American Journal of Computational Mathematics, **2013, 3, 73-79**
http://dx.doi.org/10.4236/ajcm.2013.31012 Published Online March 2013 (http://www.scirp.org/journal/ajcm)

An Algebra of Fuzzy (*m*, *n*)-Semihyperrings

S. E. Alam, Sultan Aljahdali, Nisar Hundewale
College of Computers and Information Technology, Taif University, Taif, KSA
Email: eqbal_mit2k2@hotmail.com

Received November 3, 2012; revised December 23, 2012; accepted December 31, 2012

ABSTRACT

We propose a new class of algebraic structure named as (*m*, *n*)-semihyperring which is a generalization of usual *semihyperring*. We define the basic properties of (*m*, *n*)-semihyperring like identity elements, weak distributive (*m*, *n*)-semihyperring, zero sum free, additively idempotent, hyperideals, homomorphism, inclusion homomorphism, congruence relation, quotient (*m*, *n*)-semihyperring etc. We propose some lemmas and theorems on homomorphism, congruence relation, quotient (*m*, *n*)-semihyperring, etc. and prove these theorems. We further extend it to introduce the relationship between fuzzy sets and (*m*, *n*)-semihyperrings and propose fuzzy hyperideals and homomorphism theorems on fuzzy (*m*, *n*)-semihyperrings and the relationship between fuzzy (*m*, *n*)-semihyperrings and the usual (*m*, *n*)-semihyperrings.

Keywords: (*m*, *n*)-Semihyperring; Hyperoperation; Hyperideal; Homomorphism; Congruence Relation; Fuzzy (*m*, *n*)-Semihyperring

1. Introduction

A semihyperring is essentially a semiring in which addition is a hyperoperation [1]. Semihyperring is in active research for a long time. Vougiouklis [2] generalize the concept of hyperring $(\mathcal{R},\oplus,\square)$ by dropping the reproduction axiom where $\oplus$ and $\square$ are associative hyper operations and $\square$ distributes over $\oplus$ and named it as semihyperring. Chaopraknoi, Hobuntud and Pianskool [3] studied semihyperring with zero. Davvaz and Poursalavati [4] introduced the matrix representation of polygroups over hyperring and also over semihyperring. Semihyperring and its ideals are studied by Ameri and Hedayati [5].

Zadeh [6] introduced the notion of a fuzzy set that is used to formulate some of the basic concepts of algebra. It is extended to fuzzy hyperstructures, nowadays fuzzy hyperstructure is a fascinating research area. Davvaz introduced the notion of fuzzy subhypergroups in [7], Ameri and Nozari [8] introduced fuzzy regular relations and fuzzy strongly regular relations of fuzzy hyperalgebras and also established a connection between fuzzy hyperalgebras and algebras. Fuzzy subhypergroup is also studied by Cristea [9]. Fuzzy hyperideals of semihyperrings are studied by [1,10,11].

The generalization of Krasner hyperring is introduced by Mirvakili and Davvaz [12] that is named as Krasner (*m*, *n*) hyperring. In [13] Davvaz studied the fuzzy hyperideals of the Krasner (*m*, *n*)-hyperring. Generalization of hyperstructures are also studied by [1,14-16].

In this paper, we introduce the notion of the generalization of usual semihyperring and called it as (*m*, *n*)-semihyperring and set fourth some of its properties, we also introduce fuzzy (*m*, *n*)-semihyperring and its basic properties and the relation between fuzzy (*m*, *n*)-semihyperring and its associated (*m*, *n*)-semihyperring.

The paper is arranged in the following fashion:

Section 2 describes the notations used and the general conventions followed. Section 3 deals with the definitions of (*m*, *n*)-semihyperring, weak distributive (*m*, *n*)-semihyperring, hyperadditive and multiplicative identity elements, zero, zero sum free, additively idempotent and some examples of (*m*, *n*)-semihyperrings.

Section 4 describes the properties of (*m*, *n*)-semihyperring. This section deals with the definitions of hyperideals, homomorphism, congruence relation, quotient of (*m*, *n*)-semihyperring and also the theorems based on these definitions.

Section 5 deals with the fuzzy (*m*, *n*)-semihyperrings, fuzzy hyperideals and homomorphism theorems on (*m*, *n*)-semihyperrings and fuzzy (*m*, *n*)-semihyperrings.

2. Preliminaries

Let $\mathcal{H}$ be a non-empty set and $\mathcal{P}^*(\mathcal{H})$ be the set of all non-empty subsets of $\mathcal{H}$. A hyperoperation on $\mathcal{H}$ is a map $\sigma:\mathcal{H}\times\mathcal{H}\to\mathcal{P}^*(\mathcal{H})$ and the couple $(\mathcal{H},\sigma)$ is called a hypergroupoid. If *A* and *B* are non-empty subsets of $\mathcal{H}$, then we denote $A\sigma B=\bigcup_{a\in A,b\in B} a\sigma b$,

$x\sigma A = \{x\}\sigma A$ and $A\sigma x = A\sigma\{x\}$.

Let $\mathcal{H}$ be a non-empty set, $\mathcal{P}^*$ be the set of all non-empty subsets of $\mathcal{H}$ and a mapping $f:\mathcal{H}^m \to \mathcal{P}^*(\mathcal{H})$ is called an *m-ary hyperoperation* and *m* is called the *arity of hyperoperation* [14].

A hypergroupoid $(\mathcal{H},\sigma)$ is called a *semihypergroup* if for all $x,y,z\in\mathcal{H}$ we have $(x\sigma y)\sigma z = x\sigma(y\sigma z)$ which means that

$$\bigcup_{u\in x\sigma y} u\sigma z = \bigcup_{v\in y\sigma z} x\sigma v.$$

Let *f* be an *m*-ary hyperoperation on $\mathcal{H}$ and $A_1, A_2,\cdots,A_m$ subsets of $\mathcal{H}$. We define

$$f(A_1,A_2,\cdots,A_m) = \bigcup_{x_i\in A_i} f(x_1,x_2,\cdots,x_m)$$

for all $1\le i\le m$.

Definition 2.1 $(\mathcal{H},\oplus,\otimes)$ is a semihyperring which satisfies the following axioms:

1) $(\mathcal{H},\oplus)$ is a semihypergroup;
2) $(\mathcal{H},\otimes)$ is a semigroup and;
3) $\otimes$ distributes over $\oplus$,

$x\otimes(y\oplus z) = (x\otimes y)\oplus(x\otimes z)$ and $(y\oplus z)\otimes x = (y\otimes x)\oplus(z\otimes x)$ for all $x,y,z\in\mathcal{H}$ [3].

Example 2.2 *Let* $(\mathcal{H},+,\times)$ *be a semiring, we define*

1) $x\oplus y = \langle x,y\rangle$
2) $x\otimes y = x\times y$

Then $(\mathcal{H},\oplus,\otimes)$ is a semihyperring.

An element 0 of a semihyperring $(\mathcal{H},\oplus,\otimes)$ is called a *zero* of $(\mathcal{H},\oplus,\otimes)$ if $x\oplus 0 = 0\oplus x = \{x\}$ and $x\otimes 0 = 0\otimes x = 0$ [3].

The set of integers is denoted by $\mathbb{Z}$, with $\mathbb{Z}_+$ and $\mathbb{Z}_-$ denoting the sets of positive integers and negative integers respectively. Elements of the set $\mathcal{H}$ are denoted by x_i, y_i where $i\in\mathbb{Z}_+$.

We use following general convention as followed by [10,17-19]:

The sequence $x_i, x_{i+1},\cdots,x_m$ is denoted by x_i^m.

The following term:

$$f(x_1,\cdots,x_i,y_{i+1},\cdots,y_j,z_{j+1},\cdots,z_m) \quad (1)$$

is represented as:

$$f(x_1^i, y_{i+1}^j, z_{j+1}^m) \quad (2)$$

In the case when $y_{i+1} = \cdots = y_j = y$, then (2) is expressed as:

$$f\left(x_1^i, \overset{(j-i)}{y}, z_{j+1}^m\right)$$

Definition 2.3 A non-empty set $\mathcal{H}$ with an *m*-ary hyperoperation $f:\mathcal{H}^m\to\mathcal{P}^*(\mathcal{H})$ is called an *m-ary hypergroupoid* and is denoted as $(\mathcal{H},f)$. An m-ary hypergroupoid $(\mathcal{H},f)$ is called an *m-ary semihypergroup* if and only if the following associative axiom holds:

$$f\left(x_1^i, f\left(x_i^{m+i-1}\right), x_{m+i}^{2m-1}\right) = f\left(x_1^j, f\left(x_j^{m+j-1}\right), x_{m+j}^{2m-1}\right)$$

for all $i,j\in\{1,2,\cdots,m\}$ and $x_1,x_2,\cdots,x_{2m-1}\in\mathcal{H}$ [14].

Definition 2.4 Element e is called *identity element* of hypergroup $(\mathcal{H},f)$ if

$$x\in f\Big(\underbrace{e,\cdots,e}_{i-1}, x, \underbrace{e,\cdots,e}_{n-i}\Big)$$

for all $x\in\mathcal{H}$ and $1\le i\le n$ [14].

Definition 2.5 A non-empty set $\mathcal{H}$ with an *n*-ary operation *g* is called an *n-ary groupoid* and is denoted by $(\mathcal{H},g)$ [19].

Definition 2.6 An n-ary groupoid $(\mathcal{H},g)$ is called an *n-ary semigroup* if *g* is associative, *i.e.*,

$$g\left(x_1^i, g\left(x_i^{n+i-1}\right), x_{n+i}^{2n-1}\right) = g\left(x_1^j, g\left(x_j^{n+j-1}\right), x_{n+j}^{2n-1}\right)$$

for all $i,j\in\{1,2,\cdots,n\}$ and $x_1,x_2,\cdots,x_{2n-1}\in\mathcal{H}$ [19].

3. Definitions and Examples of (*m*, *n*)-Semihyperring

Definition 3.1 $(\mathcal{H},f,g)$ is an (*m*, *n*)-semihyperring which satisfies the following axioms:

1) $(\mathcal{H},f)$ is a *m*-ary semihypergroup;
2) $(\mathcal{H},g)$ is an *n*-ary semigroup;
3) g is distributive over *f i.e.*,

$$g\left(x_1^{i-1}, f\left(a_1^m\right), x_{i+1}^n\right) = f\left(g\left(x_1^{i-1},a_1,x_{i+1}^n\right),\cdots,g\left(x_1^{i-1},a_m,x_{i+1}^n\right)\right).$$

Remark 3.2 An (*m*, *n*)-semihyperring is called *weak distributive* if it satisfies Definition 3.1 1), 2) and the following:

$$g\left(x_1^{i-1}, f\left(a_1^m\right), x_{i+1}^n\right) \subseteq f\left(g\left(x_1^{i-1},a_1,x_{i+1}^n\right),\cdots,g\left(x_1^{i-1},a_m,x_{i+1}^n\right)\right).$$

Remark 3.2 is generalization of [20].

Example 3.3 Let $\mathbb{Z}$ be the set of all integers. Let the binary hyperoperation $\oplus$ and an *n*-ary operation *g* on $\mathbb{Z}$ which are defined as follows:

$$x_1\oplus x_2 = \{x_1,x_2\}$$

and

$$g(x_1,x_2,\cdots,x_n) = \prod_{i=1}^n x_i.$$

Then $(\mathbb{Z},\oplus,g)$ is called a $(2,n)$*-semihyperring*.

Example 3.3 is generalization of Example 1 of [1].

Definition 3.4 Let *e* be the *hyper additive identity element* of hyperoperation *f* and e' be *multiplicative identity element* of operation *g* then

$$x\in f\Big(\underbrace{e,\cdots,e}_{i-1}, x, \underbrace{e,\cdots,e}_{m-i}\Big)$$

for all $x \in \mathcal{H}$ and $1 \le i \le m$ and

$$y = g\left(\underbrace{e', \cdots, e'}_{j-1}, y, \underbrace{e', \cdots, e'}_{n-j}\right)$$

for all $y \in \mathcal{H}$ and $1 \le j \le n$.

Definition 3.5 An element **0** of an (*m*, *n*)-semihyperring $(\mathcal{H}, f, g)$ is called a *zero* of $(\mathcal{H}, f, g)$ if

$$f\left(\underbrace{\mathbf{0}, \cdots, \mathbf{0}}_{m-1}, x\right) = f\left(x, \underbrace{\mathbf{0}, \cdots, \mathbf{0}}_{m-1}\right) = x$$

for all $x \in \mathcal{H}$.

$$g\left(\underbrace{\mathbf{0}, \cdots, \mathbf{0}}_{n-1}, y\right) = g\left(y, \underbrace{\mathbf{0}, \cdots, \mathbf{0}}_{n-1}\right) = \mathbf{0}$$

for all $y \in \mathcal{H}$.

Remark 3.6 Let $(\mathcal{H}, f, g)$ be an (*m*, *n*)-semihyperring and *e* and e' be hyper additive identity and multiplicative identity elements respectively, then we can obtain the additive hyper operation and multiplication as follows:

$$\langle x, y \rangle = f\left(x, \underbrace{e, \cdots, e}_{m-2}, y\right)$$

and $x \times y = g\left(x, \underbrace{e', \cdots, e'}_{n-2}, y\right)$ for all $x, y \in \mathcal{H}$.

Definition 3.7 Let $(\mathcal{H}, f, g)$ be an (*m*, *n*)-semihyperring.

1) (*m*, *n*)-semihyperring $(\mathcal{H}, f, g)$ is called *zero sum free* if and only if $\mathbf{0} \in f(x_1, x_2, \cdots, x_m)$ implies $x_1 = x_2 = \cdots = x_m = \mathbf{0}$.

2) (*m*, *n*)-semihyperring $(\mathcal{H}, f, g)$ is called *additively idempotent* if $(\mathcal{H}, f)$ be a *m*-ary semihypergroup, *i.e.* if $f(x, x, \cdots, x) \in x$.

4. Properties of (*m*, *n*)-Semihyperring

Definition 4.1 Let $(\mathcal{H}, f, g)$ be an (*m*, *n*)-semihyperring.

1) An *m*-ary sub-semihypergroup $\mathcal{R}$ of $\mathcal{H}$ is called an (*m*, *n*)-*sub-semihyperring* of $\mathcal{H}$ if $g(a_1^n) \in \mathcal{R}$, for all $a_1, a_2, \cdots, a_n \in \mathcal{R}$.

2) An *m*-ary sub-semihypergroup $\mathcal{I}$ of $\mathcal{H}$ is called

a) a left hyperideal of $\mathcal{H}$ if $g(a_1^{n-1}, i) \in \mathcal{I}$, $\forall a_1, a_2, \cdots, a_{n-1} \in \mathcal{H}$ and $i \in \mathcal{I}$.

b) a right hyperideal of $\mathcal{H}$ if $g(i, a_1^{n-1}) \in \mathcal{I}$, $\forall a_1, a_2, \cdots, a_{n-1} \in \mathcal{H}$ and $i \in \mathcal{I}$.

If $\mathcal{I}$ is both left and right hyperideal then it is called as an hyperideal of $\mathcal{H}$.

c) a left hyperideal $\mathcal{I}$ of an (*m*, *n*)-semihyperring of $\mathcal{H}$ is called *weak left hyperideal* of $\mathcal{H}$ if for $i \in \mathcal{I}$ and $x_1, x_2, \cdots, x_{m-1} \in \mathcal{H}$ then $f(i, x_1^{m-1}) \subseteq \mathcal{I}$ or $f(x_1^{m-1}, i) \subseteq \mathcal{I}$ implies $x_1, x_2, \cdots, x_{m-1} \in \mathcal{I}$.

Definition 4.1 is generalization of [21].

Proposition 4.2 A left hyperideal of an (*m*, *n*)-semihyperring is an (*m*, *n*)-sub-semihyperring.

Definition 4.3 Let $(\mathcal{H}, f, g)$ and $(\mathcal{S}, f', g')$ be two (*m*, *n*)-semihyperrings. The mapping $\sigma: \mathcal{H} \to \mathcal{S}$ is called a *homomorphism* if following condition is satisfied for all $x_1, x_2, \cdots, x_m$, $y_1, y_2, \cdots, y_n \in \mathcal{H}$.

$$\sigma(f(x_1, x_2, \cdots, x_m)) = f'(\sigma(x_1), \sigma(x_2), \cdots, \sigma(x_m))$$

and

$$\sigma(g(y_1, y_2, \cdots, y_n)) = g'(\sigma(y_1), \sigma(y_2), \cdots, \sigma(y_n)).$$

Remark 4.4 Let $(\mathcal{H}, f, g)$ and $(\mathcal{S}, f', g')$ be two (*m*, *n*)-semihyperrings. The mapping $\sigma: \mathcal{H} \to \mathcal{S}$ for all $x_1, x_2, \cdots, x_m$, $y_1, y_2, \cdots, y_n \in \mathcal{H}$ is called an *inclusion homomorphism* if following relations hold:

$$\sigma(f(x_1, x_2, \cdots, x_m)) \subseteq f'(\sigma(x_1), \sigma(x_2), \cdots, \sigma(x_m))$$

and

$$\sigma(g(y_1, y_2, \cdots, y_n)) \subseteq g'(\sigma(y_1), \sigma(y_2), \cdots, \sigma(y_n))$$

Remark 4.4 is generalization of [7].

Theorem 4.5 Let $(\mathcal{R}, f, g)$, $(\mathcal{S}, f', g')$ and $(\mathcal{T}, f'', g'')$ be (*m*, *n*)-semihyperrings. If mappings $\sigma: (\mathcal{R}, f, g) \to (\mathcal{S}, f', g')$ and $\delta: (S, f', g') \to (T, f'', g'')$ are homomorphisms, then $\sigma \circ \delta: (R, f, g) \to (T, f'', g'')$ is also a homomorphism.

Proof. Omitted as obvious.

Definition 4.6 Let $\cong$ be an equivalence relation on the (*m*, *n*)-semihyperring $(\mathcal{H}, f, g)$ and A_i and B_i be the subsets of $\mathcal{H}$ for all $1 \le i \le m$. We define $A_i \cong B_i$ for all $a_i \in A_i$ there exists $b_i' \in B_i$ such that $a_i \cong b_i'$ holds true and for all $b_i \in B_i$ there exists $a_i' \in A_i$ such that $a_i' \cong b_i$ holds true [22].

An equivalence relation $\cong$ is called a *congruence relation* on $\mathcal{H}$ if following hold:

1) for all $a_1, a_2, \cdots, a_m$, $b_1, b_2, \cdots, b_m \in \mathcal{H}$; if $\{a_i\} \cong \{b_i\}$ then $\{f(a_1^m)\} \cong \{f(b_1^m)\}$, where $1 \le i \le m$ and,

2) for all $x_1, x_2, \cdots, x_n$, $y_1, y_2, \cdots, y_n \in \mathcal{H}$; if $x_j \cong y_j$ then $g(x_1^n) \cong g(y_1^n)$, where $1 \le j \le n$ [23].

Lemma 4.7 Let $(\mathcal{H}, f, g)$ be an (*m*, *n*)-semihyperring and $\cong$ be the congruence relation on $\mathcal{H}$ then

1) if $\{x\} \cong \{y\}$ then

$$\{f(x, a_1^{m-1})\} \cong \{f(y, a_1^{m-1})\}$$

for all $x, y, a_1, a_2, \cdots, a_m \in \mathcal{H}$

2) if $x \cong y$ then following holds:

$$g(a_1^{i-1}, x, a_{i+1}^n) \cong g(a_1^{i-1}, y, a_{i+1}^n)$$

for all $x, y, a_1, a_2, \cdots, a_n \in \mathcal{H}$

Proof.

1) Given that

$$\{x\} \cong \{y\} \quad (3)$$

for all $x, y \in \mathcal{H}$. Let e be the hyper additive identity element, then (3) can be represented as follows:

$$f\left(x,\underbrace{e,\cdots,e}_{m-1}\right) \cong f\left(y,\underbrace{e,\cdots,e}_{m-1}\right) \quad (4)$$

do *f* hyperoperation on both sides of (4) with a_1 to get

$$f\left(f\left(x,\underbrace{e,\cdots,e}_{m-1}\right),a_1,\underbrace{e,\cdots,e}_{m-2}\right) \cong f\left(f\left(y,\underbrace{e,\cdots,e}_{m-1}\right),a_1,\underbrace{e,\cdots,e}_{m-2}\right) \quad (5)$$

$$f\left(f\left(x,a_1,\underbrace{e,\cdots,e}_{m-2}\right),\underbrace{e,\cdots,e}_{m-1}\right) \cong f\left(f\left(y,a_1,\underbrace{e,\cdots,e}_{m-2}\right),\underbrace{e,\cdots,e}_{m-1}\right) \quad (6)$$

$$\left\{f\left(x,a_1,\underbrace{e,\cdots,e}_{m-2}\right)\right\} \cong \left\{f\left(y,a_1,\underbrace{e,\cdots,e}_{m-2}\right)\right\} \quad (7)$$

do *f* hyperoperation on both sides of (7) with a_2 to get the following equation:

$$f\left(f\left(x,a_1,\underbrace{e,\cdots,e}_{m-2}\right),a_2,\underbrace{e,\cdots,e}_{m-2}\right) \cong f\left(f\left(y,a_1,\underbrace{e,\cdots,e}_{m-2}\right),a_2,\underbrace{e,\cdots,e}_{m-2}\right) \quad (8)$$

$$f\left(f\left(x,a_1,a_2,\underbrace{e,\cdots,e}_{m-3}\right),\underbrace{e,\cdots,e}_{m-1}\right) \cong f\left(f\left(y,a_1,a_2,\underbrace{e,\cdots,e}_{m-3}\right),\underbrace{e,\cdots,e}_{m-1}\right) \quad (9)$$

$$\left\{f\left(x,a_1,a_2,\underbrace{e,\cdots,e}_{m-3}\right)\right\} \cong \left\{f\left(f\left(y,a_1,a_2,\underbrace{e,\cdots,e}_{m-3}\right),\underbrace{e,\cdots,e}_{m-1}\right)\right\} \quad (10)$$

Similarly we can do *f* hyperoperation till a_{m-1} to get the following result:

$$\left\{f\left(x,a_1,a_2,\cdots,a_{m-1}\right)\right\} \cong \left\{f\left(y,a_1,a_2,\cdots,a_{m-1}\right)\right\} \quad (11)$$

Which can also be represented as:

$$\left\{f\left(x,a_1^{m-1}\right)\right\} \cong \left\{f\left(y,a_1^{m-1}\right)\right\} \quad (12)$$

2) Given that

$$x \cong y \quad (13)$$

for all $x, y \in \mathcal{H}$. Let e' be the multiplicative identity element

$$g\left(x,\underbrace{e',\cdots,e'}_{n-1}\right) \cong g\left(y,\underbrace{e',\cdots,e'}_{n-1}\right) \quad (14)$$

do *g* hyperoperation on both sides of (14) with a_1 to get

$$g\left(g\left(x,\underbrace{e',\cdots,e'}_{n-1}\right),a_1,\underbrace{e',\cdots,e'}_{n-2}\right) \cong g\left(g\left(y,\underbrace{e',\cdots,e'}_{n-1}\right),a_1,\underbrace{e',\cdots,e'}_{n-2}\right) \quad (15)$$

$$g\left(g\left(x,a_1,\underbrace{e',\cdots,e'}_{n-2}\right),\underbrace{e',\cdots,e'}_{n-1}\right) \cong g\left(g\left(y,a_1,\underbrace{e',\cdots,e'}_{n-2}\right),\underbrace{e',\cdots,e'}_{n-1}\right) \quad (16)$$

$$g\left(x,a_1,\underbrace{e',\cdots,e'}_{n-2}\right) \cong g\left(y,a_1,\underbrace{e',\cdots,e'}_{n-2}\right) \quad (17)$$

do *g* hyperoperation on both sides of (17) with a_2 to get the following equation:

$$g\left(g\left(x,a_1,\underbrace{e',\cdots,e'}_{n-2}\right),a_2,\underbrace{e',\cdots,e'}_{n-2}\right) \cong g\left(g\left(y,a_1,\underbrace{e',\cdots,e'}_{n-2}\right),a_2,\underbrace{e',\cdots,e'}_{n-2}\right) \quad (18)$$

$$g\left(g\left(x,a_1,a_2,\underbrace{e',\cdots,e'}_{n-3}\right),\underbrace{e',\cdots,e'}_{n-1}\right) \cong g\left(g\left(y,a_1,a_2,\underbrace{e',\cdots,e'}_{n-3}\right),\underbrace{e',\cdots,e'}_{n-1}\right) \quad (19)$$

$$g\left(x,a_1,a_2,\underbrace{e',\cdots,e'}_{n-3}\right) \cong g\left(g\left(y,a_1,a_2,\underbrace{e',\cdots,e'}_{n-3}\right),\underbrace{e',\cdots,e'}_{n-1}\right) \quad (20)$$

Similarly we can do *g* operation till a_{n-1} to get the following result:

$$g\left(x,a_1^{n-1}\right) \cong g\left(y,a_1^{n-1}\right).$$

Theorem 4.8 Let $(\mathcal{H}, f, g)$ be an (*m*, *n*)-semihyperring and $\cong$ be the congruence relation on $\mathcal{H}$. Then if $\{a_i\} \cong \{b_i\}$ and $\{x_j\} \cong \{y_j\}$ for all $a_i, b_i, x_j, y_j \in \mathcal{H}$ and $i, j \in \{1, m\}$ then the following is obtained: for all $1 \le k \le m$

$$\left\{f\left(a_1^k, x_{k+1}^m\right)\right\} \cong \left\{f\left(b_1^k, y_{k+1}^m\right)\right\}$$

Proof. Can be proved similar to Lemma 4.7.

Definition 4.9 Let $\cong$ be a congruence on $\mathcal{H}$. Then the quotient of $\mathcal{H}$ by $\cong$, written as $\mathcal{H}/\cong$, is the algebra whose universe is $\mathcal{H}/\cong$ and whose fundamental operation satisfy

$$f^{\mathcal{H}/\cong}(x_1,x_2,\cdots,x_m)=f^{\mathcal{H}}(x_1,x_2,\cdots,x_m)/\cong$$

where $x_1,x_2,\cdots,x_m\in\mathcal{H}$ [23].

Theorem 4.10 Let $(\mathcal{H},f,g)$ be an (*m*, *n*)-semihyperring and $\cong$ be the equivalence relation and strongly regular on $\mathcal{H}$ then $(\mathcal{H}/\cong,f,g)$ is also an (*m*, *n*)-semihyperring.

Definition 4.11 Let $(\mathcal{H},f,g)$ be an (*m*, *n*)-semihyperring and $\cong$ be the congruence relation. The natural map $v_{\cong}:\mathcal{H}\to\mathcal{H}/\cong$ is defined by $v_{\cong}(a_i)=a_i/\cong$ and $v_{\cong}(b_j)=b_j/\cong$ where $a_i,b_j\in\mathcal{H}$ for all $1\le i\le m$, $1\le j\le n$.

Theorem 4.12 Let ρ and σ be two congruence relations on (*m*, *n*)-semihyperring $(\mathcal{H},f,g)$ such that $\rho\subseteq\sigma$. Then

$$\sigma/\rho=\{(\rho(x),\rho(y))\in\mathcal{H}/\rho\times\mathcal{H}/\rho:(x,y)\in\sigma\}$$

is a congruence on $\mathcal{H}/\rho$ and $(\mathcal{H}/\rho)/(\sigma/\rho)\cong\mathcal{H}/\sigma$.

Proof. Similar to [24], we can deduce that σ/ρ is an equivalence relation on $\mathcal{H}/\rho$. Suppose $(a_i\rho)(\sigma/\rho)(b_i\rho)$ for all $1\le i\le m$ and $(c_j\rho)(\sigma/\rho)(d_j\rho)$ for all $1\le j\le n$. Since σ is congruence on $\mathcal{H}$ therefore $f(a_1^m)\sigma f(b_1^m)$ and $g(c_1^n)\sigma g(d_1^n)$ which implies $f(a_1^m)\rho(\sigma/\rho)f(b_1^m)\rho$ and $g(c_1^n)\rho(\sigma/\rho)g(d_1^n)\rho$ respectively, therefore σ/ρ is a congruence on $\mathcal{H}/\rho$.

Theorem 4.13 The natural map from an (*m*, *n*)-semihyperring $(\mathcal{H},f,g)$ to the quotient $(\mathcal{H}/\cong,f,g)$ of the (*m*, *n*)-semihyperring is an onto homomorphism.

Definition 4.11 and Theorem 4.13 is generalization of [23].

Proof. let $\cong$ be the congruence relation on (*m*, *n*)-semihyperring $(\mathcal{H},f,g)$ and the natural map be $v_{\cong}:\mathcal{H}\to\mathcal{H}/\cong$. For all $a_i\in\mathcal{H}$, where $1\le i\le m$ following holds true:

$$\begin{aligned}&v_{\cong}f^{\mathcal{H}}(a_1,a_2,\cdots,a_m)\\&=f^{\mathcal{H}}(a_1,a_2,\cdots,a_m)/\cong\\&=f^{\mathcal{H}/\cong}(a_1/\cong,a_2/\cong,\cdots,a_m/\cong)\\&=f^{\mathcal{H}/\cong}(v_{\cong}a_1,v_{\cong}a_2,\cdots,v_{\cong}a_m)\end{aligned}$$

In a similar fashion we can deduce for g, for all $b_j\in\mathcal{H}$, where $1\le j\le n$:

$$\begin{aligned}&v_{\cong}g^{\mathcal{H}}(b_1,b_2,\cdots,b_n)\\&=g^{\mathcal{H}}(b_1,b_2,\cdots,b_n)/\cong\\&=g^{\mathcal{H}/\cong}(b_1/\cong,b_2/\cong,\cdots,b_n/\cong)\\&=g^{\mathcal{H}/\cong}(v_{\cong}b_1,v_{\cong}b_2,\cdots,v_{\cong}b_n)\end{aligned}$$

So $v_{\cong}$ is onto homomorphism.

Proof is similar to [23].

5. Fuzzy (*m*, *n*)-Semihyperring

Let $\mathcal{R}$ be a non-empty set. Then

1) A fuzzy subset of $\mathcal{R}$ is a function $\mu:\mathcal{R}\to[0,1]$;

2) For a fuzzy subset μ of $\mathcal{R}$ and $t\in[0,1]$, the set $\mu_t=\{x\in\mathcal{R}\mid\mu(x)\ge t\}$ is called the *level subset* of μ [1,6,13,25].

Definition 5.1 A fuzzy subset μ of an (*m*, *n*)-semihyperring $(\mathcal{H},f,g)$ is called a *fuzzy* (*m*, *n*)-*sub-semihyperring* of $\mathcal{H}$ if following hold true:

1) $\min\{\mu(x_1),\mu(x_2),\cdots,\mu(x_m)\}$
$\le\inf_{z\in f(x_1,x_2,\cdots,x_m)}\mu(z)$,

for all $x_1,x_2,\cdots,x_m\in\mathcal{H}$

2) $\min\{\mu(x_1),\mu(x_2),\cdots,\mu(x_n)\}$
$\le\mu(g(x_1,x_2,\cdots,x_n))$,

for all $x_1,x_2,\cdots,x_n\in\mathcal{H}$.

Definition 5.2 A fuzzy subset μ of an (*m*, *n*)-semihyperring $(\mathcal{H},f,g)$ is called a *fuzzy hyperideal* of $\mathcal{H}$ if the following hold true:

1) $\min\{\mu(x_1),\mu(x_2),\cdots,\mu(x_m)\}$
$\le\inf_{z\in f(x_1,x_2,\cdots,x_m)}\mu(z)$,

for all $x_1,x_2,\cdots,x_m\in\mathcal{H}$,

2) $\mu(x_1)\le\mu(g(x_1,x_2,\cdots,x_n))$, for all $x_1,x_2,\cdots,x_n\in\mathcal{H}$,

3) $\mu(x_2)\le\mu(g(x_1,x_2,\cdots,x_n))$, for all $x_1,x_2,\cdots,x_n\in\mathcal{H}$,

$\vdots$

4) $\mu(x_n)\le\mu(g(x_1,x_2,\cdots,x_n))$, for all $x_1,x_2,\cdots,x_n\in\mathcal{H}$.

Theorem 5.3 A fuzzy subset μ of an (*m*, *n*)-semihyperring $(\mathcal{H},f,g)$ is a fuzzy hyperideal if and only if every non-empty level subset is a hyperideal of $\mathcal{H}$.

Proof. Suppose subset μ is a fuzzy hyperideal of (*m*, *n*)-semihyperring $(\mathcal{H},f,g)$ and μ_t is a level subset of μ.

If $x_1,x_2,\cdots,x_m\in\mu_t$ for some $t\in[0,1]$ then from the definition of level set, we can deduce the following:

$$\mu(x_1)\ge t,\mu(x_2)\ge t,\cdots,\mu(x_m)\ge t.$$

Thus, we say that:

$$\min\{\mu(x_1),\mu(x_2),\cdots,\mu(x_m)\}\ge t$$

Thus:

$$\begin{aligned}&\inf_{z\in f(x_1,x_2,\cdots,x_m)}\mu(z)\\&\ge\min\{\mu(x_1),\mu(x_2),\cdots,\mu(x_m)\}\ge t.\end{aligned}\quad(21)$$

So, we get the following:

$\mu(z) \geq t$, for all $z \in f(x_1, x_2, \cdots, x_m)$.

Therefore, $f(x_1, x_2, \cdots, x_m) \subseteq \mu_t$.

Again, suppose that $x_1, x_2, \cdots, x_n \in \mathcal{H}$ and $x_i \in \mu_t$, where $1 \leq i \leq n$. Then, we find that $\mu(x_i) \geq t$.

So, we obtain the following:

$$\begin{aligned} t \leq \mu_{x_i} &\leq \mu\left(g(x_1, x_2, \cdots, x_n)\right) \\ &\rightarrow g\left(x_1^{i-1}, \mu_t, x_{i+1}^n\right) \subseteq \mu_t \end{aligned} \tag{22}$$

Thus, we find that μ_t is a hyperideal of $\mathcal{H}$.

On the other hand, suppose that every non-empty level subset μ_t is a hyperideal of $\mathcal{H}$.

Let $t_0 = \min\{\mu_{x_1}, \mu_{x_2}, \cdots, \mu_{x_n}\}$, for all $x_1, x_2, \cdots, x_n \in \mathcal{H}$.

Then, we obtain the following:

$$\mu(x_1) \geq t_0, \mu(x_2) \geq t_0, \cdots, \mu(x_n) \geq t_0$$

Thus,

$$x_1, x_2, \cdots, x_n \in \mu_{t_0}$$

We can also obtain that:

$$f(x_1, x_2, \cdots, x_m) \subseteq \mu_{t_0}.$$

Thus,

$$\begin{aligned} &\min\{\mu(x_1), \mu(x_2), \cdots, \mu(x_m)\} \\ &= t_0 \leq \inf_{z \in f(x_1, x_2, \cdots, x_m)} \mu(z). \end{aligned} \tag{23}$$

Again, suppose that $\mu(x_1) = t_1$. Then $x \in \mu_{t_1}$.

So, we obtain:

$$g(x_1, x_2, \cdots, x_n) \in \mu_{t_1} \rightarrow t_1 \leq \mu\left(g(x_1, x_2, \cdots, x_n)\right)$$

Thus, $\mu(x_1) \leq \mu\left(g(x_1, x_2, \cdots, x_n)\right)$.

Similarly, we obtain $\mu(x_i) \leq \mu\left(g(x_1, x_2, \cdots, x_n)\right)$, for all $i \in \{1, n\}$.

Thus, we can check all the conditions of the definition of fuzzy hyperideal.

This proof is a generalization of [1].

Theorem 5.3 is a generalization of [1,11,26].

Jun, Ozturk and Song [27] have proposed a similar theorem on hemiring.

Theorem 5.4 Let $\mathcal{I}$ be a non-empty subset of an (*m*, *n*)-semihyperring $(\mathcal{H}, f, g)$. Let μ_I be a fuzzy set defined as follows:

$$\mu_I(x) = \begin{cases} s & \text{if } x \in \mathcal{I}, \\ t & \text{otherwise}, \end{cases}$$

where $0 \leq t < s \leq 1$. Then μ_I is a fuzzy left hyper ideal of $\mathcal{H}$ if and only if $\mathcal{I}$ is a left hyper ideal of $\mathcal{H}$.

Following Corollary 5.5 is generalization of [1].

Corollary 5.5 Let μ be a fuzzy set and its upper bound be t_0 of an (*m*, *n*)-semihyperring $(\mathcal{H}, f, g)$. Then the following are equivalent:

1) μ is a fuzzy hyperideal of $\mathcal{H}$.

2) Every non-empty level subset of μ is a hyperideal of $\mathcal{H}$.

3) Every level subset μ_t is a hyperideal of $\mathcal{H}$ where $t \in [0, t_0]$.

Definition 5.6 Let $(\mathcal{R}, f', g')$ and $(\mathcal{S}, f'', g'')$ be fuzzy (*m*, *n*)-semihyperrings and φ be a map from $\mathcal{R}$ into $\mathcal{S}$. Then φ is called homomorphism of fuzzy (*m*, *n*)-semihyperrings if following hold true:

$$\varphi\left(f'(x_1, x_2, \cdots, x_m)\right) \leq f''\left(\varphi(x_1), \varphi(x_2), \cdots, \varphi(x_m)\right)$$

and

$$\varphi\left(g'(y_1, y_2, \cdots, y_n)\right) \leq g''\left(\varphi(y_1), \varphi(y_2), \cdots, \varphi(y_n)\right)$$

for all $x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n \in \mathcal{R}$.

Theorem 5.7 Let $(\mathcal{R}, \mu_{f'}, \mu_{g'})$ and $(\mathcal{S}, \mu_{f''}, \mu_{g''})$ be two fuzzy (*m*, *n*)-semihyperrings and $(\mathcal{R}, f', g')$ and $(\mathcal{S}, f'', g'')$ be associated (*m*, *n*)-semihyperring. If $\varphi: \mathcal{R} \rightarrow \mathcal{S}$ is a homomorphism of fuzzy (*m*, *n*)-semihyperrings, then φ is homomorphism of the associated (*m*, *n*)-semihyperrings also.

Definition 5.6 and Theorem 5.7 are similar to the one proposed by Leoreanu-Fotea [16] on fuzzy (*m*, *n*)-ary hyperrings and (*m*, *n*)-ary hyperrings and Ameri and Nozari [8] proposed a similar Definition and Theorem on hyperalgebras.

6. Conclusion

We proposed the definition, examples and properties of (*m*, *n*)-semihyperring. (*m*, *n*)-semihyperring has vast application in many of the computer science areas. It has application in cryptography, optimization theory, fuzzy computation, Baysian networks and Automata theory, listed a few. In this paper we proposed Fuzzy (*m*, *n*)-semihyperring which can be applied in different areas of computer science like image processing, artificial intelligence, etc. We found some of the interesting results: the natural map from an (*m*, *n*)-semihyperring to the quotient of the (*m*, *n*)-semihyperring is an onto homomorphism. It is also found that if ρ and σ are two congruence relations on (*m*, *n*)-semihyperring $(\mathcal{H}, f, g)$ such that $\rho \subseteq \sigma$, then σ/ρ is a congruence on $\mathcal{H}/\rho$ and $(\mathcal{H}/\rho)/(\sigma/\rho) \cong \mathcal{H}/\sigma$. We found many interesting results in fuzzy (*m*, *n*)-semihyperring as well, like, a fuzzy subset μ of an (*m*, *n*)-semihyperring $(\mathcal{H}, f, g)$ is a fuzzy hyperideal if and only if every non-empty level subset is a hyperideal of $\mathcal{H}$. We can use (*m*, *n*)-semihyperring in cryptography in our future work.

7. Acknowledgements

The first author is indebted to Prof. Shrisha Rao of IIIT Bangalore for encouraging him to do research in this area. A few basic definitions were presented when the first

author was a master's student under the supervision of Prof. Shrisha Rao at IIIT Bangalore.

REFERENCES

[1] B. Davvaz, "Fuzzy Hyperideals in Ternary Semihyperrings," *Iranian Journal of Fuzzy Systems*, Vol. 6, No. 4, 2009, pp. 21-36.

[2] T. Vougiouklis, "On Some Representations of Hypergroups," *Annales Scientifiques de l'Universite de Clermont, Serie Mathematique*, Vol. 26, 1990, pp. 21-29.

[3] S. Chaopraknoi, S. Hobuntud and S. Pianskool, "Admitting a Semihyperring with Zero of Certain Linear Transformation Subsemigroups of $L_R(V,W)$ Part (ii)," *Journal of Mathematics*, 2008, pp. 45-58.

[4] B. Davvaz and N. S. Poursalavati, "On Polygroup Hyperrings and Representations of Polygroups," *Journal of the Korean Mathematical Society*, Vol. 36, No. 6, 1999, pp. 1021-1031.

[5] R. Ameri and H. Hedayati, "On *k*-Hyperideals of Semihyperrings," *Journal of Discrete Mathematical Sciences and Cryptography*, Vol. 10, No. 1, 2007, pp. 41-54.

[6] L. A. Zadeh, "Fuzzy Sets," *Information and Control*, Vol. 8, No. 3, 1965, pp. 338-353. doi:10.1016/S0019-9958(65)90241-X

[7] B. Davvaz, "Fuzzy Hv-Groups," *Fuzzy Sets and Systems*, Vol. 101, No. 1, 1999, pp. 191-195. doi:10.1016/S0165-0114(97)00071-7

[8] R. Ameri and T. Nozari, "Fuzzy Hyperalgebras," *Computers and Mathematics with Applications*, Vol. 61, No. 2, 2011, pp. 149-154. doi:10.1016/j.camwa.2010.08.059

[9] I. Cristea, "On the Fuzzy Subhypergroups of Some Particular Complete Hypergroups(I)," *World Applied Sciences Journal*, Vol. 7, 2009, pp. 57-63.

[10] B. Davvaz and W. A. Dudek, "Fuzzy n-ary Groups as a Generalization of Rosenfield's Fuzzy Groups," *Journal of Multiple-Valued Logic and Soft Computing*, Vol. 15, No. 5-6, 2009, pp. 471-488.

[11] R. Ameri and H. Hedayati, "Homomorphism and Quotient of Fuzzy *k*-Hyperideals," *Ratio Mathematica*, Vol. 20, 2010.

[12] S. Mirvakili and B. Davvaz, "Relations on Krasner (m, n)-Hyperrings," *European Journal of Combinatorics*, Vol. 31, No. 3, 2010, pp. 790-802. doi:10.1016/j.ejc.2009.07.006

[13] B. Davvaz, "Fuzzy Krasner (m, n)-Hyperrings," *Computers and Mathematics with Applications*, Vol. 59, No. 12, 2010, pp. 3879-3891.

[14] B. Davvaz and T. Vougiouklis, "n-ary Hypergroups," *Iranian Journal of Science and technology*, Vol. 30, 2006, pp. 165-174.

[15] B. Davvaz, P. Corsini and V. L. Fotea, "Fuzzy n-ary Subpolygroups," *Computers and Mathematics with Applications*, Vol. 57, 2009, pp. 141-152.

[16] V. L. Fotea, "A New Type of Fuzzy n-ary Hyperstructures," *Information Sciences*, Vol. 179, No. 15, 2009, pp. 2710-2718. doi:10.1016/j.ins.2009.03.017

[17] S. E. Alam, S. Rao and B. Davvaz, "(m, n)-Semirings and a Generalized Fault Tolerance Algebra of Systems," *General Mathematics*, 2010.

[18] W. A. Dudek and V. V. Mukhin, "On Topological n-ary Semigroups," *Quasigroups and Related Systems*, Vol. 3, 1996, pp. 73-88.

[19] W. A. Dudek, "Idempotents in n-ary Semigroups," *Southeast Asian Bulletin of Mathematics*, Vol. 25, No. 1, 2001, pp. 97-104. doi:10.1007/s10012-001-0097-y

[20] H. Hedayati and R. Ameri, "Construction of *k*-Hyperideals by P-Hyperoperations," *Journal of Applied Mathematics*, Vol. 15, 2005, pp. 75-89.

[21] R. Ameri and M. M. Zahedi, "Hyperalgebraic Systems," *Italian Journal of Pure and Applied Mathematics*, Vol. 6, 1999, pp. 21-32.

[22] M. K. Sen and U. Dasgupta, "Some Aspects of GH-Rings," *Journal of Annals of the Alexandru Ioan Cuza University—Mathematics*, 2010.

[23] S. Burris and H. P. Sankappanavar, "A Course in Universal Algebra of Graduate Texts in Mathematics," Springer-Verlag, Berlin, 1981. doi:10.1007/978-1-4613-8130-3

[24] S. Kar and B. K. Maity, "Congruences on Ternary Semigroups," *Journal of the Chungcheong Mathematical Society*, Vol. 20, No. 3, 2007.

[25] X. Ma, J. Zhan, B. Davvaz and B. Y. Jun, "Some Kinds of $(\in, \in \vee q)$-Interval-Valued Fuzzy Ideals of BCI-Algebras," *Information Sciences*, Vol. 178, No. 19, 2008, pp. 3738-3754. doi:10.1016/j.ins.2008.06.006

[26] H. Hedayati and R. Ameri, "Fuzzy k-Hyperideals," *International Journal of Pure and Applied Mathematical Sciences*, Vol. 2, No. 2, 2005, pp. 247-256.

[27] B. Y. Jun, A. M. Ozturk and Z. S. Song, "On Fuzzy h-Ideals in Hemirings," *Information Sciences*, Vol. 162, No. 3-4, 2004, pp. 211-226. doi:10.1016/j.ins.2003.09.007

