 - e \not\subseteq G$, v_{10} has to be adjacent to just one vertex of $\{v_1, v_4\}$. Similarly, v_{11} has to be adjacent to just one vertex of $\{v_1, v_5\}$. Since $K_4 - e \not\subseteq G$, v_1 is adjacent to at most one vertex of $\{v_{10}, v_{11}\}$. By symmetry it is sufficient to consider that $v_4v_{10}, v_1v_{11} \in E(G)$ or $v_4v_{10}, v_5v_{11} \in E(G)$. If $v_4v_{10}, v_1v_{11} \in E(G)$, then the proof is same as Case 3 (see $_{G17.4}$ in **Figure 2.5**). So it remains that $v_4v_{10}, v_5v_{11} \in E(G)$.

By Claim 1, we have $v_4v_5 \notin E(G)$ and v_1 is non-adjacent to any vertex of $\{v_6, v_7\}$. Since $K_4 - e \not\subseteq G$, $G\langle N(v)\rangle$ is isomorphic to one graph of $\{4K_1, 2K_1 \cup K_2, 2K_2\}$. If $G\langle N(v)\rangle$ is isomorphic to $4K_1$, then u_1, u_2, u_3, u_4, v_8 and v_{10} would yield a K_6 in $\overline{G}$, a contradiction. Hence $G\langle N(v)\rangle$ is isomorphic to $2K_1 \cup K_2$ or $2K_2$.

Case 4.1. Suppose that $G\langle N(v)\rangle \cong 2K_1 \cup K_2$, say $u_3u_4 \in E(G)$. If each vertex of $\{u_1, u_2, u_3, u_4\}$ is adjacent neither to v_4 nor to v_5, then u_1, u_2, u_3 (or u_4), v_4, v_5 and v_{12} would yield a K_6 in $\overline{G}$, a contradiction. Hence there is at least one edge between vertex sets $\{u_1, u_2, u_3, u_4\}$ and $\{v_4, v_5\}$. Assume that there is at least one edge between vertex sets $\{u_1, u_2\}$ and $\{v_4, v_5\}$, say $u_1v_4 \in E(G)$. Then since $d(u_1) \geq 4$ and $K_4 - e \not\subseteq G$, u_1 has to be adjacent to one vertices of $\{v_3, v_5, v_7\}$. In any case, there exists at least one vertex of $\{v_1, v_6\}$ whose degree is at most 3, a contradiction. Hence there is no edge between vertex sets $\{u_1, u_2\}$ and $\{v_4, v_5\}$. Then there is at least one edge between vertex sets $\{u_3, u_4\}$ and $\{v_4, v_5\}$, say $u_3v_4 \in E(G)$. If $u_4v_5 \notin E(G)$, then u_1, u_2, u_4, v_4, v_5 and v_{12} would yield a K_6 in $\overline{G}$, a contradiction too. Hence we have $u_4v_5 \in E(G)$ (see $G_{17.5}$ in **Figure 2.5**). There also exists at least one vertex of $\{v_1, v_6, v_7\}$ whose degree is at most 3, a contradiction.

Case 4.2. Suppose that $G\langle N(v)\rangle \cong 2K_2$, say $u_1u_2, u_3u_4 \in E(G)$. Since $d(v_1) \geq 4$ and $K_4 - e \not\subseteq G$, v_1 has to be adjacent to one vertex of $\{u_1, u_2\}$ and one vertex of $\{u_3, u_4\}$, say $v_1u_1, v_1u_3 \in E(G)$. Then since $K_4 - e \not\subseteq G$, v_1 is adjacent neither to u_2 nor to u_4. If there is no edge between vertex sets $\{u_2, u_4\}$ and $\{v_4, v_5\}$, then $u_2, u_4, v_4,$

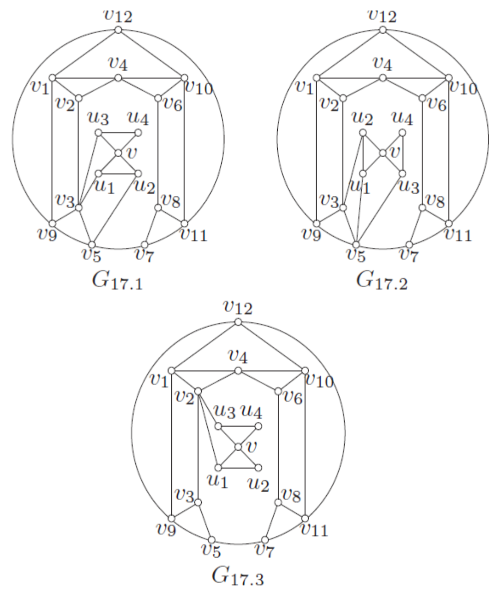

Figure 2.4. The graphs $G_{17.1}$–$G_{17.3}$.

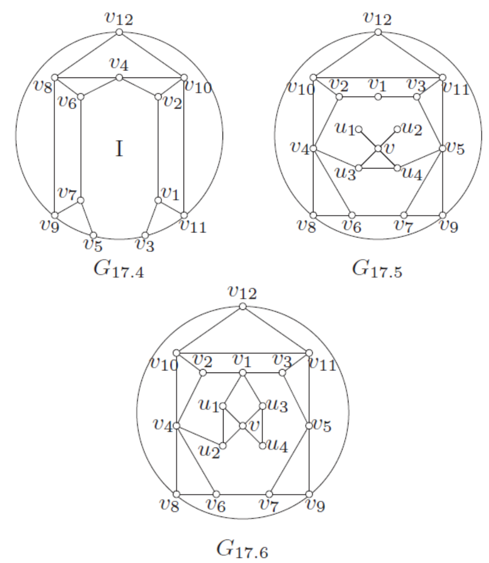

Figure 2.5. The graphs $G_{17.4}$–$G_{17.6}$.

v_5, v_1 and v_{12} would yield a K_6 in $\overline{G}$, a contradiction. Hence there is at least one edge between vertex sets $\{u_2, u_4\}$ and $\{v_4, v_5\}$, say $u_2v_4 \in E(G)$. Then since $d(u_2) \geq 4$ and $K_4 - e \nsubseteq G$, u_2 has to be adjacent to at least one vertex of $\{v_3, v_5, v_7\}$(see $G_{17.6}$ in **Figure 2.5**). In any case, we have $d(v_6) = 3$, a contradiction.

By an argument similar to the above proof, we can prove that $4K_3 \nsubseteq H$. However, the proof of $4K_3 \nsubseteq H$ is more complicated than Case 3 or 4, and it is available from the authors. Hence the assumption does not hold.

3. The Main Results

Lemma 3.1. There is no $(K_4-e,K_6;17)$-P-graph.

Proof. Assume that there is a $(K_4-e, K_6; 17)$-P-graph G. Let v be a vertex of degree $\delta(G)$ and $H = G(V(G)-N[v])$. Since $PR(K_4 - e, K_5) = 14$, by Lemma 2.1, it follows $\delta(G) \geq 3$. By Lemma 2.2, $q(G) \leq \lfloor 12(17-2)/5 \rfloor = 36$ implying $\delta(G) \leq 4$. By Lemma 2.5, we have $\delta(G) \neq 4$. It is forced that $\delta(G) = 3$, thus $p(H) = 13$.

Let $N(v)=\{u_1, u_2, u_3\}$. Since $K_4-e \nsubseteq G$, we have $|E(G(N(v)))| \leq 1$. Without loss of generality, let u_1u_2, $u_1u_3 \notin E(G)$. Since $d(u_1) \geq 3$ and $K_4 - e \nsubseteq G$, $N[v]$ can not lie in any triangle of H. By Lemma 2.4(b), H is isomorphic to $G_{13\text{-}0}$ or $G_{13\text{-}0}+v_3v_4$. If $H \cong G_{13\text{-}0,}$ by symmetry it is sufficient to consider that $N[v]$ lie in region I or II. If H $\cong G_{13\text{-}0} + v_3v_4$, by symmetry it is sufficient to consider that $N[v]$ lie in region I, II, III or IV (see **Figure 2.3**). If $N[v]$ lie in region I, then u_1,u_2,v_3,v_5,v_{10} and v_{12} would yield a K_6 in $\overline{G}$, a contradiction. If $N[v]$ lie in region II, then u_1, u_2, v_4, v_6, v_9 and v_{11} would yield a K_6 in $\overline{G}$, a contradiction. If $N[v]$ lie in region III or IV, then u_1, u_2, v_2, v_7, v_8 and v_{13} would yield a K_6 in $\overline{G}$, a contradiction too.

Theorem 3.2. If $l \geq 3$, then $PR(K_4-e, K_l) \geq 3l+\lfloor(l-1)/4\rfloor-2$.

Proof. Note that $G_{13\text{-}0}$ shown in **Figure 2.3** is a $(K_4 - e, K_5; 13)$-P-graph. Let G be a graph which is a union of $\lfloor(l-1)/4\rfloor$ copies of $G_{13\text{-}0}$ and $(l-4\times\lfloor(l-1)/4\rfloor-1)$ copies of a triangle, then $K_4 - e \nsubseteq G$. Since $K_5 \nsubseteq \overline{G}_{13\text{-}0}$, G contains independent set of size at most $l - 1$. Hence G is a $(K_4 - e, K_l; n)$-P-graph, where $n=3l+\lfloor(l-1)/4\rfloor-3$.

By Lemma 3.1 and Theorem 3.2, taking $l = 6$, we have

Theorem 3.3. $PR(K_4 - e, K_6) = 17$.

Furthermore, we have the following conjecture,

Conjecture 3.4. If $l \geq 3$, then $PR(K_4-e, K_l) = 3l+\lfloor(l-1)/4\rfloor-2$.

By Bielak and Gorgol [1], Sun et al. [5]and Theorem 3.3, the conjecture is true for $l \leq 6$.

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