

# Simplified Optimization Routine for Tuning Robust Fractional Order Controllers

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## ABSTRACT

Fractional order controllers have been used intensively over the last decades in controlling different types of processes. The main methods for tuning such controllers are based on a frequency domain approach followed by optimization routine, generally in the form of the Matlab *fminsearch*, but also evolving to more complex routines, such as the genetic algorithms. An alternative to these time consuming optimization routines, a simple graphical method has been proposed. However, these graphical methods are not suitable for all combinations of the imposed performance specifications. To preserve their simplicity, but also to make these graphical methods generally applicable, a modified graphical method using a very straightforward and simple optimization routine is proposed within the paper. Two case studies are presented, for tuning fractional order PI and PD controllers.

**Keywords:** Fractional Order Controllers; Graphical Tuning; Simplified Optimization Routine

## 1. Introduction

Fractional order PIDs (FO-PIDs) have been employed in various engineering fields ranging applications in a wide variety of domains. The fractional order PID controller is in fact a generalization of the classical integer order PID. In the fractional order PID control algorithm, the error signal is integrated and differentiated to any order, rather than to an integer order as with the traditional PIDs. The fractional order PIDs have two supplementary parameters compared to the traditional PIDs. It is for this reason, that the fractional order PIDs have the potential to meet more design specifications than the traditional PIDs and hence to increase the performance and robustness of closed loop systems [1-4]. A couple of interesting methods have been proposed for tuning such FO-PIDs with the great majority centered upon Matlab's *fminsearch* or graphical approaches [1, 5-7]. The current trend nowadays is directed to the latter methods, since they require less computational and time resources. Nevertheless, if no exact solution exists, the current graphical methods fail at the tuning of the FO-PID controller.

The purpose of this paper is to design an improved graphical method for tuning FO-PI and FO-PD controllers, based upon an optimization routine that selects the best possible tuning option even in the case of no exact solution. For exemplification, two case studies are considered. The first case study implies the design of FO-PI control for a simple first order process. The second case

study consists in the design of a FO-PD controller for a second order process with integrator effect. Simulation results in both case studies show that the fractional order controllers tuned using the proposed algorithm can meet all performance specifications. To exemplify the optimized graphical methods for tuning fractional order controllers, the first case study has no exact solution, while the second case study has an exact solution.

The paper is organized as follows. Section 2 contains the main contribution of the present paper, with a description of the fractional order PI and PD optimized graphical tuning algorithms, while Section 3 presents the two case studies. The final section contains the concluding remarks.

## 2. Optimization Routine for Tuning Fractional Order Controllers

The transfer function of the fractional order PI (FO-PI) controller is given by:

$$H_{FO-PI}(s) = k_p \left( 1 + \frac{k_i}{s^\mu} \right) \quad (1)$$

while the transfer function for the fractional order PD (FO-PD) controller is given by:

$$H_{FO-PD}(s) = k_p (1 + k_d s^\lambda) \quad (2)$$

where  $k_p$ ,  $k_i$  and  $k_d$  are the proportional, integral and derivative gains and  $\mu \in \mathfrak{R}$  and  $\lambda \in \mathfrak{R}$  are the fractional

order. If  $\mu = 1$ , then the FO-PI controller in (1) is reduced to a traditional PI controller:

$$H_{FO-PI}(s) = k_p \left( 1 + \frac{k_i}{s} \right) \quad (3)$$

and the FO-PD is reduced to the classical PD controller by setting  $\lambda = 1$  in (2):

$$H_{FO-PD}(s) = k_p (1 + k_d s) \quad (4)$$

A proper tuning of the FO-PI and FO-PD controllers in (1) and (3), as well as of the PI and PD controllers in (3) and (4), respectively, implies determining the values for the parameters, three in the case of the FO-PI and FO-PD controllers and two in the case of the traditional PI and PD controllers. For tuning FO-PI and FO-PD controllers, in order to uniquely determine the three parameters -  $\mu$ ,  $k_p$  and  $k_i$  in the case of the FO-PI and  $\lambda$ ,  $k_p$  and  $k_d$  in the case of the FO-PD- three equations are used that describe the performance of the closed loop system. The general approach regarding the tuning of fractional order controllers is based on frequency domain performance specifications [8-10], which refer to imposing a gain crossover frequency, a phase margin and robustness to open loop gain variations.

For a general process transfer function  $H_p(s)$ , the open loop system when  $s \rightarrow j\omega$  is written as:

$$H_{open-loop}(j\omega) = H_{FO-PI}(j\omega)H_p(j\omega) \quad (5)$$

where  $\omega$  is the frequency.

In order for the open loop system to attain an imposed gain crossover frequency  $\omega_{gc}$ , then the following relation must hold:

$$\left| H_{open-loop}(j\omega_{gc}) \right| = 1 \quad (6)$$

where  $|\cdot|$  denotes the modulus of the complex function.

The open loop phase margin,  $\phi_m$ , is also computed at the gain crossover frequency as:

$$\angle H_{open-loop}(j\omega_{gc}) = -\pi + \phi_m \quad (7)$$

where  $\angle$  denotes the phase of the complex function.

Finally, the last performance specification, robustness to gain variations, implies that the phase of the open loop system at the gain crossover frequency should be flat:

$$\frac{d(\angle H_{open-loop}(j\omega_{gc}))}{d\omega_{gc}} = 0 \quad (8)$$

## 2.1. Optimization Routine for Tuning Fractional Order PI Controllers

The transfer function of the FO-PI controller, in the frequency domain, may be written as:

$$H_{FO-PI}(j\omega) = k_p \left[ 1 + k_i \omega^{-\mu} \left( \cos \frac{\pi\mu}{2} - j \sin \frac{\pi\mu}{2} \right) \right] \quad (9)$$

in which

$$\frac{1}{s^\mu} \rightarrow \frac{1}{(j\omega)^\mu} = \omega^{-\mu} \left( \cos \frac{\pi\mu}{2} - j \sin \frac{\pi\mu}{2} \right) \quad (10)$$

Equations (6), (7) and (8) imply a certain behavior of the closed loop system, according to the specified values for the gain crossover frequency and the phase margin, and may further be used to determine all three values for the  $k_p$ ,  $k_i$  and  $\mu$  parameters of the FO-PI controller:

$$\left| k_p \left[ 1 + k_i \omega_{gc}^{-\mu} \left( \cos \frac{\pi\mu}{2} - j \sin \frac{\pi\mu}{2} \right) \right] \right| = 1 / \left| H_p(j\omega_{gc}) \right| \quad (11)$$

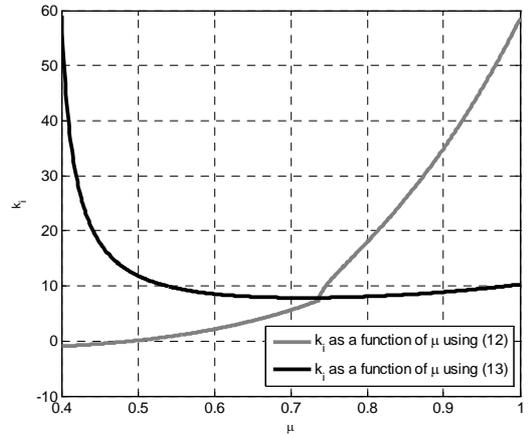
$$\frac{k_i \sin \left( \frac{\pi \cdot \mu}{2} \right)}{\omega_{gc}^\mu + k_i \cos \left( \frac{\pi \cdot \mu}{2} \right)} = \text{tg} \left( \pi - \phi_m + \angle H_p(j\omega_{gc}) \right) \quad (12)$$

$$\frac{\mu k_i \omega_{gc}^{-\mu-1} \sin \frac{\pi\mu}{2}}{1 + 2k_i \omega_{gc}^{-\mu} \cos \frac{\pi\mu}{2} + k_i^2 \omega_{gc}^{-2\mu}} = - \frac{d \angle H_p(j\omega_{gc})}{d\omega_{gc}} \quad (13)$$

where  $H_p(s)$  is the process transfer function.

Using solely equations (12) and (13),  $k_i$  and  $\mu$  may be determined uniquely, while (11) may be then used to determine  $k_p$ . The simplest method for computing the FO-PI parameter values is based on a graphical approach [1, 5-7], which implies that  $k_i$  is computed and plotted as a function of  $\mu$  using equations (12) and (13). The intersection point of the resulting two curves yields the final values for  $k_i$  and  $\mu$ . Consequently,  $k_p$  is determined using (11) and the previously graphically selected values for  $k_i$  and  $\mu$ . Such an example is given in **Figure 1**.

Although this method proves to be highly efficient and simple, the graphical approach is based upon the intersection of the curves resulting from (12) and (13). Such



**Figure 1. Selection of  $k_i$  and  $\mu$  according to the intersection point of the curves.**

an intersection point depends upon the imposed criteria for the gain crossover frequency and the phase margin. For a specified set of gain crossover frequency and phase margin, such an intersection point might not exist. Thus, the existing graphical methods cannot be used to compute the parameters and optimization algorithms need to be used instead.

In order to facilitate the use of the simplicity of the graphical methods in tuning the FO-PI controllers and to avoid complex optimization algorithms, a simple approach is proposed that combines the graphical methods with a very simple optimization routine. The idea behind the optimization routine consist in plotting the two curves for  $k_i$  as a function of  $\mu$  and selecting the values that minimize the distance between the two plotted curves. The proposed tuning algorithm is given below:

```

for  $\mu = 0 : 1$ 
  compute  $k_i$  using (12)
  store result in vector  $k_{i1}$ 
  compute  $k_i$  using (13)
  store result in vector  $k_{i2}$ 
end
plot  $k_{i1}$  as a function of  $\mu$ 
plot  $k_{i2}$  as a function of  $\mu$ 
compute absolute value of  $distance = k_{i1} - k_{i2}$ 
determine  $optim = \min(distance)$ 
return  $\mu_{optim}$  corresponding to  $optim$ 
compute  $k_i$  using (13) and  $\mu_{optim}$ 
compute  $k_p$  using (11)

```

The algorithm for computing PI controllers is based upon setting  $\mu = 1$  and computing  $k_i$  using either (12) or (13) and  $k_p$  using (11). Since, the PI controller in (3) has only two design parameters, the tuning of the PI controller may be done using any combination of two performance criteria in (11), (12) or (13). Thus, imposing (11) and (12) means that (13) will not necessarily be ensured, which is the main drawback of traditional PI controllers as compared to FO-PI controllers.

## 2.2. Optimization Routine for Tuning Fractional Order PD Controllers

The tuning of the FO-PD controller is achieved in a similar manner to the FO-PI. The transfer function of the FO-PD controller, in the frequency domain, may be written as:

$$H_{FO-PD}(j\omega) = k_p \left[ 1 + k_d \omega^\lambda \left( \cos \frac{\pi\lambda}{2} + j \sin \frac{\pi\lambda}{2} \right) \right] \quad (14)$$

in which

$$s^\lambda \rightarrow (j\omega)^\lambda = \omega^\lambda \left( \cos \frac{\pi\lambda}{2} + j \sin \frac{\pi\lambda}{2} \right) \quad (15)$$

Similar to the FO-PI situation, equations (6), (7) and (8)

may be used to determine the three parameters of the FO-PD controller,  $k_p$ ,  $k_d$  and  $\lambda$ :

$$\left| k_p \left[ 1 + k_d \omega_{gc}^\lambda \left( \cos \frac{\pi\lambda}{2} + j \sin \frac{\pi\lambda}{2} \right) \right] \right| = \left| \frac{1}{H_p(j\omega_{gc})} \right| \quad (16)$$

$$\frac{k_d \sin \left( \frac{\pi \cdot \lambda}{2} \right)}{\omega_{gc}^{-\lambda} + k_d \cos \left( \frac{\pi \cdot \lambda}{2} \right)} = \text{tg} \left( \pi - \phi_m + \angle H_p(j\omega_{gc}) \right) \quad (17)$$

$$\frac{\lambda k_d \omega_{gc}^{\lambda-1} \sin \frac{\pi\lambda}{2}}{1 + 2k_d \omega_{gc}^\lambda \cos \frac{\pi\lambda}{2} + k_d^2 \omega_{gc}^{2\lambda}} = - \frac{d \angle H_p(j\omega_{gc})}{d \omega_{gc}} \quad (18)$$

Then, (17) and (18) may be employed to determine using the optimized graphical algorithm the parameters  $k_d$  and  $\lambda$ , and then,  $k_p$  may be computed directly using (16), as described below:

```

for  $\lambda = 0 : 1$ 
  compute  $k_d$  using (17)
  store result in vector  $k_{d1}$ 
  compute  $k_d$  using (18)
  store result in vector  $k_{d2}$ 
end
plot  $k_{d1}$  as a function of  $\lambda$ 
plot  $k_{d2}$  as a function of  $\lambda$ 
compute absolute value of  $distance = k_{d1} - k_{d2}$ 
determine  $optim = \min(distance)$ 
return  $\lambda_{optim}$  corresponding to  $optim$ 
compute  $k_d$  using (18) and  $\lambda_{optim}$ 
compute  $k_p$  using (16)

```

The algorithm for computing PD controllers is based upon setting  $\lambda = 1$  and computing  $k_d$  using either (17) or (18) and  $k_p$  using (16). Since, the PD controller in (4) has only two design parameters; the tuning of the PD controller may be done using any combination of two performance criteria in (16), (17) or (18). Thus, imposing (16) and (17) means that (18) will not necessarily be ensured, which is the main drawback of traditional PD controllers as compared to FO-PD controllers.

## 3. Case Studies

### 3.1. Tuning an FO-PI Controller for a First Order Process

The process transfer function is given by:

$$H_p(s) = \frac{27.5}{0.26s + 1} \quad (19)$$

For a gain crossover frequency of  $\omega_{cg} = 15$  rad/s and a phase margin of  $\phi_m = 70^\circ$ , the curves in **Figure 1** are obtained. Thus, the existing graphical methods may be used to determine the final values for the FO-PI control-

ler parameters. Imposing slightly different performance criteria, such as  $\omega_{cg}=30$  rad/s,  $\varphi_m=70^\circ$  and robustness to gain uncertainties, the two curves in **Figure 2** are obtained.

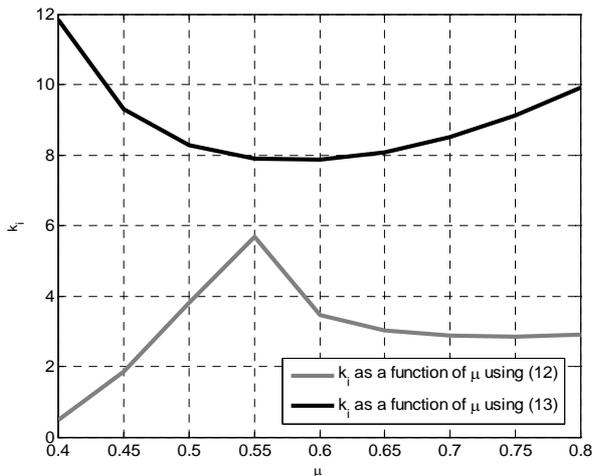
For these particular performance criteria, the two plots for  $k_i$  do not intersect. Nevertheless, using the algorithm proposed in Section 2, the minimum distance between the two curves is computed, yielding  $\mu = 0.55$  and  $k_i = 5.69$ . Finally, using (11) the remaining FO-PI parameter is computed as  $k_p=0.1677$ .

The resulting (FO-PI) is:

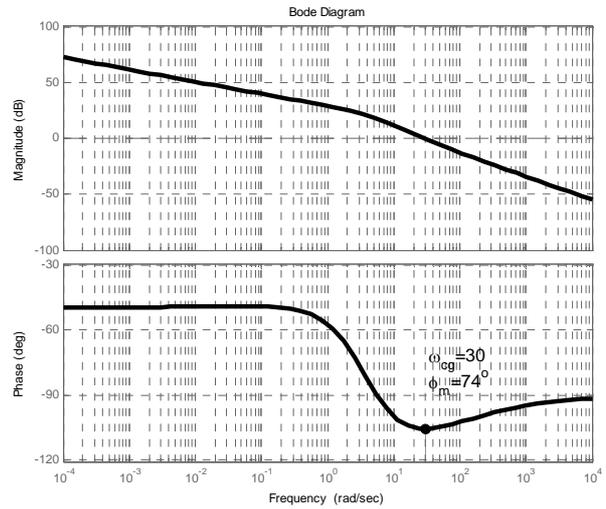
$$H_{FO-PI}(s) = 0.1677 \left( 1 + \frac{5.69}{s^{0.55}} \right) \quad (20)$$

**Figure 3** shows that the Bode plot of the open loop system with a FO-PI controller. It can be seen that the phase margin is slightly increased from  $70^\circ$  as imposed to  $74^\circ$ . This is due to the optimization algorithm, in which the final value for  $k_i$  is chosen in order to meet the robustness criteria, rather than the phase margin criteria. However, thanks to the optimal choice for the fractional order  $\mu$ , the phase margin criteria obtained does not vary significantly from the one imposed in the design phase. The Bode plot also indicates that the modulus crosses the zero axes at  $30$ rad/s, as imposed in the design specifications. Most importantly, it can be seen, that changing the open loop gain will not reduce the phase margin, but rather increase it, which means that the overshoot of the closed loop system will not vary significantly from the nominal value. Hence, the closed loop system should behave robustly despite uncertainties in the gain variations.

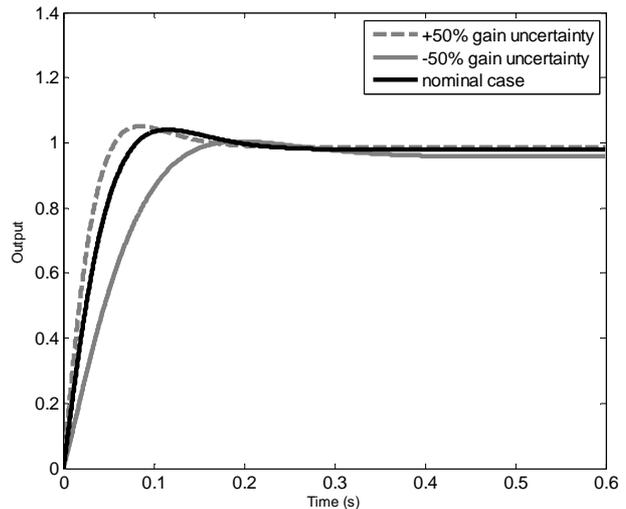
The closed loop results considering  $\pm 50\%$  gain uncertainty are given in **Figure 4**. It can be seen that the FO-PI controller maintains the overshoot below  $5\%$ , while the settling time varies slightly between  $0.15$ - $0.25$  seconds.



**Figure 2.** Plots of  $k_i$  as a function of  $\mu$  using (12) and (13) for  $\omega_{cg} = 25$  rad/s and  $\varphi_m = 70^\circ$



**Figure 3.** Open loop Bode diagram using FO-PI controller.



**Figure 4.** Closed loop results with FO-PI controller considering  $\pm 50\%$  process gain variation.

### 3.2. Tuning an FO-PD Controller for a Second Order Process

The process transfer function is given as:

$$H_p(s) = \frac{1}{s(s+0.5)} \quad (21)$$

Taking  $\omega_{cg} = 15$  and  $\varphi_m = 50^\circ$  and using the algorithm described in Section 2, the plots of  $k_d$  as a function of  $\mu$  are derived as given in **Figure 5**. In this case, the algorithm presented in Section 2 yields the same result as any of the existing graphical methods, since the two curves intersect. **Figure 5** finally yields a fractional order  $\lambda = 0.573$  and  $k_d = 2.59$ .

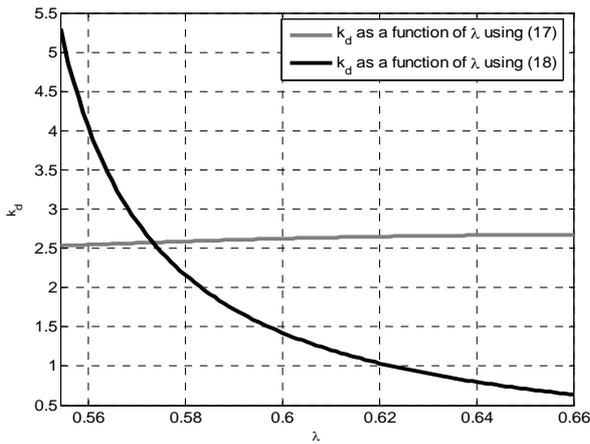
Using (16), the following value is obtained for the  $k_p$  parameter, as a function of the previously determined  $\mu$  and  $k_d$  values:  $k_p = 17.5$ .

The resulting (FO-PD) is:

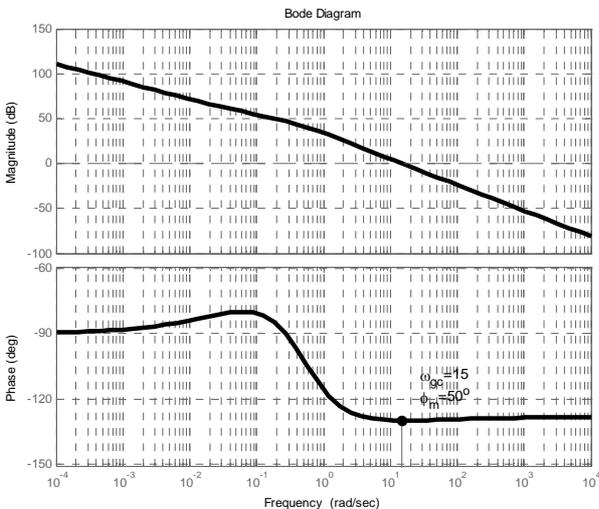
$$H_{FO-PD}(s) = 17.5(1 + 2.59s^{0.573}) \quad (22)$$

The Bode diagram of the open loop system using the previously determined FO-PD controller is given in **Figure 6**, while the closed loop system considering  $\pm 50\%$  gain uncertainty is given in **Figure 7**.

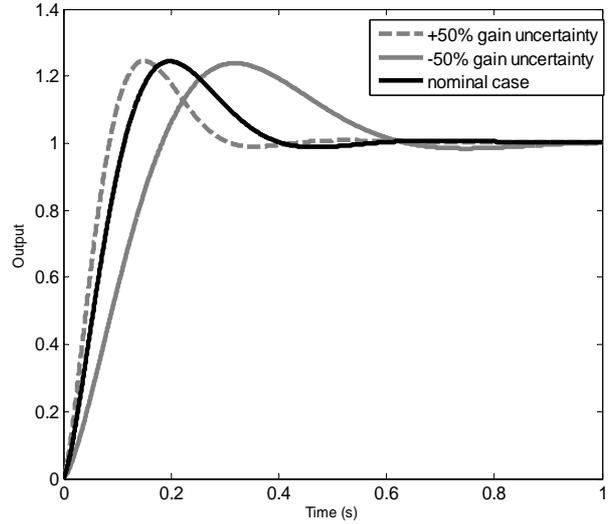
The Bode diagram in **Figure 6** shows that variations of the open loop gain will not have a negative effect on the overshoot of the closed loop system, but only on the settling time, which demonstrates that the designed fractional order PD controller ensures the robustness of the closed loop system despite gain variations. As compared to the fractional order PI controller, the solution of the PD controller at the intersection of the two curves implies that all performance specifications are met: the gain crossover frequency is exactly 15 rad/sec, as specified, the phase margin is exactly  $50^\circ$  and the phase plot is flat around the gain crossover frequency.



**Figure 5.** Selection of the fractional order  $\mu$  and  $k_d$  parameter.



**Figure 6.** Bode diagram of the open loop system with FO-PD controller.



**Figure 7.** Closed loop responses considering  $\pm 50\%$  gain uncertainty with a FO-PD controller.

As expected from the Bode plot, the overshoot is maintained in all three case scenarios at the value of 25%, while the settling time varies between 0.3-0.6 seconds.

### 4. Conclusions

The purpose of the present paper was to present a simple and efficient optimization algorithm for tuning fractional order PI and PD controllers. For specific performance criteria, the existing graphical methods may not yield an exact solution. Thus, optimization routines need to be used in order to tune the fractional order controllers. The paper shows that even in the case of no exact solution, the graphical methods may still be employed with a slight modification that implies computing and selecting the minimum distance between the possible solutions. It is shown through simulations that the fractional order controllers designed using the proposed method yield satisfactorily results in terms of closed loop performance and robustness.

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