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Numerical Solution of Nonlinear Fredholm-Volterra Integtral Equations via Piecewise Constant Function by Collocation Method

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Abstract

In this work, we present a computational method for solving nonlinear Fredholm-Volterra integral equations of the second kind which is based on replacement of the unknown function by truncated series of well known Block-Pulse functions (BPfs) expansion. Error analysis is worked out that shows efficiency of the method. Finally, we also give some numerical examples.

Keywords: Nonlinear Fredholm-Volterra Integral Equation, Block-Pulse Function, Error Analysis, Collocation Points

1. Introduction

The integral equation method is widely used for solving many problems in mathematical physics and engineering. This article proposes a computational method for solving nonlinear Fredholm-Volterra integral equations. Several numerical methods for approximating the solution of linear and nonlinear integral equations and specially Fredholm-Volterra integral equations are known [1-10]. Also, Block-Pulse functions are studied by many authors and applied for solving different problems. In presented paper, by using vector forms of BPfs, the main problem can be easily reduced to a nonlinear system of algebraic equations which can be solved by Newton's iterative method.

2. Review of Some Related Papers

Some computational methods for approximating the solution of linear and nonlinear integral equations are known. The classical method of successive approximation for Fredholm-Hammerstein integral equations was introduced in [3]. Brunner in [4] applied a collocation type method and Ordokhani in [8] applied rationalized Haar function to nonlinear Volterra-Fredholm-Hammerstein integral equations. A variation of the Nystrom method was presented in [5]. A collocation type method was developed in [6]. The asymptotic error expansion of a collocation type method for volterra-Hammerstein integral equations has been considered in [7]. Yousefi in [9] applied Legendre wavelets to a special type of nonlinear Volterra-Fredholm integral equations of the form.

$$u(t) = f(t) + \lambda_1 \int_0^t K_1(t,x)F(u(x))\mathrm{d}x + \lambda_2 \int_0^1 K_2(t,x)G(u(x))\mathrm{d}x,\ 0 \le x,\ t \le 1, \tag{1}$$

where $f(t)$, and $K_1(t,\ x)$ and $K_2(t,\ x)$ are assumed to be in $L^2(R)$ on the interval $0 \le x$, $t \le 1$. Yalcinbas in [10] used Taylor polynomials for solving Equation (1) with $F(u) = u^p$ and $G(u) = u^q$. Orthogonal functions and polynomials receive attention in dealing with various problems that one of those in integral equation. The main characteristic of using orthogonal basis is that it reduces these problems to solving a system of nonlinear algebraic equations. The aim of this work is to present a numerical method for approximating the solution of nonlinear Fredholm-Volterra integral equation of the form:

$$u(t) = f(t) + \lambda_1 \int_0^t K_1(t,x)(u(x))^m \mathrm{d}x + \lambda_2 \int_0^1 K_2(t,x)(u(x))^n \mathrm{d}x,\ 0 \le x,\ t \le 1, \tag{2}$$

where m and n are nonnegative integers and λ_1 and λ_2 are constants. For this purpose we define a k-set of BPfs as

$$B_i(t) = \begin{cases} 1, & \dfrac{i-1}{k} \le t < \dfrac{i}{k},\ \text{for all } i = 1,2,\cdots,k \\ 0, & \text{elsewhere} \end{cases} \tag{3}$$

The functions $B_r(t)$ are disjoint and orthogonal. That is,

$$B_j(t)B_i(t) = \begin{cases} 0, & i \neq j \\ B_i(t), & i = j \end{cases} \tag{4}$$

$$< B_i(t)B_j(t) >= \begin{cases} 0, & i \neq j \\ \dfrac{1}{k}, & i = j \end{cases} \tag{5}$$

A function $u(t)$ defined over the interval [0, 1) may be expanded as:

$$u(t) = \sum_{i=1}^{\infty} u_i B_i(t) . \tag{6}$$

In practice, only *k*-term of (6) are considered, where k is a power of 2, that is,

$$u(t) \cong u_k(t) = \sum_{i=1}^{k} u_i B_i(t) , \tag{7}$$

with matrix from:

$$u(t) \cong u_k(t) = u^t B(t) , \tag{8}$$

where, $\mathbf{u} = [u_1,\ u_2, \cdots, u_k]^t$ and

$$\mathbf{B}(t) = [B_1(t),\ B_2(t), \cdots, B_k(t)]^t .$$

In a similar manner, $[u(t)]^m$ can be approximated in term of BPfs

$$[u(t)]^m \cong \tilde{\mathbf{u}}^t \mathbf{B}(t) ,$$

that we need to calculate vector $\tilde{u}$ whose elements are nonlinear combination of the elements of the vector $\mathbf{u}$ For this purpose, we can write

$$u(t) = \mathbf{u}^t \mathbf{B}(t) \quad \text{and} \quad [u(t)]^m = \tilde{\mathbf{u}}^t \mathbf{B}(t) .$$

So,

$$\tilde{\mathbf{u}}^t \mathbf{B}(t) = [\mathbf{u}^t \mathbf{B}(t)]^m \tag{9}$$

now using (4) leads to

$$\mathbf{B}(t)\mathbf{B}^t(t) = \begin{pmatrix} B_1(t) & & & 0 \\ & B_2(t) & & \\ & & \ddots & \\ 0 & & & B_k(t) \end{pmatrix}$$

also from (3) we get

$0 \le t < \dfrac{1}{k}$ implies that $B_1(t) = 1$ and $B_i(t) = 0$ for $i = 2, \cdots, k$.

$\dfrac{1}{k} \le t < \dfrac{2}{k}$ implies that $B_2(t) = 1$ and $B_i(t) = 0$ for $i = 1, \cdots, k$ and $i \neq 2$.

$\dfrac{k-1}{k} \le t < 1$ implies that $B_k(t) = 1$ and $B_i(t) = 0$ for $i = 1, \cdots, k-1$. Therefore, simply we obtain

$$\int_0^1 \mathbf{B}(t)\mathbf{B}^t(t)\mathrm{d}t = \frac{1}{k}\mathbf{I} , \tag{10}$$

where, $\mathbf{I}$ is the identity matrix of order k. By incorporating these results we have

$$\tilde{\mathbf{u}}^t = \tilde{\mathbf{u}}^t \mathbf{I} = k\int_0^1 \tilde{\mathbf{u}}^t \mathbf{B}(t)\mathbf{B}^t(t)\mathrm{d}t = k\int_0^1 [\mathbf{u}^t \mathbf{B}(\mathbf{t})]^m \mathbf{B}^t(t)\mathrm{d}t .$$

Hence,

$$\begin{aligned} \tilde{\mathbf{u}}^t &= k\int_0^1 [\mathbf{u}^t \mathbf{B}(t)]^m \mathbf{B}^t(t)\mathrm{d}t \\ &= k\sum_{i=1}^{k} \int_{\frac{i-1}{k}}^{\frac{i}{k}} [\mathbf{u}^t \mathbf{B}(t)]^m \mathbf{B}^t(t)\mathrm{d}t \\ &= k\sum_{i=1}^{k} \int_{\frac{i-1}{k}}^{\frac{i}{k}} [\mathbf{u}^t \mathbf{B}(t)]^{m-1} \mathbf{u}^t [\mathbf{B}(t)\mathbf{B}^t(t)]\mathrm{d}t \end{aligned} \tag{11}$$

So using (11) leads to

$$\tilde{\mathbf{u}}^t = k\int_0^{\frac{1}{k}} \left([u_1, u_2, \ldots, u_k] \begin{pmatrix} 1 \\ 0 \\ \ddots \\ 0 \end{pmatrix} \right)^{m-1} .$$

$$[u_1,\ u_2, \cdots, u_k] \begin{pmatrix} 1 & & & & 0 \\ & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix} \mathrm{d}t$$

$$+ k\int_{\frac{1}{k}}^{\frac{2}{k}} \left([u_1,\ u_2, \cdots, u_k] \begin{pmatrix} 0 \\ 1 \\ 0 \\ \ddots \\ 0 \end{pmatrix} \right)^{m-1} .$$

$$[u_1,\ u_2, \cdots, u_k] \begin{pmatrix} 0 & & & & 0 \\ & 1 & & & \\ & & 0 & & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix} \mathrm{d}t + \cdots$$

$$+ k\int_{\frac{k-1}{k}}^{1} \left([u_1,\ u_2, \cdots, u_k] \begin{pmatrix} 0 \\ 0 \\ \ddots \\ 0 \\ 1 \end{pmatrix} \right)^{m-1} .$$

$$[u_1,\ u_2, \cdots, u_k] \begin{pmatrix} 0 & & & & 0 \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ 0 & & & & 1 \end{pmatrix} \mathrm{d}t$$

$$= k\int_0^{\frac{1}{k}} u_1^{m-1}[u_1 0,\cdots,0]\mathrm{d}t + k\int_{\frac{1}{k}}^{\frac{2}{k}} u_2^{m-1}[0,\ u_2,\cdots,0]\mathrm{d}t + \cdots$$

$$+ k\int_{\frac{k-1}{k}}^{1} u_m^{m-1}[0,\cdots,0,\ u_m]\mathrm{d}t = [u_1^m,\ u_2^m,\cdots,u_m^m]$$

Now for evaluating the integral $\int_0^t \mathbf{B}(t)\mathbf{B}^t(t)\mathrm{d}t$ at the collocation points

$$t_j = \frac{j-\frac{1}{2}}{k},\quad j = 1,\ 2,\cdots,k, \tag{12}$$

we may proceed as follows

$$\int_0^{t_j}\mathbf{B}(t)\mathbf{B}^t(t)dt = \int_0^{\frac{j-1/2}{k}}\mathbf{B}(t)\mathbf{B}^t(t)\mathrm{d}t \int_0^{\frac{1}{k}}\mathbf{B}(t)\mathbf{B}^t(t)\mathrm{d}t + \int_{\frac{1}{k}}^{\frac{2}{k}}\mathbf{B}(t)\mathbf{B}^t(t)\mathrm{d}t + \cdots + \int_{\frac{j-2}{k}}^{\frac{j-1}{k}}\mathbf{B}(t)\mathbf{B}^t(t)\mathrm{d}t + \int_{\frac{j-1}{k}}^{\frac{j-1/2}{k}}\mathbf{B}(t)\mathbf{B}^t(t)\mathrm{d}t$$

$$= \begin{pmatrix} \int_0^{\frac{1}{k}} 1\mathrm{d}t & & 0 \\ & 0 & \\ & & \ddots \\ 0 & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & 0 \\ & \int_{\frac{1}{k}}^{\frac{2}{k}} 1\mathrm{d}t & \\ & 0 & \ddots \\ 0 & & 0 \end{pmatrix} + \cdots$$

$$+ \begin{pmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & 0 & & \\ & & \int_{\frac{j-2}{k}}^{\frac{j-1}{k}} 1\mathrm{d}t & & \\ & & 0 & & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & 0 & & \\ & & 0 & & \\ & & \int_{\frac{j-1}{k}}^{\frac{j-1/2}{k}} 1\mathrm{d}t & & \\ & & 0 & & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix} = \frac{1}{k}\mathbf{D}^j \tag{13}$$

where,

$$\mathbf{D}^j = Diag[1,\ 1,\cdots,\frac{1}{2},\ 0,\cdots,0]_{k\times k},$$

in fact, the diagonal matrix $\mathbf{D}^j$, $j = 1,\ 2,\cdots,k$ is defined as follows :

$$\mathbf{D}^j_{mn} = \begin{cases} 1, & m = n = 1, 2, \cdots j-1, \\ \frac{1}{2}, & m = n = j, \\ 0, & m = n = j+1,\cdots,k. \end{cases}$$

Also, $K(x,\ t) \in L^2[0,\ 1)^2$ may be approximated as:

$$K(x,\ t) \cong \sum_{i=1}^{k}\sum_{j=1}^{k} K_{ij}B_i(x)B_j(t),$$

or in matrix form

$$K(x,\ t) \cong \mathbf{B}^t(x)\mathbf{K}\mathbf{B}(t), \tag{14}$$

where $\mathbf{K} = [K_{ij}]_{1\le i,\, j\le k}$ and $K_{ij} = k^2\int_0^1\int_0^1 K(x,\ t)B_i(x)B_j(t)\mathrm{d}x\mathrm{d}t$.

3. Solution of the Nonlinear Fredholm-Volterra Integral Equations

In order to use BPfs for solving nonlinear Fredholm-Voterra integral equations given in Equation (2), we first approximate the $u(t)$, $f(t)$, $(u(x))^m$, $(u(x))^n$, $K_1(t,\ x)$ and $K_2(t,\ x)$ with respect to BPfs

$$u(t) \cong \mathbf{B}^t(t)\mathbf{u} \tag{15}$$

$$f(t) \cong \mathbf{B}^t(t)\mathbf{f} \tag{16}$$

$$(u(x))^m \cong \tilde{\mathbf{u}}_1^t\mathbf{B}(x) \tag{17}$$

$$(u(x))^n \cong \tilde{\mathbf{u}}_2^t\mathbf{B}(x) \tag{18}$$

$$K_1(t,\ x) \cong \mathbf{B}^t(t)\mathbf{K}_1\mathbf{B}(x) \tag{19}$$

$$K_2(t,\ x) \cong \mathbf{B}^t(t)\mathbf{K}_2\mathbf{B}(x) \tag{20}$$

where k-vectors $\mathbf{u}$, $\mathbf{f}$, $\tilde{\mathbf{u}}_1$, $\tilde{\mathbf{u}}_2$, and $k\times k$ matrices $\mathbf{K}_1$ and $\mathbf{K}_2$ are BPfs coefficients of $u(t)$, $f(t)$, $(u(x))^m$, $(u(x))^n$, $K_1(t,\ x)$ and $K_2(t,\ x)$ respectively. For solving Equation (2), we substitute (15-20) into (2), therefore

$$\mathbf{B}^t(t)\mathbf{u} = \mathbf{B}^t(t)\mathbf{f} + \lambda_1\mathbf{B}^t(t)\mathbf{K}_1\int_0^t\mathbf{B}(x)\mathbf{B}^t(x)\mathrm{d}x\tilde{\mathbf{u}}_1 + \lambda_2\mathbf{B}^t(t)\mathbf{K}_2\int_0^1\mathbf{B}(x)\mathbf{B}^t(x)\mathrm{d}x\tilde{\mathbf{u}}_2, \tag{21}$$

We now collocate Equation (21) at k points t_i, $j = 1,\ 2,\cdots,k$ defined by (12) as

$$\mathbf{B}^t(t_j)\mathbf{u} = \mathbf{B}^t(t_j)\mathbf{f} + \lambda_1\mathbf{B}^t(t_j)\mathbf{K}_1\int_0^{t_j}\mathbf{B}(x)\mathbf{B}^t(x)\mathrm{d}x\tilde{\mathbf{u}}_1 + \lambda_2\mathbf{B}^t(t_j)\mathbf{K}_2\int_0^1\mathbf{B}(x)\mathbf{B}^t(x)\mathrm{d}x\tilde{\mathbf{u}}_2 \tag{22}$$

by using (10) and (13) and the fact that $\mathbf{B}(t_j) = \mathbf{e}_j$ where, $\mathbf{e}_j$ is the j-th column of the identity matrix of order k, Equation (22) may then be restated as

$$u_j = f_i + \frac{\lambda_1}{k}\mathbf{e}_j^t\mathbf{K}_1\mathbf{D}^j\tilde{\mathbf{u}}_1 + \frac{\lambda_2}{k}\mathbf{e}_j^t\mathbf{K}_2\tilde{\mathbf{u}}_2,\quad j = 1,\ 2,\cdots,k. \tag{23}$$

Equation (23) gives k nonlinear equations which can

Table 1.

t	Exact	Approximate for $k=8$	Approximate for $k=16$
0.1	−1.99	−1.9847	−1.9876
0.2	−1.96	−1.9505	−1.9532
0.3	−1.91	−1.8857	−1.8905
0.4	−1.84	−1.7905	−1.8122
0.5	−1.75	−1.7650	−1.7666
0.6	−1.64	−1.6650	−1.6589
0.7	−1.51	−1.5091	−1.5080
0.8	−1.36	−1.3205	−1.3342
0.9	−1.19	−1.1103	−1.1297

Table 2.

t	Exact	Approximate for $k=8$	Approximate for $k=16$
0.1	0.0998	0.0625	0.0936
0.2	0.1986	0.1866	0.2070
0.3	0.2955	0.3078	0.2776
0.4	0.3894	0.4242	0.3952
0.5	0.4794	0.5139	0.5067
0.6	0.5646	0.5339	0.5596
0.7	0.6442	0.6353	0.6514
0.8	0.7173	0.7268	0.7243
0.9	0.7833	0.8069	0.7874

be solved for the elements $\tilde{\mathbf{u}}_1$ using Newton's iterative method.

4. Error in BPfs Approximation

Theorem. If a differentiable function $u(t)$ with bounded first derivative on (0, 1) is represented in a series of BPfs over subinterval $[\frac{i-1}{k}, \frac{i}{k})$, we have $\|e(t)\| = O(\frac{1}{k})$, where $e(t) = u_k(t) - u(t)$.

Proof. See [1].

5. Illustrative Examples

Consider the following nonlinear volterra-Fredholm integral equations.

Example 1.

$$u(t) = \frac{-1}{30}t^6 + \frac{1}{3}t^4 - t^2 + \frac{5}{3}t - \frac{5}{4} + \int_0^t (t-x)[u(x)]^2 dx + \int_0^1 (t+x)[u(x)]dx,$$

$$0 \le t, \quad x \le 1.$$

We applied the method presented in this paper for solving Equation (2) with $k=8$ and $k=16$.

The computational results together with the exact solution $u(t) = t^2 - 2$ are given in **Table 1**.

Example 2.

$$u(t) = \sin t + \frac{1}{8}\sin 2t - \frac{1}{4}t + \int_0^t \frac{1}{2}[u(x)]^2 \mathrm{d}x, \; 0 \le t, \; x \le 1.$$

The computational results with $k=8$ and $k=16$ together with the exact solution $u(t) = \sin t$ are given in **Table 2**.

6. Conclusions

The aim of present work is to apply a method for solving the nonlinear Volterra-Fredholm integral equations. The properties of the Block Pulse functions together with the collocation method are used to reduce the problem to the solution of nonlinear algebraic equations. Example 1 is solved in [2] using Chebyshev expansion method (Cem), comparing the results shows Cem is more accurate than BPfs method But, it seems the number of calculations of BPfs method is lower. Also, the benefits of this method are low cost of setting up the equations due to properties of BPfs mentioned in Section 2. In addition, the nonlinear system of algebraic equations is sparse. Finally, this method can be easily extended and applied to nonlinear Volterra-Fredholm integral equations of the form Equation (1). Illustrative examples are included to demonstrate the validity and applicability of the technique.

7. References

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