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Nonstationary Wavelets Related to the Walsh Functions

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ABSTRACT

Using the Walsh-Fourier transform, we give a construction of compactly supported nonstationary dyadic wavelets on the positive half-line. The masks of these wavelets are the Walsh polynomials defined by finite sets of parameters. Application to compression of fractal functions are also discussed.

Keywords: Walsh Functions; Nonstationary Dyadic Wavelets; Fractal Functions; Adapted Multiresolution Analysis

1. Introduction

As usual, let $\mathbb{R}_+ = [0, +\infty)$ be the positive half-line, $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$ be the set of all nonnegative integers, and let $\mathbb{N} = \{1, 2, \cdots\}$ be the set of all positive integers. The first examples of orthogonal wavelets on $\mathbb{R}_+$ related to the Walsh functions and the corresponding wavelets on the Cantor dyadic group have been constructed in [1]; then, in [2] and [3], a multifractal structure of this wavelets is observed and conditions for wavelets to generate an unconditional basis in L^q-spaces for all $1 < q < \infty$ have been found. These investigations are continued in [4-10] where among other subjects the algorithms to construct orthogonal and biorthogonal wavelets associated with the generalized Walsh functions are studied. In the present paper, using the Walsh-Fourier transform, we construct nonstationary dyadic wavelets on $\mathbb{R}_+$ (cf. [11-13], [14, Ch.8]).

Let us denote by $[x]$ the integer part of x. For every $x \in \mathbb{R}_+$, we set

$$x_j = \left[2^j x\right](\mathrm{mod}\,2), x_{-j} = \left[2^{1-j} x\right](\mathrm{mod}\,2),\ j \in \mathbb{N},$$

where $x_j, x_{-j} \in \{0,1\}$. Then

$$x = \sum_{j<0} x_j 2^{-j-1} + \sum_{j>0} x_j 2^{-j}.$$

The dyadic addition on $\mathbb{R}_+$ is defined as follows

$$x \oplus y = \sum_{j<0} \left|x_j - y_j\right| 2^{-j-1} + \sum_{j<0} \left|x_j - y_j\right| 2^{-j}.$$

Further, we introduce the notations

$$\chi(x,\omega) = (-1)^{\sigma(x,\omega)}, \sigma(x,\omega) = \sum_{j=1}^{\infty} x_j \omega_{-j} + x_{-j}\omega_j,$$

where $x, \omega \in \mathbb{R}_+$. Then the Walsh function w_k of order k is $w_k(x) = \chi(x,k)$ (see, e.g., [15]).

The Walsh-Fourier transform of every function f that belongs to $L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ is defined by

$$\hat{f}(\omega) = \int_0^{\infty} f(x)\chi(x,\omega)\,\mathrm{d}x, \quad \omega \in \mathbb{R}_+.$$

and extent to the whole space $L^2(\mathbb{R}_+)$ in a standard way. The intervals

$$\Delta_k^{(n)} = \left[k 2^{-n}, (k+1)2^{-n}\right),\ k \in \mathbb{Z}_+,$$

are called the *dyadic intervals of range* n. The dyadic topology on $\mathbb{R}_+$ is generated by the collection of dyadic intervals. A subset E of $\mathbb{R}_+$ which is compact in the dyadic topology will be called *W-compact*.

For any $j \in \mathbb{Z}_+$ we define φ_j and ψ_j by the following algorithm:

Step 1. For each $j \in \mathbb{N}$ choose $n_j \in \mathbb{N}$, and $b_k^{(j)} \in \mathbb{C}$, $k = 0, 1 \cdots, 2^{n_j} - 1$, such that

$$b_0^{(j)} = 1, \left|b_k^{(j)}\right|^2 + \left|b_{k+2^{n_j-1}}^{(j)}\right|^2 = 1 \tag{1}$$

for all $k = 0, 1, \cdots, 2^{n_j - 1} - 1$.

Step 2. Define the masks

$$m_0^{(j)}(\omega) = \frac{1}{2} \sum_{k=0}^{2^{n_j}-1} c_k^{(j)} w_k(\omega) \tag{2}$$

with the coefficients

$$c_k^{(j)} = \frac{1}{2^{n_j - 1}} \sum_{l=0}^{2^{n_j}-1} b_l^{(j)} w_l\left(2^{-n_j} k\right),\ k = 0, 1, \cdots, 2^{n_j} - 1,$$

so that $m_0^{(j)}(\omega) = b_l^{(j)}$ for all $\omega \in \Delta_l^{(j)}$ (cf. [15, Sect. 9.7]).

Step 3. For each $j \in \mathbb{Z}_+$ put

$$\hat{\varphi}_j(\omega) = 2^{-j/2} \prod_{l=j+1}^{\infty} m_0^{(l)}\left(2^{-l}\omega\right), \tag{3}$$

so that

$$\varphi_j(x) = \frac{1}{\sqrt{2}} \sum_{k=0}^{2^{n_{j+1}}-1} c_k^{(j+1)} \varphi_{j+1}\left(x \oplus 2^{-j-1}k\right). \quad (4)$$

Step 4. Define ψ_j by the formula

$$\psi_j(x) = \frac{1}{\sqrt{2}} \sum_{k=0}^{2^{n_{j+1}}-1} (-1)^{k+1} c_{k\oplus 1}^{(j+1)} \varphi_{j+1}\left(x \oplus \frac{k}{2^{j+1}}\right). \quad (5)$$

Further, let us define subspaces $\{V_j\}$ and $\{W_j\}$ in $L^2(\mathbb{R}_+)$ as follows

$$V_j = \overline{\operatorname{span}\{\varphi_{j,k} : k \in \mathbb{Z}_+\}},$$

$$W_j = \overline{\operatorname{span}\{\psi_{j,k} : k \in \mathbb{Z}_+\}}$$

for all $j \in \mathbb{Z}_+$.

We say that a polynomial m satisfies the *modified Cohen condition* if there exists a *W*-compact subset E of $\mathbb{R}_+$ such that

$$\operatorname{int} E \ni 0, \mu(E) = 1, E \equiv [0,1)(\operatorname{mod} \mathbb{Z}_+)$$

and

$$\inf_{j \in \mathbb{N}} \inf_{\omega \in E} \left| m\left(2^{-j}\omega\right)\right| > 0. \quad (6)$$

Theorem. Suppose that the masks $m_0^{(n)}$ satisfy the modified Cohen condition with a subset E and there exists $j_0 \in \mathbb{N}$ such that

$$m_0^{(n)}(\omega) = 1 \quad \text{for all} \quad \omega \in \left[0, 2^{-j_0}\right), \; n \in \mathbb{N}. \quad (7)$$

Then for any $j \in \mathbb{Z}_+$ the following properties hold:

a) $\varphi_j, \psi_j \in L^2(\mathbb{R}_+)$ and $\operatorname{supp} \varphi_j \subset [0,1]$;

b) $\{\varphi_{j,k} : k \in \mathbb{Z}_+\}$ and $\{\psi_{j,k} : k \in \mathbb{Z}_+\}$ are orthonormal basis in V_j and W_j, respectively;

c) $V_j \subset V_{j+1}$, $V_j \oplus W_j = V_{j+1}$.

Moreover, we have

$$\overline{\bigcup_{j=0}^{\infty} V_j} = L^2(\mathbb{R}_+).$$

Corollary. The system

$$\{\varphi_0(\cdot \oplus k) : k \in \mathbb{Z}_+\} \cup \{\psi_{j,k} : j, k \in \mathbb{Z}_+\}$$

is an orthonormal basis in $L^2(\mathbb{R}_+)$.

We prove this theorem in the next section. Then using the notion of an adapted multiresolution analysis suggested by Sendov [12], we discuss an application of the nonstationary dyadic wavelets to compression of the Weierstrass function and the Swartz function.

2. Proof of the Theorem

At first we prove the orthonormality of $\{\varphi_{j,k}\}_{k \in \mathbb{Z}_+}$. In view of

$$\langle \varphi_{j,0}, \varphi_{j,n} \rangle = \langle \hat{\varphi}_{j,0}, \hat{\varphi}_{j,n} \rangle = \int_0^\infty \left|\hat{\varphi}_j(\omega)\right|^2 w_n\left(2^{-j}\omega\right) \mathrm{d}\omega,$$

let us show that

$$\int_0^\infty \left|\varphi_j(\omega)\right|^2 w_n\left(2^{-j}\omega\right) \mathrm{d}\omega = \delta_{0,n}, \quad n \in \mathbb{Z}_+.$$

Denote by $\mathbf{1}_E$ the characteristic function of E. For each j we define

$$\hat{\varphi}_j^{(s)}(\omega) = 2^{-j/2} \prod_{l=j+1}^{s} m_0^{(l)}\left(2^{-l}\omega\right) \mathbf{1}_E\left(2^{-s}\omega\right)$$

for $s = j+1, j+2, \cdots$ Since $0 \in \operatorname{int} E$ and, for all $j \in \mathbb{Z}_+$, $m_0^{(j)}(\omega) = 1$ in some neighbourhood of zero, we obtain from Equation (3)

$$\lim_{k \to \infty} \hat{\varphi}_j^{(k)}(\omega) = \hat{\varphi}_j(\omega) \quad \text{for all} \quad \omega \in \mathbb{R}_+. \quad (8)$$

Let

$$I_j^{(k)}[n] := \int_0^\infty \left|\hat{\varphi}_j^{(k)}(\omega)\right|^2 w_n\left(2^{-j}\omega\right) \mathrm{d}\omega,$$

where $k > j$, $n \in \mathbb{Z}_+$. Letting $\zeta = 2^{-s}\omega$, we have

$$I_j^{(s)}[k] = 2^{s-j} \int_E \prod_{l=j+1}^{s} \left|m_0^{(l)}\left(2^{s-l}\zeta\right)\right|^2 w_k\left(2^{s-j}\zeta\right) \mathrm{d}\zeta$$

$$= 2^{s-j} \int_0^1 \left|m_0^{(k)}(\zeta)\right|^2 \prod_{l=j+1}^{s-1} \left|m_0^{(l)}\left(2^{s-l}\zeta\right)\right|^2 w_k\left(2^{s-j}\right) \mathrm{d}\zeta$$

$$= 2^{s-j} \int_0^{1/2} \left(\left|m_0^{(k)}(\zeta)\right|^2 + \left|m_0^{(k)}(\zeta + 1/2)\right|^2\right)$$

$$\times \prod_{l=j+1}^{s-1} \left|m_0^{(l)}\left(2^{s-l}\zeta\right)\right|^2 w_k\left(2^{s-j}\zeta\right) \mathrm{d}\zeta,$$

that yields $I_j^{(s)}[k] = I_j^{(s-1)}[k]$. By induction, we obtain

$$I_j^{(s)}[k] = I_j^{(s-1)}[k] = \cdots = I_j^{(j+1)}[k] = \delta_{0,k}.$$

According to Equation (8), by Fatou's lemma, we have

$$\int_0^\infty \left|\hat{\varphi}_j(\omega)\right|^2 \mathrm{d}\omega \le \lim_{s \to \infty} \int_0^\infty \left|\hat{\varphi}_j^{(s)}(\omega)\right|^2 \mathrm{d}\omega = \lim_{s \to \infty} I_j^{(s)}[0] = 1. \quad (9)$$

Consequently, $\varphi_j \in L^2(\mathbb{R}_+)$ and, in view of Equation (5), $\psi_j \in L^2(\mathbb{R}_+)$. It is known that if $\hat{f} \in L^1(\mathbb{R}_+)$ is constant on dyadic intervals of range n, then $\operatorname{supp} f \subset \left[0, 2^n\right]$ (see [16, Sect. 6.2]). Therefore, each function $\hat{\varphi}_j$ is constant on $[k, k+1)$, $k \in \mathbb{Z}_+$, which implies $\operatorname{supp} \varphi_j \subset [0,1]$.

In view of Equation (7), there exists $j_0 \in \mathbb{N}$ such that

$$m_0^{(j)}\left(2^{-j}\omega\right) = 1 \quad \text{for all} \quad j > j_0, \; \omega \in E.$$

Hence, for $\omega \in E$,

$$\hat{\varphi}_j(\omega) = 2^{-j/2} \prod_{l=j+1}^{j_0} m_0^{(l)}\left(2^{-l}\omega\right).$$

It follows from Equation (6) that for some $c_1 > 0$

$$\left|m_0^{(j)}\left(2^{-j}\omega\right)\right| \ge c_1 \quad \text{for} \quad j \in \mathbb{N},\ \omega \in E .$$

Since

$$c_1^{j-j_0}\left|\hat{\varphi}_j(\omega)\right| \ge 2^{-j/2}\mathbf{1}_E(\omega), \quad \omega \in \mathbb{R}_+ .$$

We have

$$\left|\hat{\varphi}_j^{(s)}(\omega)\right| \le c_1^{j-j_0}\prod_{l=j+1}^{s}\left|m_0^{(l)}\left(2^{-l}\omega\right)\right|\left|\hat{\varphi}_j\left(2^{-s}\omega\right)\right| .$$

or, taking into account Equation (3),

$$\left|\hat{\varphi}_j^{(s)}(\omega)\right| \le c_1^{j-j_0}\left|\hat{\varphi}_j(\omega)\right|, \quad \omega \in \mathbb{R}_+$$

for $s > j$, $j \in \mathbb{Z}_+$.

Applying the dominated convergence theorem we obtain

$$\begin{aligned}&\int_0^\infty\left|\hat{\varphi}_j(\omega)\right|^2 w_k\left(2^{-j}\omega\right)\mathrm{d}\omega\\&=\lim_{s\to\infty}\int_0^\infty\left|\hat{\varphi}_j^{(s)}(\omega)\right|^2 w_k\left(2^{-j}\omega\right)\mathrm{d}\omega\\&=\delta_{0,k},\end{aligned}$$

which means that $\left\{\varphi_{j,k}\right\}_{k\in\mathbb{Z}_+}$ is an orthonormal system.

Now, let us prove an orthonormality of $\left\{\psi_{j,k}\right\}_{k\in\mathbb{Z}_+}$.

For any $k\in\mathbb{Z}_+$ denote $\mathrm{d}_k^{(j)} = (-1)^{k+1}c_{k\oplus 1}^{(j)}$. Then

$$_{j,k}(x) = \frac{1}{\sqrt{2}}\sum_{l\in\mathbb{Z}_+}\mathrm{d}_{l\oplus 2k}^{(j+1)}\varphi_{j+1,l}(x) . \qquad (10)$$

Since

$$\psi\sum_{l\in\mathbb{Z}_+}\mathrm{d}_l^{(j)}\mathrm{d}_{l\oplus 2k}^{(j)} = 2\delta_{0,k} ,$$

We have

$$\begin{aligned}\left\langle\psi_{j,k},\psi_{j,k'}\right\rangle &= \frac{1}{2}\sum_{l,s\in\mathbb{Z}_+}\mathrm{d}_{l\oplus 2k}^{(j+1)}\mathrm{d}_{s\oplus 2k'}^{(j+1)}\left\langle\varphi_{j+1,l},\varphi_{j+1,s}\right\rangle\\&=\delta_{k,k'}.\end{aligned}$$

Then from Equation (10)

$$V_j \subset V_{j+1}, \quad W_j \subset V_{j+1} . \qquad (11)$$

Let us define

$$m_1^{(j)}(\omega) := \frac{1}{2}\sum_{k=0}^{2^{n_j}-1}\mathrm{d}_k^{(j)}w_k(\omega) .$$

Denote $\omega' = 2^{-j-1}\omega$. Under the unitarity of the matrices

$$\begin{pmatrix} m_0^{(j)}(\omega') & m_0^{(j)}(\omega'+1/2)\\ m_1^{(j)}(\omega') & m_1^{(j)}(\omega'+1/2)\end{pmatrix},$$

We can write

$$\begin{aligned}\hat{\varphi}_{j+1}(\omega) &= \hat{\varphi}_{j+1}(\omega)\\&\times\Big\{\Big[\left|m_0^{(j+1)}(\omega')\right|^2+\left|m_1^{(j+1)}(\omega')\right|^2\Big]\\&+\Big[m_0^{(j+1)}(\omega')\overline{m_0^{(j+1)}(\omega'+1/2)}\\&+m_1^{(j+1)}(\omega')\overline{m_1^{(j+1)}(\omega'+1/2)}\Big]\Big\}\\&=\Big[\overline{m_0^{(j+1)}(\omega')}+\overline{m_0^{(j+1)}(\omega'+1/2)}\Big]\\&\times m_0^{(j+1)}(\omega')\hat{\varphi}_{j+1}(\omega)\\&+\Big[\overline{m_1^{(j+1)}(\omega')}+\overline{m_1^{(j+1)}(\omega'+1/2)}\Big]\\&\times m_1^{(j+1)}(\omega')\hat{\varphi}_{j+1}(\omega)\\&=\sqrt{2}\sum_{l\in\mathbb{Z}_+}\overline{c}_{2l}^{(j+1)}w_{2l}\left(2^{-j-1}\omega\right)\hat{\varphi}_j(\omega)\\&+\sqrt{2}\sum_{l\in\mathbb{Z}_+}\overline{d}_{2l}^{(j+1)}w_{2l}\left(2^{-j-1}\omega\right)\hat{\psi}_j(\omega).\end{aligned}$$

Using the inverse Fourier-Walsh transform, we have

$$\varphi_{j+1}(x) = \sqrt{2}\sum_{l\in\mathbb{Z}_+}\left(\overline{c}_{2l}^{(j+1)}\varphi_{j,l}(x)+\overline{\mathrm{d}}_{2l}^{(j+1)}\psi_{j,l}(x)\right)$$

or,

$$\varphi_{j+1,k}(x) = \sqrt{2}\sum_{l\in\mathbb{Z}_+}\left(\overline{c}_{k\oplus 2l}^{(j+1)}\varphi_{j,l}(x)+\overline{\mathrm{d}}_{k\oplus 2l}^{(j+1)}\psi_{j,l}(x)\right).$$

With Equation (11) it yields $V_j \oplus W_j = V_{j+1}$

To conclude the proof it remains to show that

$$\overline{\bigcup_{j=0}^{\infty}V_j} = L_2(\mathbb{R}_+) . \qquad (12)$$

Note, that by Equation (7) for any $\omega\in\mathbb{R}_+$ there exist $j\in\mathbb{Z}_+$ such that $\hat{\varphi}_j(\omega) = 2^{-j/2}$ and, consequently,

$$\bigcup_{j=0}^{\infty}\operatorname{supp}\hat{\varphi}_j = \mathbb{R}_+ . \qquad (13)$$

For any $x\in\mathbb{R}_+$ the subspace $\overline{\bigcup_{j=0}^{\infty}V_j}$ is invariant with respect to the shift $f(\cdot)\mapsto f(\cdot\oplus x)$. Actually, an arbitrary $x\in\mathbb{R}_+$ can be approximated by fractions $2^{-j}l$, with arbitrary large j. Besides, each V_j is invariant with respect to the shifts $2^{-j}l$. By Equation (4) it is clear that $V_j \subset V_{j+1}$.

Let $f\in\bigcup_{j=0}^{\infty}V_j$. There exist j_1 such that $f\in V_{j_1}$ and hence $f\left(\cdot\oplus 2^{-j}l\right)\in V_j$ for all $j\ge j_1$. The continuity of $\left\|f(\cdot\oplus x)\right\|$ implies that $f(\cdot\oplus x)\in\overline{\bigcup_{j=0}^{\infty}V_j}$. If $g\in\overline{\bigcup_{j=0}^{\infty}V_j}$, then approximating g with f from $\bigcup_{j=0}^{\infty}V_j$ and using the invariance of a norm with respect to the shift, we obtain $g(\cdot\oplus x)\in\overline{\bigcup_{j=0}^{\infty}V_j}$.

Denote by $\left(\overline{\bigcup_{j=0}^{\infty} V_j}\right)^{\wedge}$ the set of all $\hat{f}$ such that $f \in \overline{\bigcup_{j=0}^{\infty} V_j}$. By the Weiner's theorem we can write $\left(\overline{\bigcup_{j=0}^{\infty} V_j}\right)^{\wedge} = L_2(\Omega)$, for some measurable $\Omega \subset \mathbb{R}_+$. It is clearly that $\bigcup_{j=0}^{\infty} \operatorname{supp} \hat{\varphi}_j \subset \Omega$ and, in view of Equation (13), we have $\Omega = \mathbb{R}_+$. Hence, the Equation (12) holds. The theorem is proved.

3. Numerical Experiments

For any $N \in \mathbb{N}$, let $\Delta_j(N) := \left[0, (2N-1)2^{-j}\right]$, $j \in \mathbb{Z}_+$. According to [12] an adapted multiresolution analysis (AMRA) of rank N in $L^2(\mathbb{R})$ is a collection of closed subspaces $V_j \subset L^2(\mathbb{R})$, $j \in \mathbb{Z}_+$, which satisfies the following conditions:

1) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}_+$;

2) $\overline{\bigcup_{j=0}^{\infty} V_j} = L^2(\mathbb{R})$;

3) For every $j \in \mathbb{Z}_+$ there is a function φ_j in $L^2(\mathbb{R})$ with a finite support $\Delta_j(N)$ such that $\left\{\varphi_j\left(\cdot - k2^{-j}\right): k \in \mathbb{Z}\right\}$ is an orthonormal basis of V_j;

4) For every $j \in \mathbb{Z}_+$ there exists a filter

$$\mathbf{c}(j) = \{c_k(j)\}_{k=0}^{2N-1}$$

such that

$$\varphi_{j-1}(x) = \sum_{k=0}^{2N-1} c_k(j)\varphi_j\left(x - k2^{-j}\right), \quad j \in \mathbb{N}. \tag{14}$$

The sequence $\{\varphi_j\}$ from condition (4) is called a scaling sequence for given an AMRA. The corresponding a wavelet sequence $\{\psi_j\}$ can be defined by

$$\psi_{j-1}(x) = \sum_{k=0}^{2N-1} (-1)^k c_{2N-k-1}(j)\varphi_j\left(x - k2^{-j}\right). \tag{15}$$

Denote by W_j the orthogonal complement of V_{j-1} in V_j. It is known that, under some conditions, the system $\left\{\psi_j\left(\cdot - k2^{-j}\right): k \in \mathbb{Z}\right\}$ is an orthonormal basis of W_j (for more details, see, e.g., [14, Sect. 8.1]). Moreover, if f_A denotes the projection of a function $f \in L^2(\mathbb{R})$ on the subset $A \subset L^2(\mathbb{R})$, then

$$\|f\|^2 = \left\|f_{V_0}\right\|^2 + \sum_{j=0}^{\infty} \left\|f_{W_j}\right\|^2$$

and

$$\left\|f_{V_j}\right\|^2 = \left\|f_{V_{j-1}}\right\|^2 + \left\|f_{W_{j-1}}\right\|^2. \tag{16}$$

Let us denote

$$h_k(j) = c_k(j)/\sqrt{2}$$

and

$$g_k(j) = (-1)^k h_{1-k}(j).$$

For a given array

$$\mathbf{A}(j) = \left\{a_{j,0}, a_{j,1} \cdots, a_{j,2^j-1}\right\},$$

the direct non-stationary discrete wavelet transform

$$a_{j-1,k} = \sum_{l \in \mathbb{Z}} h_{l-2k}(j) a_{j,l}, \; d_{j-1,k} = \sum_{l \in \mathbb{Z}} g_{l-2k}(j) a_{j,l},$$

maps it into

$$\mathbf{A}(j-1) = \left\{a_{j-1,0}, a_{j-1,1} \cdots, a_{j-1,2^{j-1}-1}\right\}$$

and

$$\mathbf{D}(j-1) = \left\{a_{j-1,0}, a_{j-1,1} \cdots, a_{j-1,2^{j-1}-1}\right\}.$$

The inverse transform is defined as follows

$$a_{j,l} = \sum_{k \in \mathbb{Z}} h_{l-2k}(j) a_{j-1,l} + g_{l-2k}(j) d_{j-1,l}$$

and reconstructs $\mathbf{A}(j)$ by $\mathbf{A}(j-1)$ and $\mathbf{D}(j-1)$. According to [12] in order to choose the filter $\mathbf{c}(j)$ to maximize $\left\|f_{V_{j-1}}\right\|^2$ in Equation (16), we must solve the following problem.

Problem 1. Let $U_N^{(1)}$ be the subset of the $2N$-dimensional Euclidean space $\mathbb{R}^{2N}$, which consists of the points $u = (u_0, u_1, \cdots, u_{2N-1})$ satisfying the conditions

$$\sum_{k=0}^{2N-1} u_k^2 = 1, \; \sum_{k=0}^{2N-l-1} u_k u_{2l+k} = 0. \tag{17}$$

for $l = 0, 1 \cdots, N-1$. Find a point u^* for which

$$\sum_{m,k=0}^{2N-1} u_m^* u_k^* F_{m,k} = \sup_{u \in U_N^{(1)}} \left\{\sum_{m,k=0}^{2N-1} u_m u_k F_{m,k}\right\}, \tag{18}$$

where $\left\|F_{m,k}\right\|$ is a $2N \times 2N$ symmetric matrix.

Problem 1 has a solution since U_N is a compact. But, as noted in [12], the numerical solution of this problem is not trivial even for $N = 2$.

Concerning the standard Haar and Daubechies (with 4 coefficients) discrete transforms see, e.g., [17]; we will denote them as SWTH and SWTD, respectively. We write NSWTH for the simplest case of a multiresolution analysis of rank 1 which is considered in [12, Sect. 3] (see also [13]). The nonstationary Daubechies discrete wavelet transform which corresponds an AMRA of rank N are defined in [12] and we will use the symbol NSWTDN to denote this transform (see NSWTD1 and NSWTD2 in the tables below).

Method A associated with one of the mentioned above discrete wavelet transforms (cf. [17, Chap.7]) consists of the following steps:

Step 1. Apply the discrete wavelet transform j times to an input array $\mathbf{A}(j)$ and get the sequence

$$\mathbf{A}(0), \mathbf{D}(0), \mathbf{D}(1), \cdots, \mathbf{D}(j-1).$$

Step 2. Allocate a certain percentage of the wavelet coefficients with lagest absolute value (we choose 10%) and nullify the remaining coefficients.

Step 3. Apply the inverse wavelet transform to the modified arrays of the wavelet coefficients.

Step 4. Calculate $\left\|\mathbf{A}(j) - \tilde{\mathbf{A}}(j)\right\|_2$, where $\mathbf{A}(j)$ is a reconstructed array.

In *Method B* the second step is replaced on the uniform quantization and the forth step is replaced on the calculation of the entropy of a vector, obtained in the third step.

We recall that $\mathbf{y} = \{y_1, \cdots, y_m\}$ is a *vector uniform quantization* for given vector $\mathbf{x} = (x_1, \cdots, x_m)$, if

$$y_j = \begin{cases} 0, \left|x_j\right| < \Delta, \\ \Delta\left[\dfrac{x_j}{\Delta}\right] + \operatorname{sign}(x_j)\dfrac{\Delta}{2}, \left|x_j\right| \geq \Delta, \end{cases}$$

where Δ is the length of the quantization interval.

The value Δ will be calculated by

$$\Delta = \left(\max_{1 \leq j \leq m} x_j - \min_{1 \leq j \leq m} x_j\right) \Big/ 50.$$

The Shannon entropy of $\mathbf{x}$ is defined by the formula

$$H(\mathbf{x}) = -\sum_{j=1}^{m} p_j \log_2(p_j),$$

where p_j is frequency of the value x_j.

Let us consider a similar approach associated with the following problem:

Problem 2. Let $N = 2^{n-1}$. Denote by $U_N^{(2)}$ the set of all points $u = (u_0, u_1, \cdots, u_{2N-1}) \in \mathbb{R}^{2N}$ such that

$$(u_l)^2 + (u_{l+N})^2 = 1, l = 0, 1, \cdots, N-1.$$

For every $u \in U_N^{(2)}$ we define

$$c_k(u) = \frac{1}{N}\sum_{j=0}^{2N-1} u_j w_j\left(k/(2N)\right)$$

for $k = 0, 1, \cdots, 2N-1$. Find a point u^* for which

$$\sum_{m,k=0}^{2N-1} c_m(u^*) c_k(u^*) F_{m,k} = \sup_{u \in U_N^{(2)}} \left\{\sum_{m,k=0}^{2N-1} c_m(u) c_k(u) F_{m,k}\right\}, \tag{19}$$

where $\left\|F_{m,k}\right\|$ is a $2N \times 2N$ symmetric matrix.

Given an array $\mathbf{A}(j) = \left\{a_{j,0}, a_{j,1} \cdots, a_{j,2^j-1}\right\}$, we define the matrix $\left\|F_{m,k}\right\|$ in Problem 1 and Problem 2 by

$$F_{m,k} = \sum_{s \in \mathbb{Z}} a_{j,2s+m} a_{j,2s+m}$$

and

$$F_{m,k} = \sum_{s \in \mathbb{Z}_+} a_{j,2s \oplus m} a_{j,2s \oplus m},$$

respectively. Here $a_{j,s} = 0$ for $s \notin \left\{0, 1, \cdots, 2^j - 1\right\}$. Suppose that u^* is a solution of Equation (19). Then the direct and inverse nonstationary discrete dyadic wavelet transforms are defined by

$$a_{j-1,k} = \sum_{l \in \mathbb{Z}_+} h_{l \oplus 2k}^{(j)} a_{j,l}, \quad d_{j-1,k} = \sum_{l \in \mathbb{Z}_+} g_{l \oplus 2k}^{(j)} a_{j,l},$$

$$a_{j,l} = \sum_{k \in \mathbb{Z}_+} h_{l \oplus 2k}^{(j)} a_{j-1,l} + g_{l \oplus 2k}^{(j)} d_{j-1,l},$$

where $h_k^{(j)} = c_k(u^*)/\sqrt{2}$ and $g_k^{(j)} = (-1)^k h_{1 \oplus k}^{(j)}$. We

Table 1. Values of the square error corresponding to Method A.

	SWTH	NSWTH	NSWTL1	SWTD	NSWTD1	NSWTD2	NSWTL2
$\mathcal{S}$	0.166547	0.123983	0.123980	0.248311	0.167071	0.128120	0.122886
$\mathcal{W}_{0.9,3}$	15.823238	14.802541	14.802635	14.290849	14.807025	14.275246	14.022471
$\mathcal{W}_{0.9,5}$	16.813738	15.932313	15.932307	15.378600	15.171461	14.782221	15.130797
$\mathcal{W}_{0.9,7}$	15.887306	13.631379	13.631383	15.595433	16.649683	12.724437	12.674001

Table 2. Values of the entropy obtained by Method B.

	SWTH	NSWTH	NSWTL1	SWTD	NSWTD1	NSWTD2	NSWTL2
$\mathcal{S}$	0.320865	0.327626	0.310639	0.863949	0.299818	0.304681	0.241210
$\mathcal{W}_{0.9,3}$	4.486757	3.810555	3.772764	4.152313	3.822598	3.525294	3.466450
$\mathcal{W}_{0.9,5}$	4.688737	3.874187	3.848227	4.224801	4.106692	3.766994	3.700762
$\mathcal{W}_{0.9,7}$	4.392570	3.371864	3.344916	4.001358	4.435942	3.232151	3.197167

denote these discrete transforms as NSWTL1 if $N = 1$ and as NSWTL2 if $N = 2$.

Let us recall that the Weierstrass function is defined as

$$\mathcal{W}_{\alpha,\beta}(x) = \sum_{n=1}^{\infty} \alpha^n \cos\left(\beta^n \pi x\right),\ 0 < \alpha < 1,\ \beta \geq \frac{1}{\alpha},$$

and the Swartz function is defined as

$$\mathcal{S}(x) = \sum_{n=1}^{\infty} \frac{h\left(2^n x\right)}{4^n},$$

where $h(x) = [x] - \sqrt{x - [x]}$. We will consider arrays $\mathbf{A}(8)$ with elements $a_{8,k} = \mathcal{W}_{\alpha,\beta}(k/128)$ or $a_{8,k} = \mathcal{S}(k/256)$, $k = 0, \cdots, 255$. Then we use the Matlab function fminsearch to solve the optimization problems in Equations (18) and (19). The results of these numerical experiments are presented in **Tables 1** and **2**. We see that in several cases the introduced nonstationary dyadic wavelets have an advantage over the classical Haar and Daubechies wavelets.

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