American Journal of Computational Mathematics, **2013, 3, 27-31**
http://dx.doi.org/10.4236/ajcm.2013.31A005 Published Online April 2013 (http://www.scirp.org/journal/ajcm)

Overlapping Nonmatching Grid Method for the Ergodic Control Quasi Variational Inequalities

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Received February 16, 2013; revised March 17, 2013; accepted March 31, 2013

ABSTRACT

In this paper, we provide a maximum norm analysis of an overlapping Schwarz method on nonmatching grids for a quasi-variational inequalities related to ergodic control problems studied by M. Boulbrachène [1], where the "discount factor" (*i.e.*, the zero order term) is set to 0, we use an overlapping Schwarz method on nonmatching grid which consists in decomposing the domain in two sub domains, where the discrete alternating Schwarz sequences in sub domains converge to the solution of the ergodic control IQV for the zero order term. For $\alpha \in]0,1[$ and under a discrete maximum principle we show that the discretization on each sub domain converges quasi-optimally in the L^∞ norm to 0.

Keywords: Quasi Variational Inequalities; Ergodic Control; Schwarz Method; Finite Element Method

1. Introduction

The Schwarz alternating method can be used to solve elliptic boundary value problems on domains which consist of two or more overlapping sub domains.

The solution is approximated by an infinite sequence of functions which results from solving a sequence of elliptic boundary value problems in each of the sub domain.

In this paper, we are interested in the error analysis in the maximum norm for the obstacle problem in the context of overlapping nonmatching grids: we consider a domain Ω which is the union of two overlapping subdoains where each sub domain has its own triangulation. This kind of discretizations is very interesting as they can be applied to solving many practical problems which cannot be handled by global discretizations. They are earning particular attention of computational experts and engineers as they allow the choice of different mesh sizes and different orders of approximate polynomials in different sub domains according to the different properties of the solution and different requirements of the practical problems.

We study a new approach for the finite element approximation for the ergodic problem where the obstacle is related to a solution. We consider a domain which the union of two overlapping sub-domains where each sub domain has its own generated triangulation. The grid points on the sub-domain boundaries need not much the grid points from the other sub-domains. Under a discrete maximum principle, we show that the discretization on each sub-domain converges quasi-optimally in the L^∞ norm.

In the first section we study the Schwarz method for the ergodic control Quasi-variational inequalities; we state the continuous alternating Schwarz sequence for quasi-variational inequalities, and define their respective finite element counterparts in the context of overlapping grids.

In Section 2, we give a simple proof for the main result concerning error estimates in the L^∞ norm for the problem studied, taking into account the combination of Jinping Zeng & Shuzi Zhou [2] geometrical convergence and P. L. Lions, B. Perthame [3] quasi-variational inequalities and ergodic impulse control.

2. Schwarz Method for the Ergodic Control Quasi-Variational Inequalities

We begin by down a classical results related to ergodic control quasi-variational inequalities [1-18].

It is well known that impulse control problems for reflected diffusion process may be solved by considering the solution of quasi variational inequalities (QVI) (see Bensoussan [4], A. Bensoussan and J. L. Lions [5]). A typical example is the following:

$$\begin{cases} a(u_\alpha, v-u_\alpha)+(u_\alpha, v-u_\alpha)\ge (f, v-u_\alpha) \\ u_\alpha \in H^1(\Omega), u_\alpha \le Mu_\alpha; v\in H^1(\Omega), v\le Mu_\alpha \end{cases} \quad (1)$$

where Ω is given bounded smooth open set in $\mathbb{R}^N$, $\alpha > 0$, f is given function, the cost function Mu represents the obstacle of impulse control defined by:

$$M\varphi(x) = k + \inf(x+\xi); \quad \xi \ge 0, x+\xi \in \Omega, \text{where } k>0 \quad (2)$$

where k is a positive number, M is an operator defined on $C(\bar{\Omega})$ and assumed to map into itself, that is,

$$M\varphi(x) \le M\tilde{\varphi}(x) \quad \text{whenever} \quad \varphi(x)\le\tilde{\varphi}(x) \quad (3)$$

$a(u,v)=\int_\Omega \nabla u\cdot\nabla v \mathrm{d}x$; and (.,.) denotes the inner product on Ω.

It has been proved that the long run average cost for this problem solves the ergodic QVI. More precisely, denoting by:

$$\langle w\rangle = \frac{1}{\Omega}\int_\Omega w\mathrm{d}x, w_\alpha = u_\alpha - \langle u_\alpha\rangle \quad \text{and} \quad \lambda_\alpha = \alpha\langle u_\alpha\rangle.$$

P. L. Lions and B. Perthame [3] proved that the solution $(w_\alpha, \lambda_\alpha)$ of the QVI :

$$\begin{cases} a(w_\alpha, v-w_\alpha)+(w_\alpha, v-w_\alpha)\ge (f-\lambda_\alpha, v-w_\alpha) \\ w_\alpha \in H^1(\Omega), w_\alpha \le Mw_\alpha; v\in H^1(\Omega), v\le Mw_\alpha, \langle w_\alpha\rangle = 0 \end{cases} \quad (4)$$

converges to the solution of the ergodic control QVI :

$$\begin{cases} a(w_0, v-w_0)+(w_0, v-w_0)\ge (f-\lambda_0, v-w_0) \\ w_0 \in H^1(\Omega), w_0 \le Mw_0; v\in H^1(\Omega), v\le Mw_0, \langle w_0\rangle = 0 \end{cases} \quad (5)$$

As stated in the following theorem.

Theorem 1 [1]: As α goes to 0^+, λ_α converges uniformly in $C(\bar{\Omega})$ to some constant λ_0, and w_α converges uniformly in $C(\bar{\Omega})$ and strongly in $H^1(\Omega)$ to w_0. Moreover (λ_0, w_0) is the unique solution of the quasi variational inequality of the ergodic control problem (5).

Problem Position

Let α be fixed in the open interval $]0,1[$ and set. Then, one can easily see that problem (1) is equivalent to the following QVI:

$$\begin{cases} b(u_\alpha, v-u_\alpha)\ge (f+\gamma u_\alpha, v-u_\alpha) \\ u_\alpha \in H^1(\Omega), u_\alpha \le Mu_\alpha; \\ v\in H^1(\Omega), v\le Mu_\alpha \end{cases} \quad (6)$$

where

$$b(u,v) = a(u,v)+(u,v), \quad \text{and} \quad \gamma = 1-\alpha \quad (7)$$

Thanks to [5], (1) or (6) has a unique solution. Also, notice that, as the bilinear form (7) is independent of α, the left hand side of (6) is independent of α too.

3. The Schwarz Method for the Obstacle

We decompose Ω into two overlapping polygonal sub domains Ω_1 and Ω_2, such that

$$\Omega = \Omega_1 \cup \Omega_2 \quad (8)$$

and u_α satisfies the local regularity condition

$$u_\alpha/\Omega_i \in W^{2,p}(\Omega_i), 2\le p<\infty \quad (9)$$

We denote by $\partial\Omega_i$ the boundary condition of Ω_i and $\Gamma_i = \partial\Omega_i \cap \Omega_i$ the intersection of $\bar{\Gamma}_i$ and $\bar{\Gamma}_j$ is assumed to be empty.

3.1. The Schwarz Sequences for Problem (4)

We denote by:

$$w_\alpha^{n+1} = u_\alpha^{n+1} - \langle u_\alpha^{n+1}\rangle \quad \text{and} \quad \lambda_\alpha^{n+1} = \alpha^{n+1}\langle u_\alpha^{n+1}\rangle, n\ge 0 \quad (10)$$

Choosing $w_\alpha^0 = w_0$ such that w_0 the unique solution of:

$$\begin{cases} a(w_0, v-w_0)+(w_0, v-w_0)\ge (f-\lambda_0, v-w_0) \\ w_0 \in H^1(\Omega), w_0 \le Mw_0; \\ v\in H^1(\Omega), v\le Mw_0, \langle w_0\rangle = 0 \end{cases} \quad (11)$$

We respectively define $w_{\alpha i}^{n+1}$ the alternating Schwarz sequences on Ω_i such that:

$$\begin{cases} a(w_{\alpha i}^{n+1}, v-w_{\alpha i}^{n+1})+(w_{\alpha i}^{n+1}, v-w_{\alpha i}^{n+1})\ge (f_i-\lambda_{\alpha i}^{n+1}, v-w_{\alpha i}^{n+1}) \\ w_{\alpha i}^{n+1} = w_{\alpha i}^n \text{ on } \Gamma_i, w_{\alpha i}^{n+1}\le Mw_{\alpha i}^n; \\ v\in H^1(\Omega), v\le Mw_{\alpha i}^n, \langle w_{\alpha i}^{n+1}\rangle = 0 \end{cases} \quad (12)$$

where $i=1,2$ and $f_i = f/\Omega_i$.

3.2. The Continuous Schwarz Sequences for Principal Problem

We consider the following problem:

$$\begin{cases} b(u_\alpha, v-u_\alpha)\ge (f+\gamma u_\alpha, v-u_\alpha) \\ u_\alpha \in H^1(\Omega), u_\alpha \le Mu_\alpha; v\in H^1(\Omega), v\le Mu_\alpha \end{cases} \quad (13)$$

Choosing $u_\alpha^0 = u_0$, solution of:

$$\begin{cases} a(u_0, v-u_0)\ge (f, v-u_0) \\ u_0 \in H^1(\Omega), u_0 \le Mu_0; \\ v\in H^1(\Omega), v\le Mu_0 \end{cases} \quad (14)$$

We respectively define the alternating Schwarz sequences $(u_{\alpha 1}^{n+1})$ on Ω_1 such that: $\Omega = \Omega_1 \cup \Omega_2$ and;

$$\begin{cases} b\left(u_{\alpha 1}^{n+1}, v-u_{\alpha 1}^{n+1}\right) \geq \left(f+\gamma u_{\alpha 1}^{n+1}, v-u_{\alpha 1}^{n+1}\right), \forall v \in H^1(\Omega) \\ u_{\alpha 1}^{n+1} \leq Mu_{\alpha 1}^{n}, Mu_{\alpha 1}^{n} > 0 \\ u_{\alpha 1}^{n+1} = u_{\alpha 2}^{n} \text{ on } \Gamma_1, v = u_{\alpha 2}^{n} \text{ on } \Gamma_1 \end{cases} \quad (15)$$

$$\begin{cases} b\left(u_{\alpha 2}^{n+1}, v-u_{\alpha 2}^{n+1}\right) \geq \left(f+\gamma u_{\alpha 2}^{n+1}, v-u_{\alpha 2}^{n+1}\right), \forall v \in H^1(\Omega) \\ u_{\alpha 2}^{n+1} \leq Mu_{\alpha 2}^{n}, Mu_{\alpha 2}^{n} > 0 \\ u_{\alpha 2}^{n+1} = u_{\alpha 1}^{n+1} \text{ on } \Gamma_2, v = u_{\alpha 1}^{n+1} \text{ on } \Gamma_2 \end{cases} \quad (16)$$

$$\begin{aligned} & b_i(u_\alpha, v) \\ & = a(u_\alpha, v) + \gamma \int_{\Omega_i} u_\alpha \cdot v \mathrm{d}x, i = 1,2 \end{aligned} \quad (17)$$

Lemma 1 [3]: for each $n \geq 0$;

$$Mu_{\alpha 1}^{n+1} \left(\text{respec } Mu_{\alpha 2}^{n+1}\right) \in C\left(\bar{\Omega}\right)$$

4. The Discret Problem

We suppose for simplicity that Ω is polyhedral. Let r_h be a regular, quasi uniform triangulation of Ω into n-simplexes of diameter less than h.

We denote by V_h the standard piecewise linear finite element espace, we consider the discrete variational inequality:

$$\begin{cases} a\left(u_{\alpha h}, v_h - u_{\alpha h}\right) + \left(u_{\alpha h}, v_h - u_{\alpha h}\right) \geq \left(f, v_h - u_{\alpha h}\right) \\ u_{\alpha h} \in V_h, u_{\alpha h} \leq r_h M u_{\alpha h}; v_h \in V_h, v_h \leq r_h M u_{\alpha h} \end{cases} \quad (18)$$

Thanks to ([6]), QVI (18) has a unique solution.

4.1. The Discrete Maximum Principle [7]

We assume that the matrix A with generic coefficient is a M-matrix.

$$B_{ij}\left(\varphi_i, \varphi_j\right), 1 \leq i, j \leq m(h) \quad (19)$$

As for the continuous problem, it is easy to see that $u_{\alpha h}$, the solution of (18), is also solution to the following (QVI):

$$\begin{cases} b\left(u_{\alpha h}, v_h - u_{\alpha h}\right) \geq \left(f + \gamma u_{\alpha h}, v_h - u_{\alpha h}\right) \\ u_{\alpha h} \in V_h, u_{\alpha h} \leq r_h M u_{\alpha h}; v_h \in V_h, v_h \leq r_h M u_{\alpha h} \end{cases} \quad (20)$$

Theorem 2 [1]: Let u_α and $u_{\alpha h}$ (the discrete solution). Then, there exists a constant independent of α the both and *h* such that

$$\left\|u_\alpha - u_{\alpha h}\right\|_\infty \leq C \cdot \alpha^{-2} h^2 \left|\log h\right|^3$$

Theorem 3 [1]: Under conditions of theorem 1 and 2, we have

$$\lim_{h \to 0} \left\|w_{\alpha h} - w_0\right\|_\infty = 0$$
$$\lim_{h \to 0} \left|\lambda_{\alpha h} - \lambda_0\right| = 0$$

Note that $w_{\alpha h}$ and $\lambda_{\alpha h}$ is the finite element approximation of w_α and λ_α respectively.

4.2. The Discrete Schwarz Sequences

Let $V_{hi} = V_h(\Omega_i)$ be the space of continuous piecewise linear function on τ_{hi} which vanish on $\partial\Omega \cap \partial\Omega_i$.

For $w \in C(\Gamma_i)$, we define

$$V_{hi}^{(w)} = \left\{v \in V_h / v = 0 \text{ on } \partial\Omega \cap \partial\Omega_i, v = \pi_{hi}(w) \text{ on } \Gamma_i\right\}$$

where π_{hi} denotes the interpolation operator on Γ_i.

For $i = 1,2$. Let τ_{hi} be a standard regular finite element triangulation in Ω_i, h_i being the mesh size.

We suppose that the two triangulations are mutually independent on $\Omega_1 \cup \Omega_2$ a triangle belonging to one triangulation does not necessarily belong to the other.

Choosing $u_{\alpha h}^0 = u_{0h}$, such that u_{0h} is a solution of the following inequation:

$$\begin{cases} a\left(u_{0h}, v_h - u_{0h}\right) \geq \left(f, v_h - u_{0h}\right) \\ u_{0h} \in V_h, u_{0h} \leq r_h M u_{0h}; v_h \in V_h, v_h \leq r_h M u_{0h} \end{cases} \quad (21)$$

We define the alternating Schwarz sequences $\left(u_{\alpha 1h}^{n+1}\right)$ on Ω_1 such that:

$$\begin{cases} b_1\left(u_{\alpha 1h}^{n+1}, v_h - u_{\alpha 1h}^{n+1}\right) \geq \left(f_1 + \gamma u_{\alpha 1h}^{n+1}, v_h - u_{\alpha 1h}^{n+1}\right), v_h \in V_h \\ u_{\alpha 1h}^{n+1} \leq r_h M u_{\alpha 1h}^{n}, M u_{\alpha 1h}^{n} > 0 \\ u_{\alpha 1h}^{n+1} = u_{\alpha 2h}^{n} \text{ on } \Gamma_1, v_h = u_{\alpha 2h}^{n} \text{ on } \Gamma_1 \end{cases} \quad (22)$$

And $\left(u_{\alpha 2h}^{n+1}\right)$ on Ω_2 such that:

$$\begin{cases} b_2\left(u_{\alpha 2h}^{n+1}, v_h - u_{\alpha 2h}^{n+1}\right) \geq \left(f_2 + \gamma u_{\alpha 2h}^{n+1}, v_h - u_{\alpha 2h}^{n+1}\right), \\ v_h \in V_h, u_{\alpha 2h}^{n+1} \leq r_h M u_{\alpha 2h}^{n}, M u_{\alpha 2h}^{n} > 0 \\ u_{\alpha 1h}^{n+1} = u_{\alpha 1h}^{n+1} \text{ on } \Gamma_2, v_h = u_{\alpha 1h}^{n+1} \text{ on } \Gamma_2 \end{cases} \quad (23)$$

Notation: We will adapt the following notations:

$$\begin{aligned} |\cdot|_i &= \|\cdot\|_{L^\infty(\Gamma_i)}, i = 1,2 \\ \|\cdot\|_i &= \|\cdot\|_{L^\infty(\Omega_i)}, i = 1,2 \end{aligned} \quad (24)$$

5. L^∞-Error Analysis

Lemma 2 [8]: $B = \left(b_{ij}\right)_{i,j=\{1,\cdots,N\}}$ is M-matrix such that

$$\left(b_{ij}\right) = b_{ij}\left(\varphi_i, \varphi_j\right)$$

then exists constants k_1, k_2 such that

$$k_1 = \sup\left\{w_h(x), x \in \eta_1\right\} \in (0,1)$$

and $k_2 = \sup\left\{w_h(x), x \in \eta_2\right\} \in (0,1)$

$$\sup_{\eta_1}\left|u_{\alpha 1h}-u_{\alpha 1h}^{n+1}\right|_1 \le k_1 \sup_{\eta_1}\left|u_{\alpha 1h}-u_{01h}^{0}\right|_1$$
$$\sup_{\eta_2}\left|u_{\alpha 2h}-u_{\alpha 2h}^{n+1}\right|_2 \le k_2 \sup_{\eta_2}\left|u_{\alpha 2h}-u_{02h}^{0}\right|_2 \quad (25)$$

Theorem 4 [8]: $\left(u_{\alpha 1h}^{n+1}\right),\left(u_{\alpha 2h}^{n+1}\right); n\ge 0$ produced by Schwarz alternating method converges geometrically to u_α the solution of obstacle problem, more precisely, there exist $k_1,k_2\in(0,1)$ which depend only respectively of (Ω_1,η_1) and (Ω_2,η_2), such that

$$\sup_{\bar\Omega_1}\left|u_{\alpha 1h}-u_{\alpha 1h}^{n+1}\right|_1 \le k_1^{n+1}k_2^{n+1} \sup_{\eta_1}\left|u_{\alpha 1h}-u_{01h}^{0}\right|_1$$
$$\sup_{\bar\Omega_2}\left|u_{\alpha 2h}-u_{\alpha 2h}^{n+1}\right|_2 \le k_1^{n+1}k_2^{n} \sup_{\eta_2}\left|u_{\alpha 2h}-u_{02h}^{0}\right|_2 \quad (26)$$

Theorem 5: Let $h=\max(h_1,h_2)$. Then, there exist two constants C and $k, 0<k<1$, independent of both h and n such that:

$$\left\|u_{\alpha 1}-u_{\alpha 1h}^{n+1}\right\|_\infty \le C\cdot\alpha^{-2}h^2\left|\log h\right|^4$$

and

$$\left\|u_{\alpha 2}-u_{\alpha 2h}^{n+1}\right\|_\infty \le C\cdot\alpha^{-2}h^2\left|\log h\right|^4.$$

Proof:

$$\left\|u_{\alpha 1}-u_{\alpha 1h}^{n+1}\right\|_\infty$$
$$\le \left\|u_{\alpha 1}-u_{\alpha 1h}\right\|_\infty + \left\|u_{\alpha 1h}-u_{\alpha 1h}^{n+1}\right\|_\infty$$
$$\le C\cdot\alpha^{-2}h^2\left|\log h\right|^3 + k_1^{n+1}\left\|u_{\alpha 1h}-u_{01h}^{0}\right\|_1$$

$$\left\|u_{\alpha 1}-u_{\alpha 1h}^{n+1}\right\|_\infty$$
$$\le C_1\cdot\alpha^{-2}h^2\left|\log h\right|^3 + k_1^{n+1}\left[\left\|u_{\alpha 1}-u_{\alpha 1h}\right\|_1+\left\|u_{\alpha 1}-u_{01h}^{0}\right\|_1\right]$$
$$\le C_1\cdot\alpha^{-2}h^2\left|\log h\right|^3 + k_1^{n+1}\left\|u_{\alpha 1}-u_{\alpha 1h}\right\|_1 + k_1^{n+1}\left\|u_{\alpha 1}-u_{01h}^{0}\right\|_1$$
$$\le C_1\cdot\alpha^{-2}h^2\left|\log h\right|^3 + \left(k_1^{n+1}\right)C_2\cdot\alpha^{-2}h^2\left|\log h\right|^3.$$

We obtain

$$\left\|u_{\alpha 1}-u_{\alpha 1h}^{n+1}\right\|_\infty \le C\cdot\alpha^{-2}h^2\left|\log h\right|^4.$$

The case $i=2$ is similar.

Theorem 6: Let $\left(w_{\alpha ih}^{n+1}\right)$ and $\left(\lambda_{\alpha ih}^{n+1}\right)$, the discretes alternating Schwarz sequences, we have:

$$\lim_{h\to 0}\left\|w_{\alpha ih}^{n+1}-w_{0i}\right\|_\infty = 0, i=1,2$$
$$\lim_{h\to 0}\left|\lambda_{\alpha ih}^{n+1}-\lambda_{0i}\right| = 0, i=1,2.$$

Proof:

1)

$$\left\|w_{\alpha 1h}^{n+1}-w_{01}\right\|_1 \le \left\|w_{\alpha 1h}^{n+1}-w_{\alpha 1h}\right\|_1 + \left\|w_{\alpha 1h}-w_{01}\right\|_1$$
$$\le \left\|w_{\alpha 1h}^{n+1}-w_{\alpha 1h}\right\|_1 + \left\|w_{\alpha 1h}-w_{01h}\right\|_1 + \left\|w_{01h}-w_{01}\right\|_1$$

$$\lim \alpha\to 0, \left\|w_{\alpha 1h}-w_{01h}\right\|_1 \to 0$$

$$\left\|w_{\alpha 1h}^{n+1}-w_{01}\right\|_1 \le \left\|w_{\alpha 1h}^{n+1}-w_{\alpha 1h}\right\|_1 + \left\|w_{01h}-w_{01}\right\|_1$$
$$\le \left\|u_{\alpha 1h}^{n+1}-\left\langle u_{\alpha 1h}^{n+1}\right\rangle - u_{\alpha 1h} + \left\langle u_{\alpha 1h}\right\rangle\right\|_1$$
$$+\left\|u_{01h}-\left\langle u_{01h}\right\rangle - u_{01} + \left\langle u_{01}\right\rangle\right\|_1$$
$$\le \left\|u_{\alpha 1h}^{n+1}-u_{\alpha 1h}\right\|_1 + meas(\Omega)^{-1}\left\|u_{\alpha 1h}^{n+1}-u_{\alpha 1h}\right\|_1$$
$$+\left\|u_{01h}-u_{01}\right\|_1 + meas(\Omega)^{-1}\left\|u_{01h}-u_{01}\right\|_1$$
$$\le 4C\cdot\alpha^{-2}h^2\left|\log h\right|^4$$

For: $\alpha=\sqrt{h}$ and $h\to 0$.

The case i = 2 is similar.

$\lim_{h\to 0}\left\|w_{\alpha ih}^{n+1}-w_{0i}\right\|_i = 0, i=1,2$.

2)

$$\left|\lambda_{\alpha 1h}^{n+1}-\lambda_{01}\right|$$
$$\le \left|\lambda_{\alpha 1h}^{n+1}-\lambda_{\alpha 1}\right| + \left|\lambda_{\alpha 1}-\lambda_{01}\right|$$
$$\le \left|\alpha^{n+1}\left\langle u_{\alpha 1h}^{n+1}\right\rangle - \alpha\left\langle u_{\alpha 1}\right\rangle\right| + \left|\lambda_{1\alpha}-\lambda_{01}\right|.$$

Hence $\left|\alpha^{n+1}\right|\le|\alpha|$: for $\alpha\in]0,1[$

$$\left|\lambda_{\alpha 1h}^{n+1}-\lambda_{01}\right|$$
$$\le \alpha\left|u_{\alpha 1h}^{n+1}-u_{\alpha 1}\right| + \left|\lambda_{1\alpha}-\lambda_{01}\right|$$
$$\le \alpha\cdot C\cdot\alpha^{-2}h^2\left|\log h\right|^4 + \left|\lambda_{1\alpha}-\lambda_{01}\right|$$
$$\le C\cdot\alpha^{-1}h^2\left|\log h\right|^4 + \left|\lambda_{1\alpha}-\lambda_{01}\right|$$

$\lim\alpha\to 0$, $\left|\lambda_{1\alpha}-\lambda_{01}\right|\to 0$, For: $\alpha=\sqrt{h}$ and $h\to 0$.

The case $i=2$ is similar.

$$\lim_{h\to 0}\left|\lambda_{\alpha ih}^{n+1}-\lambda_{0i}\right| = 0, i=1,2.$$

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