

Error Estimation and Assessment of an Approximation in a Wavelet Collocation Method

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ABSTRACT

This article describes how to assess an approximation in a wavelet collocation method which minimizes the sum of squares of residuals. In a research project several different types of differential equations were approximated with this method. A lot of parameters must be adjusted in the discussed method here. For example one parameter is the number of collocation points. In this article we show how we can detect whether this parameter is too small and how we can assess the error sum of squares of an approximation. In an example we see a correlation between the error sum of squares and a criterion to assess the approximation.

Keywords: ODE; Sinc Collocation; Shannon Wavelet; Wavelet Collocation; Error Estimation

1. Introduction

In the wavelet theory a scaling function ϕ is used, which has properties that are defined in the MSA (multi scale analysis). Through the MSA we know, we can construct an orthonormal basis of a closed subspace V_j , where V_j belongs to a sequence of subspaces with the following property:

$$\dots \subset V_{-1} \subset V_0 \subset V_1 \subset \dots \subset L^2(R),$$

$\{\phi_{j,k}(t)\}_{k \in \mathbb{Z}}$ is an orthonormal basis of V_j with

$$\phi_{j,k}(t) = 2^{j/2} \phi(2^j t - k).$$

We use the following approximation function:

$$y_j(t) := \sum_{k=k_{\min}}^{k_{\max}} c_k \cdot \phi_{j,k}(t), \text{ with } \phi \in C^1(R).$$

k_{\min} and k_{\max} depend on the approximation interval $[t_0, t_{\text{end}}]$.

Now we can approximate the solution of an initial value problem $y' = f(y, t)$ and $y(t_0) = y_0$ by minimizing the following function ($\|\cdot\|$ is the Euklid norm)

$$Q(c) = \sum_{i=1}^m \|y'_j(t_i) - f(y_j(t_i), t_i)\|^2 + \|y_j(t_0) - y_0\|^2. \quad (1)$$

For $m = |k_{\max} - k_{\min}|$ we get an equivalent problem:

$$y'_j(t_i) = f(y_j(t_i), t_i), \text{ with } i = 1, 2, \dots, m \text{ and } y_j(t_0) = y_0.$$

The advantage of calculating c by minimizing Q is that we can choose more collocation points t_i as shown in the following example. In that case we apply the least squares method to calculate c . Many simulations had shown that if Q_{\min} was very small then the approximation y_j would be good. An even better criterion for a good approximation y_j is Q_a (see (3)). Moreover, the equations have been ill-conditioned in several examples.

Analogously we could use boundary conditions instead of the initial conditions. This method can be even used analogously for PDEs, ODEs of higher order or DAEs, which have the form

$$F(y', y, t) = 0.$$

If $y' = f(y, t)$ is an ODE system, then we use the approximation function:

$$y_j(t) = \left(\sum_{k=k_{\min}}^{k_{\max}} c_{k,1} \phi_{j,k}(t), \sum_{k=k_{\min}}^{k_{\max}} c_{k,2} \phi_{j,k}(t), \dots, \sum_{k=k_{\min}}^{k_{\max}} c_{k,n_f} \phi_{j,k}(t) \right)^T$$

For the i -th component of the solution y , we use the notation y_j as usual. We use for the i -th component of y_j the notation $y_j^{(i)}$, in order not to lead to a confusion

with the approximation y_j out of V_j , so it will be always distinguished whether the approximation y_j or the i -th component of y is used.

We use the collocation points t_i , with $t_i = t_0 + i \cdot h$ and

$$h = \frac{t_{\text{end}} - t_0}{m} \quad (m \geq |k_{\text{max}} - k_{\text{min}}|). \quad (2)$$

Simulations have shown that even with $m < |k_{\text{max}} - k_{\text{min}}|$ we get good approximations.

For the assessment of the approximation we use the value Q_a , with

$$Q_a = \sum_{i=1}^{m_a} \|y'_j(\tau_i) - f(y_j(\tau_i), \tau_i)\|^2 + \|y_j(t_0) - y_0\|^2, \quad (3)$$

$\tau_i = t_0 + i \cdot h/a$, $m_a = a \cdot m$ and $a > 1$ is an integer. For big a we should weight Q_a with $1/a$.

Remarks 1:

1) We get

$$Q_{\text{min}}^{(k_{\text{max}}^{(1)})} \leq Q_{\text{min}}^{(k_{\text{max}}^{(2)})}$$

for $k_{\text{max}}^{(1)} > k_{\text{max}}^{(2)}$, because of:

$$Q_{\text{min}}^{(k_{\text{max}}^{(1)})}(c_{k_{\text{min}}}, \dots, c_{k_{\text{max}}^{(1)}}) = Q_{\text{min}}^{(k_{\text{max}}^{(2)})}(c_{k_{\text{min}}}, \dots, c_{k_{\text{max}}^{(1)}})$$

Analogously for smaller k_{min} .

2) The sums in (1) and (3) could start with $i = 0$, too.

3) $\|y_j(t_0) - y_0\|^2$ in (1) could also be used as a constraint if the initial value should be fulfilled. But in all good approximations, $\|y_j(t_0) - y_0\|^2$ was very small.

In the examples we use the Shannon wavelet. Although it has no compact support and no high order, in many examples and simulations we got a much better

approximation than using other wavelets (f.e. Daubechies wavelets of order 5 to 8), even with a small n . The Meyer wavelet yields good results, too.

We even get a good extrapolation outside the interval $[t_0, t_{\text{end}}]$.

Example 1:

1) We use the following ODE

$$y' = -t \cdot y, \quad y(0) = 1.$$

The exact solution is $y(t) = e^{-1/2t^2}$.

We approximated the solution on the interval $[0,1]$ and chose $k_{\text{max}} = -k_{\text{min}}$, like in all examples.

With Q_a we could see in all our simulations, if the approximation was good. We got a linear relationship between $\ln(Q_2)$ and $\ln(sse)$. In **Figure 1** we see the graph of a linear regression (with an R squared of 0.991196) of $\ln(sse)$ against $\ln(Q_2)$ with the points $(\ln(Q_2), \ln(sse))$, which have been calculated with different j ($j = 0, 1, 2$), $k_{\text{max}} = -k_{\text{min}}$ ($k_{\text{max}} = 15, 20, 25$) and m ($m = k_{\text{max}}, 2k_{\text{max}}, 3k_{\text{max}}$) with the ODE and I of the example 1.

sse is the mean squared error

$$sse = \sum_{i=0}^{100} (y(\tilde{t}_i) - y_j(\tilde{t}_i))^2 \quad \text{with } \tilde{t}_i = i/100.$$

Now we see a regression table (**Table 1**) of $\ln(sse)$ on $\ln(Q_2)$, which shows a linear dependency in our example and the graph of the linear regression function.

Here is a graph of the regression function and the graphs of the functions y_i and $y_j - y$ for $j = 0$, $k_{\text{max}} = 15$ and $m = 30$ on the approximation interval $[0,1]$ (see **Figures 2** and **3**) and on the interval $[-1,2]$ (see **Figures 4** and **5**). In **Figures 4** and **5** we see that we get even a good extrapolation.

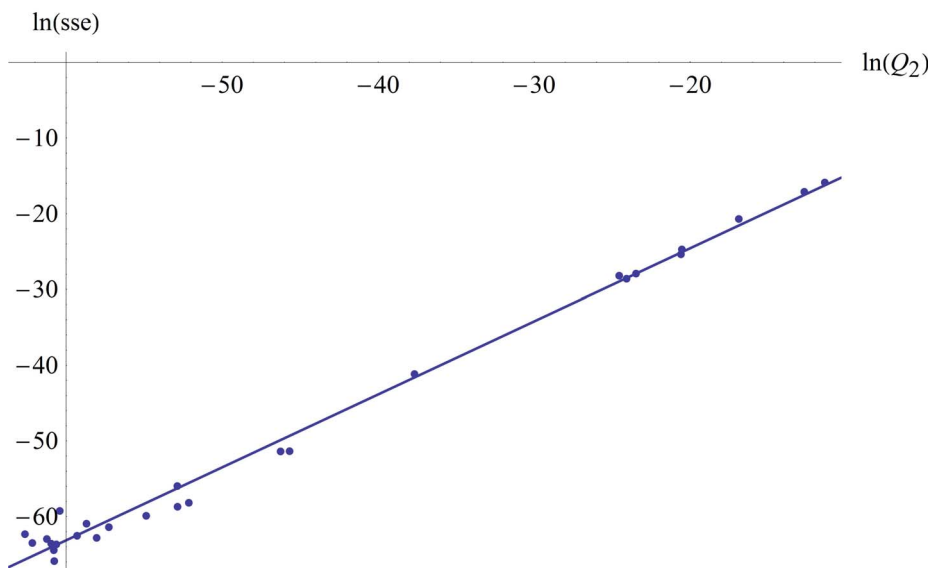


Figure 1. Linear regression plot of $\ln(sse)$ against $\ln(Q_2)$.

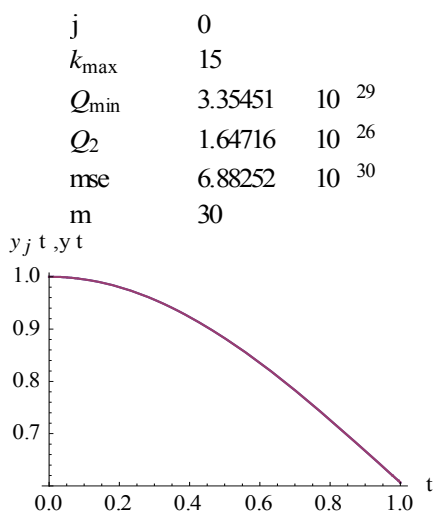


Figure 2. Graph of $y_0, k_{\max} = 15, m = 30$.

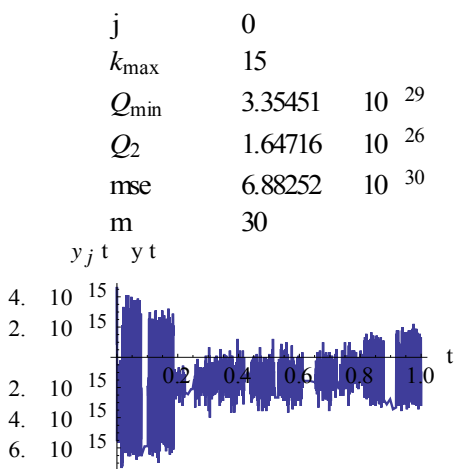


Figure 3. Graph of $y_0 - y, k_{\max} = 15, m = 30$.

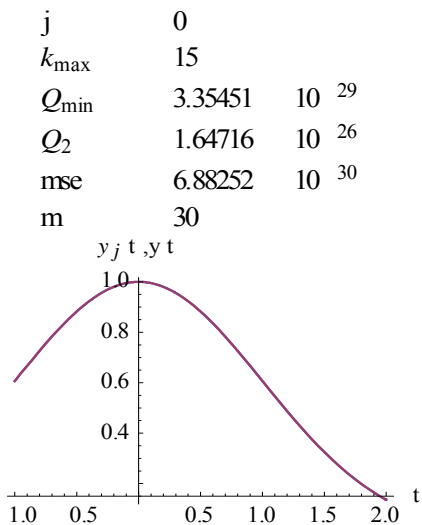


Figure 4. Graph of $y_0, k_{\max} = 15, m = 30$.

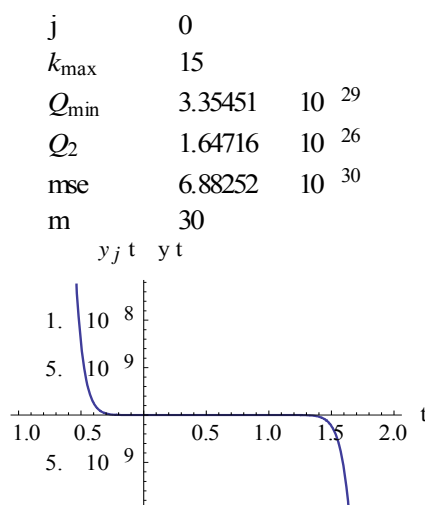


Figure 5. Graph of $y_0 - y, k_{\max} = 15, m = 30$.

Table 1. Linear regression table of $\ln(sse)$ on $\ln(Q_2)$.

	Estimate	SE	T Stat	P Value
Intercept	-5.34843	0.882147	-6.06297	2.46251×10^{-6}
Slope	0.962059	0.0181342	53.0523	3.2288×10^{-27}

2. Error Estimation and Assessment of the Approximation

In the example we used the Shannon wavelet. For this wavelet we have additional information about the error in the Fourier space from the Shannon theorem. For a good approximation with a small j the behavior of $Y(\omega)$ with growing $|\omega|$ is important, because (if y_i is an orthogonal projection from y on V_j and $k_{\max} = -k_{\min} = \infty$)

$$\begin{aligned}
 & y(t) - y_j(t) \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Y(\omega) e^{i\omega t} d\omega - \frac{1}{\sqrt{2\pi}} \int_{-2^j\pi}^{2^j\pi} Y(\omega) e^{i\omega t} d\omega \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-2^j\pi} Y(\omega) e^{i\omega t} d\omega + \frac{1}{\sqrt{2\pi}} \int_{2^j\pi}^{\infty} Y(\omega) e^{i\omega t} d\omega.
 \end{aligned}$$

With the Parseval theorem we get

$$\|y - y_j\|_{L^2} = \sqrt{\int_{-\infty}^{-2^j\pi} |Y(\omega)|^2 d\omega + \int_{2^j\pi}^{\infty} |Y(\omega)|^2 d\omega},$$

so

$$\|d_j\|_{L^2} = \sqrt{\int_{-2^{j+1}\pi}^{-2^j\pi} |Y(\omega)|^2 d\omega + \int_{2^j\pi}^{2^{j+1}\pi} |Y(\omega)|^2 d\omega}$$

With the Riemann-Lebesgue theorem we get:

$$\lim_{j \rightarrow \infty} \|d_j\|_{L^2} = 0$$

For the approximation error the decay behaviour of the detail coefficients $d_k^j = \langle y, \psi_{j,k} \rangle$ is important:

$$\begin{aligned} \|y - y_j\|_{L^2} &= \left\| \sum_{s=j}^{\infty} d_s \right\|_{L^2} = \left\| \sum_{s=j}^{\infty} \sum_{k=-\infty}^{\infty} d_k^s \psi_{s,k} \right\|_{L^2} \\ &= \sqrt{\sum_{s=j}^{\infty} \sum_{k=-\infty}^{\infty} |d_k^s|^2} \end{aligned}$$

On the other side: we have got in many simulations with the Shannon wavelet better approximations (with the described collocation method) than with higher order wavelets.

Remarks 2:

1) For a theoretical multi resolution analysis we could consider $1_I \cdot y$ instead of y , because when y is in $L^2(I)$ then $1_I \cdot y$ is in $L^2(R)$, if we need an approximation on I . Here 1_I is the indicator function of the interval I .

2) For interpolating wavelets there are a number of publications with error estimates and also for the approximation of the solutions of initial value problems and boundary value problems (for ordinary and partial differential equations) see [1,2], as well as to the sinc collocation method (see [3-5]) with special collocation points (“sinc grid points”, see [5]).

Theorem 1 (for the decay behaviour):

The wavelet ψ has the order $p, y \in L^2(R), y \in C^r(R)$ with $r < p$ and $y^{(r)}$ is Lipschitz continuous. Then exists a $c > 0$ independent from b with

$$|W_y(a, b)| \leq c |a|^{r+3/2}.$$

W_y is the wavelet transform of y with

$$W_y(a, b) = |a|^{-1/2} \cdot \int_{-\infty}^{\infty} y(t) \cdot \overline{\psi\left(\frac{t-b}{a}\right)} dt.$$

A proof is in [6]. So we get for the detail coefficients an appraisal because

$$\begin{aligned} d_k^j &= \langle y, \psi_{j,k} \rangle = \int_{IR} y(t) \overline{2^{j/2} \psi(2^j t - k)} dt \\ &= W_y(2^{-j}, k 2^{-j}) \end{aligned}$$

and so

$$|d_k^j| = |W_y(2^{-j}, k 2^{-j})| \leq c \cdot 2^{-j(r+3/2)}.$$

Now we saw that the decay of the detail coefficients depends on the order of a wavelet.

From the Gilbert-Strang Theory (see [7]) we know additionally an upper bound of the approximation error in dependency of the order p : if the wavelet is of order p then the approximation error has the order $O(2^{-j \cdot p})$ if $\|y^{(p)}\|_{L^2} < \infty$ and (if y_j is an orthogonal projection from y on V_j and $k_{\max} = -k_{\min} = \infty$)

$$\|y - y_j\|_{L^2} \leq C_\phi \cdot 2^{-jp} \cdot \|y^{(p)}\|_{L^2}.$$

If a wavelet is of order p the scaling function ϕ even has an interpolation property, because then we can construct the functions t^r with $r = 0, 1, \dots, p-1$ over a linear combination of $\phi(t-k)$ (see [7]). That’s also a property of the so called interpolating wavelets. For interpolating wavelets we find error estimations in [8] and [9].

Remarks 3:

1) Error estimations for the sinc collocation with a transformation can be found in [4] and [5].

2) Although the approximation error is depended on the order of a wavelet in many simulations the Shannon wavelet led to much better approximations than Daubechies wavelets of higher order, if the approximation function y_j was calculated by minimizing the sum of squares of residuals Q . Even when comparing the extrapolations the Shannon wavelet was significantly better.

The reason is, that we do not calculate an orthogonal projection on V_j like in the appraisal above and the function y is in general case not quadratic integrabel on R (we consider only a compact interval I).

The following appraisal takes account of the fact that we calculate the approximation function by the minimization of Q . We first need a theorem, which follows from the Gronwall-Lemma.

Theorem 2:

Assumptions: we have a initial value problem $y' = f(y, t)$ with $y(t_0) = y_0$ and

$$\begin{aligned} \|y_j(t_0) - y(t_0)\| &\leq \delta, \\ \|y'_j(t) - f(y_j(t), t)\| &\leq M \end{aligned} \tag{4}$$

and

$$\begin{aligned} \|f(y(t), t) - f(y_j(t), t)\| &\leq L \cdot \|y(t) - y_j(t)\| \\ &\text{with } L > 0. \end{aligned} \tag{5}$$

Then we get for $t \geq t_0$:

$$\|y(t) - y_j(t)\| \leq \delta \cdot e^{L(t-t_0)} + M/L \cdot (e^{L(t-t_0)} - 1)$$

For a proof see [10].

Theorem 3:

With the assumptions from Theorem 2 we get (if $y_j(t_0) = y_0$):

$$sse = \sum_{i=1}^{\tilde{m}} \|y_j(\tilde{t}_i) - y(\tilde{t}_i)\|^2 \leq (M/L)^2 \cdot \underbrace{\sum_{i=1}^{\tilde{m}} (e^{L(\tilde{t}_i-t_0)} - 1)^2}_{=C_L}$$

So we get the follow inequality for $\ln(sse)$, which is used in the example 2:

$$\ln(sse) \leq \ln(M^2) - \ln(L^2) + \ln(C_L) \tag{6}$$

$\ln(M^2)$ depends on y_j and $\ln(C_L) - \ln(L^2)$ only on the initial value problem and the collocation points. We write $\ln(M^2)$ instead of $2\ln(M)$ because in example 2 we set $\ln(M^2)$ on the x -axes so we have a comparison with example 1 where we set $\ln(Q_2)$ on the x -axes.

Remark 4:

We get with $y_j(t_0) = y_0$

$$Q_{\min} = \sum_{i=1}^m \|y'_j(t_i) - f(y_j(t_i), t_i)\|^2 \leq m \cdot M^2.$$

If additionally $\|y'_j(t_i) - f(y_j(t_i), t_i)\| = M$ for one (or more) $i \in \{1, 2, \dots, m\}$ we get:

$$M \leq \sqrt{Q_{\min}} \leq \sqrt{m} \cdot M$$

This is analogously right for Q_a instead if Q_{\min} with

$$Q_a = \sum_{i=1}^{m_a} \|y'_j(\tau_i) - f(y_j(\tau_i), \tau_i)\|^2$$

and $\tau_i = t_0 + i \cdot h/a, m_a = a \cdot m$ and an integer $a > 1$. Q_a is an upper bound for $Q_{\min} : Q_{\min} \leq Q_a$. With Q_a we could assess in all simulations the quality of an approximation and in linear regressions from $\ln(sse)$ on $\ln(Q_2)$ we got in almost all simulations a R^2 (R squared) greater than 0.99 (see next example). Only if all approximations have been bad, then R^2 was less than 0.99 (but we still have a dependency). If y_j is the exact solution, then $Q_a = 0$. Because we get not only a approximation with

points (we get a approximation function y_j) we must not calculate a second minimization for the calculation of Q_a . $\|y'_j(t_i) - f(y_j(t_i), t_i)\|$ will be in general (for $i > 0$) less than M , because we use the collocations points t_i and so $\|y'_j(t_i) - f(y_j(t_i), t_i)\|$ is very small at these points (see the next graphic). Q_{\min} was in many good simulations less than 10^{-16} .

In many simulation $\|y'_j(t_i) - f(y_j(t_i), t_i)\|$ is relative big between to collocation points (or at the edge of I if we start with $i = 1$ in the sum (1)).

In **Figure 6** we see the graph of

$$d(t) = (y'_j(t) - f(y_j(t), t))^2$$

in example 1 for $j = 2, k_{\max} = 10$ and $m = 4$. Here a too small m results in a very bad approximation.

We see that Q_{\min} could be very small with a too small m , but Q_2 is very big here. In the graph we see that d is very small at the collocation points $t_i = 0.25i$ but between them d is very big. That's the reason because we could identify with Q_2 a worse approximation in any our simulations. On the other hand a big Q_{\min} is an indicative of a too small j .

So we can approximate M here with the maximum of $\|y'_j(\tau_i) - f(y_j(\tau_i), \tau_i)\|$ at the points $\tau_i = t_0 + i \cdot h/a$ with $a = 2$ like we do it in the next example.

Now we want to apply the result from theorem 2. Furthermore we will see a correlation between an approximation of $\ln(M^2)$ and $\ln(sse)$ in this example like we saw it before between $\ln(Q_2)$ and $\ln(sse)$.

j	2
k_{\max}	10
Q_{\min}	5.27296 10^{-31}
Q_2	35.0546
mse	0.194116
m	4

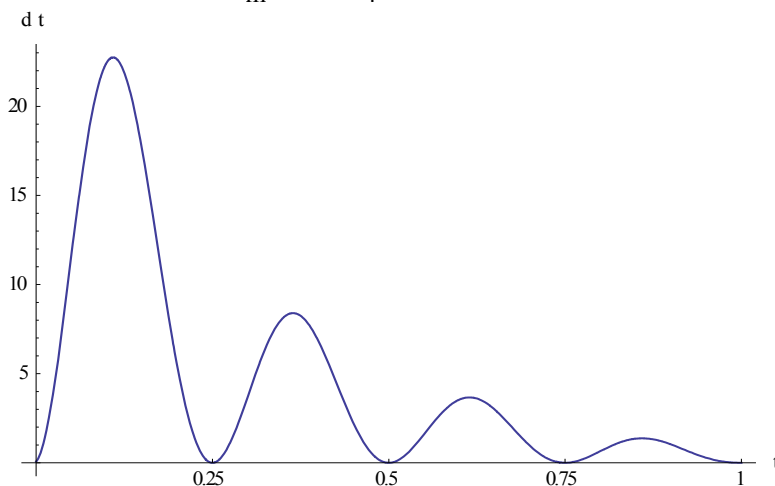


Figure 6. Graph of d .

Example 2:

We use the initial value problem and the approximations with the different parameters j , k_{\max} and m of example 1. If $\delta > 0$ than follows from theorem 2 (under the assumptions from this theorem):

$$\ln(sse) \leq \ln \left(\underbrace{\sum_{i=1}^{\tilde{m}} \left(\delta \cdot e^{L(\tilde{t}_i - t_0)} + M/L \cdot \left(e^{L(\tilde{t}_i - t_0)} - 1 \right) \right)^2}_{=O_\delta(M)} \right)$$

Here we get (see (6)):

$$O_0(M) = \ln(M^2) - \ln(L^2) + \ln(C_L)$$

We now apply a linear regression of $\ln(sse)$ on $\ln(\hat{M}^2)$ with the approximation

$$\hat{M}^2 = \max \left\| y'_j(\tau_i) - f(y_j(\tau_i), \tau_i) \right\|^2,$$

from M^2 with $\tau_i = t_0 + i \cdot h/2, i = 0, 1, 2, \dots, 200$ (the points from Q_2 beginning with $i = 0$). sse and O_δ have been calculated with the points $\tilde{t}_i = i/100$ (and the summation indices $i = 0, 1, 2, \dots, 100$).

Here is the regression table (Table 2) (with a R squared of 0.986877).

In Figure 7 we see a graph from O_0 (in red), the graph of the regression function (in blue) and the regression points $(\ln(\hat{M}^2), \ln(sse))$. $\|y_j(t_0) - y(t_0)\|$ was not considered (this means we set $\delta = 0$) because it was very small.

Here are the graphs of O_δ with $\delta = 10^{-10}$, $\delta = 10^{-15}$

and $\delta = 0$ in Figure 8. In most simulations δ was less than 10^{-16} .

Generally we can use

$$\hat{M}_a^2 = \max \left\| y'_j(\tau_i) - f(y_j(\tau_i), \tau_i) \right\|^2,$$

$\tau_i = t_0 + i \cdot h/a$ and $i = 0, 1, \dots, a \cdot m$ (with an integer $a > 1$) for an approximation of M^2 . Here we know the following relation:

$$\hat{M}_a^2 \leq Q_a$$

3. Conclusions

We defined a variable Q_a with which you can evaluate an approximation. In many simulations and in the examples of this article we saw that we get good results with $a = 2$. A linear relationship between $\ln(Q_a)$ and $\ln(sse)$ was shown in example 1. It is also shown that the approximation can be used to extrapolate outside the approximation interval.

Using Theorem 2 we derive an estimate (see theorem 3). Then it is shown how to detect a too great step size using Q_2 . In example 2 we show that the deduced estimate represents a straight line (in the coordinate system with $\ln(M^2)$ on the x -axes and $\ln(sse)$ on the y -axes), which runs approximately parallel to the regression line (it is approximately parallel because the regression function is an estimation, theoretically it must be parallel because it cannot cross the upper bound line). In a research project we got analogous results in many

Table 2. Regression table of $\ln(sse)$ on $\ln(\hat{M}^2)$.

	Estimate	SE	T Stat	P Value
Intercept	-5.27413	1.08064	-4.88058	0.0000507742
Slope	0.952356	0.0219645	43.3589	4.75173×10^{-25}

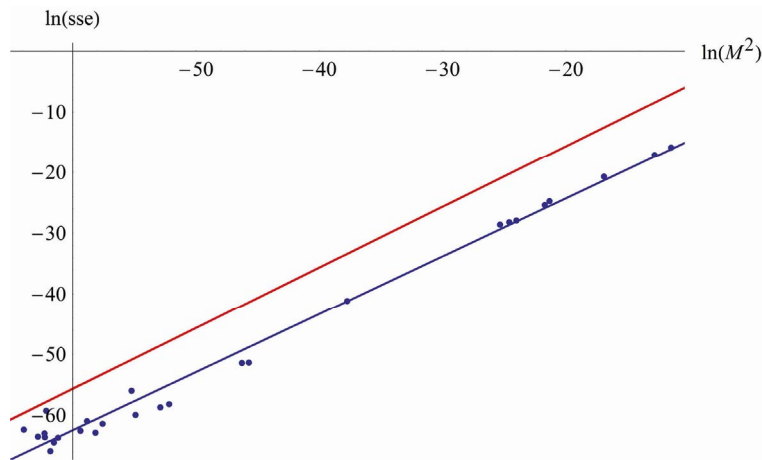


Figure 7. Linear regression plot of $\ln(sse)$ against $\ln(\hat{M}^2)$.

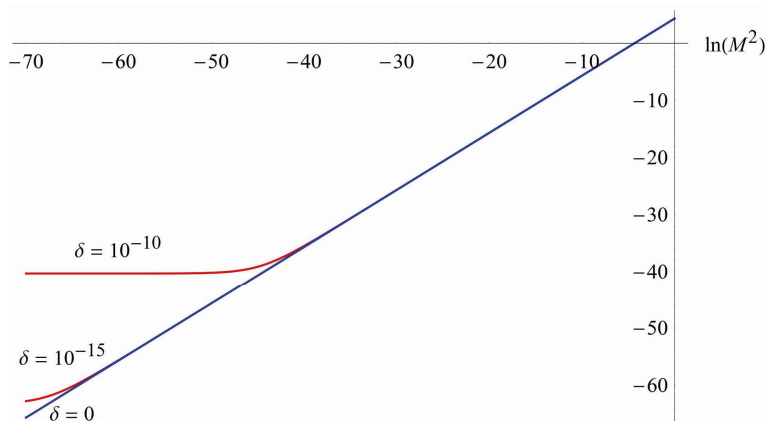


Figure 8. Graphs of O_δ with $\delta = 10^{-10}$, $\delta = 10^{-15}$ and $\delta = 0$.

simulations. even with systems and higher order odes.

It is shown that M^2 (the size of the estimate) can be approximated via \hat{M}^2 , and this approximation has Q_2 as upper bound. The regression of the points $(\ln(Q_2), \ln(sse))$ returns a slightly larger R^2 than the regression with the points $(\ln(\hat{M}^2), \ln(sse))$. As a consequence, Q_2 is well suited to assess, especially as you can estimate the approximation of M^2 with Q_2 and in Q_2 more information is included. Moreover we can compare Q_2 with Q_{\min} to assess the approximation (see **Figure 6**).

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