

On the Location of Zeros of Polynomials

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Abstract

In this paper, we prove some extensions and generalizations of the classical Eneström-Kakeya theorem.

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1. Introduction and Statement of Results

Let $P(z) := \sum_{j=0}^n a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq a_{n-2} \geq \dots \geq a_1 \geq a_0 \geq 0,$$

then according to a classical result usually known as Eneström-Kakeya theorem [11], $P(z)$ does not vanish in $|z| > 1$. Applying this result to the polynomial $P(tz)$, the following more general result is immediate.

Theorem A. If $P(z) := \sum_{j=0}^n a_j z^j$ is a polynomial of degree n such that for some $t > 0$

$$a_n t^n \geq a_{n-1} t^{n-1} \geq a_{n-2} t^{n-2} \geq \dots \geq a_1 t \geq a_0 \geq 0,$$

then $P(z)$ has all the zeros in $|z| \leq t$.

In the literature, [1-15], there exist extensions and generalizations of Eneström-Kakeya theorem. Joyal, Labelle and Rahman [9] extended this theorem to polynomials whose coefficients are monotonic but not necessarily non negative and the result was further generalized by Dewan and Bidkham [6] to read as:

Theorem B. If $P(z) := \sum_{j=0}^n a_j z^j$ is a polynomial of degree n such that for some $t > 0$ and $0 < \lambda \leq n$,

$$a_n t^n \leq a_{n-1} t^{n-1} \leq \dots \leq a_\lambda t^\lambda \geq a_{\lambda-1} t^{\lambda-1} \geq \dots \geq t a_1 \geq a_0,$$

then $P(z)$ has all the zeros in the circle

$$|z| \leq \frac{t}{|a_n|} \left\{ \left(\frac{2a_\lambda}{t^{n-\lambda}} - a_n \right) + \frac{1}{t^n} (|a_0| - a_0) \right\}.$$

Govil and Rahman [8] extended Theorem A to the polynomials with complex coefficients. As a refinement of the result of Govil and Rahman, Govil and Jain [7]

proved the following.

Theorem C. Let $P(z) := \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with complex coefficients such that for some β

$$|\arg a_k - \beta| \leq \alpha \leq \pi/2, \quad k = 0, 1, \dots, n$$

and

$$|a_n| \geq |a_{n-1}| \geq |a_{n-2}| \geq \dots \geq |a_1| \geq |a_0|,$$

then $P(z)$ has all its zeros in the ring-shaped region given by

$$R_3 \leq |z| \leq R_2.$$

Here

$$R_2 = \frac{c}{2} \left(\frac{1}{|a_n|} - \frac{1}{M_1} \right) + \left\{ \frac{c^2}{4} \left(\frac{1}{|a_n|} - \frac{1}{M_1} \right)^2 + \frac{M_1}{|a_n|} \right\}^{\frac{1}{2}},$$

$$R_3 = \frac{1}{2M_2^2} \left[-R_2^2 |b|(M_2 - |a_0|) + \{4|a_0|R_2^2 M_2^3 + R_2^4 |b|^2 (M_2 - |a_0|)^2\}^{\frac{1}{2}} \right],$$

where

$$M_1 = |a_n| R_2,$$

$$M_2 = |a_n| R_2^2 \left[R + R_2 - \frac{|a_0|}{|a_n|} (\cos \alpha + \sin \alpha) \right],$$

$$c = |a_n - a_{n-1}|,$$

$$b = a_1 - a_0$$

and

$$R = \cos \alpha + \sin \alpha + \frac{2 \sin \alpha}{|a_n|} \sum_{k=0}^{n-1} |a_k|$$

By using Schwarz's Lemma, Aziz and Mohammad [1] generalized Eneström-Kakeya theorem in a different way and proved:

Theorem D. Let $P(z) := \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with real positive coefficients. If $t_1 > t_2 \geq 0$ can be found such that

$$a_r t_1 t_2 + a_{r-1} (t_1 - t_2) - a_{r-2} \geq 0, r = 1, 2, \dots, n+1$$

where $a_{-1} = a_{n+1} = 0$, then all the zeros of $P(z)$ lie in $|z| \leq t_1$.

In this paper, we also make use of a generalized form of Schwarz's Lemma and prove some more general results which include not only the above theorems as special cases, but also lead to a standard development of

interesting generalizations of some well known results. Infact we prove

Theorem 1. Let $P(z) := \sum_{j=0}^n \alpha_j z^j$ be a polynomial of degree n such that

$\alpha_j = a_j + ib_j$ where a_j and $b_j, j = 0, 1, \dots, n$ are real numbers and for certain non negative real numbers t_1, t_2 with $t_1 \geq t_2$ and $t_1 \neq 0$

$$a_r t_1 t_2 + a_{r-1} (t_1 - t_2) - a_{r-2} \geq 0$$

$$b_r t_1 t_2 + b_{r-1} (t_1 - t_2) - b_{r-2} \geq 0, r = 1, 2, \dots, n+1$$

$$a_{-1} = a_{n+1} = 0 = b_{-1} = b_{n+1},$$

then all the zeros of $P(z)$ lie in

$$\min \left(r_2, \frac{1}{t_1} \right) \leq |z| \leq \max (r_1, t_1).$$

Here

$$r_1 = \frac{-\left(|\alpha_n| - K_1\right) \left|\alpha_n (t_1 - t_2) - \alpha_{n-1}\right| + \left\{\left(|\alpha_n| - K_1\right)^2 \left|\alpha_n (t_1 - t_2) - \alpha_{n-1}\right|^2 + 4K_1^3 t_1^2 \left|\alpha_n\right|\right\}^{\frac{1}{2}}}{2K_1 \left|\alpha_n\right|},$$

$$r_2 = \frac{-\left(|\alpha_0| t_1 t_2 - K_2\right) \left|\alpha_1 t_1 t_2 + \alpha_0 (t_1 - t_2)\right| t_1^2 + \left\{\left(|\alpha_0| t_1 t_2 - K_2\right)^2 \left|\alpha_1 t_1 t_2 + \alpha_0 (t_1 - t_2)\right|^2 t_1^4 + 4K_2^3 \left|\alpha_0\right| t_1^3 t_2\right\}^{\frac{1}{2}}}{2K_2^2},$$

where

$$K_1 = (a_n + b_n) + \left(|a_0| - a_0\right) \frac{t_2}{t_1^{n+1}} + \left(|b_0| - b_0\right) \frac{t_2}{t_1^{n+1}},$$

$$K_2 = (a_n + b_n) t_1^{n+2} + \left(|a_n| + |b_n|\right) t_1^{n+2} - (a_0 + b_0) t_1 t_2.$$

Assuming that all the coefficients $\alpha_j, j = 0, 1, \dots, n$ are real, the following result is immediate:

Corollary 1. Let $P(z) := \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with real coefficients such that for certain

non negative real numbers t_1, t_2 with $t_1 \geq t_2$ and $t_1 \neq 0$

$$a_r t_1 t_2 + a_{r-1} (t_1 - t_2) - a_{r-2} \geq 0,$$

$$r = 1, 2, \dots, n+1, \quad (a_{-1} = a_{n+1} = 0),$$

then all the zeros of $P(z)$ lie in

$$\min \left(r_2, \frac{1}{t_1} \right) \leq |z| \leq \max (r_1, t_1).$$

Here

$$r_1 = \frac{-\left(|a_n| - K_1\right) \left|a_n (t_1 - t_2) - a_{n-1}\right| + \left\{\left(|a_n| - K_1\right)^2 \left|a_n (t_1 - t_2) - a_{n-1}\right|^2 + 4K_1^3 t_1^2 \left|a_n\right|\right\}^{\frac{1}{2}}}{2K_1 \left|a_n\right|},$$

$$r_2 = \frac{-\left(|a_0| t_1 t_2 - K_2\right) \left|a_1 t_1 t_2 + a_0 (t_1 - t_2)\right| t_1^2 + \left\{\left(|a_0| t_1 t_2 - K_2\right)^2 \left|a_1 t_1 t_2 + a_0 (t_1 - t_2)\right|^2 t_1^4 + 4K_2^3 \left|a_0\right| t_1^3 t_2\right\}^{\frac{1}{2}}}{2K_2^2},$$

where

$$K_1 = a_n - a_0 \frac{t_2}{t_1^{n+1}} + \left|a_0\right| \frac{t_2}{t_1^{n+1}}, K_2 = \left|a_n\right| t_1^{n+2} + a_n t_1^{n+2} - a_0 t_1 t_2.$$

If in Corollary 1, we assume that all the coefficients are positive and $t_2 = 0$, then we have the following:

Corollary 2. Let $P(z) := \sum_{j=0}^n a_j z^j$ be a polynomial

of degree n such that for some real number $t > 0$

$$a_n t^n \geq a_{n-1} t^{n-1} \geq a_{n-2} t^{n-2} \geq \dots \geq a_0 > 0,$$

then all the zeros of $P(z)$ lie in

$$\min \left(\frac{a_0}{2a_n t^{n-1}}, \frac{1}{t} \right) \leq |z| \leq t.$$

In particular, if $t = 1$, Corollary 2 gives the following improvement of Eneström-Kakeya theorem.

Corollary 3. Let $P(z) := \sum_{j=0}^n a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq a_{n-2} \geq \dots \geq a_0 > 0,$$

then all the zeros of $P(z)$ lie in

$$\frac{a_0}{2a_n} \leq |z| \leq 1.$$

We next prove the following more general result which include many known results as special cases.

Theorem 2. Let $P(z) := \sum_{j=0}^n \alpha_j z^j$ be a polynomial of degree n such that $\alpha_j = a_j + ib_j$ where a_j and b_j , $j = 0, 1, \dots, n$ are real numbers. If $t_1 > t_2 \geq 0$ can be found such that for a certain integer λ , $0 < \lambda \leq n$

$$a_r t_1 t_2 + a_{r-1} (t_1 - t_2) - a_{r-2} \begin{cases} \geq 0, r = 2, 3, \dots, \lambda + 1 \\ \leq 0, r = \lambda + 2, \lambda + 3, \dots, n + 1 \end{cases}$$

$$b_r t_1 t_2 + b_{r-1} (t_1 - t_2) - b_{r-2} \begin{cases} \geq 0, r = 2, 3, \dots, \lambda + 1 \\ \leq 0, r = \lambda + 2, \lambda + 3, \dots, n + 1 \end{cases}$$

$$a_{n+1} = 0 = b_{n+1},$$

then all the zeros of $P(z)$ lie in

$$|z| \leq R, \tag{1}$$

where

$$R = \frac{t_1}{|\alpha_n|} \left\{ \left(\frac{2a_\lambda}{t_1^{n-\lambda}} - a_n \right) + \frac{1}{t_1^n} (|a_0| - a_0) \right\} + \frac{t_2}{|\alpha_n|} \left\{ \frac{2a_{\lambda+1}}{t_1^{n-\lambda-1}} + \frac{1}{t_1^n} (|a_0| - a_0) \right\} + \frac{t_1}{|\alpha_n|} \left\{ \left(\frac{2b_\lambda}{t_1^{n-\lambda}} - b_n \right) + \frac{1}{t_1^n} (|b_0| - b_0) \right\} + \frac{t_2}{|\alpha_n|} \left\{ \frac{2b_{\lambda+1}}{t_1^{n-\lambda-1}} + \frac{1}{t_1^n} (|b_0| - b_0) \right\}$$

Remark 1. Theorem B is a special case of Theorem 2, if we take $t_2 = 0$ and assume that all the coefficients α_j , $j = 0, 1, 2, \dots, n$ are real.

The following result follows immediately from Theorem 2 by taking $\lambda = n$ and assuming α_j , $j = 0, 1, 2, \dots, n$ to be a real.

Corollary 4. Let $P(z) := \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with real coefficients. If $t_1 > t_2 \geq 0$ can be found such that

$$a_r t_1 t_2 + a_{r-1} (t_1 - t_2) - a_{r-2} \geq 0, r = 2, 3, \dots, n + 1,$$

then all the zeros of $P(z)$ lie in

$$|z| \leq \frac{1}{|a_n| t_1^n} \left[t_1 (a_n t_1^n + |a_0| - a_0) + t_2 (|a_0| - a_0) \right].$$

Remark 2. For $t_1 = 1$ and $t_2 = 0$, Corollary 4 reduces to a result of Joyal, Labelle and Rahman [9].

We also prove the following result which is of independent interest.

Theorem 3. Let $P(z) := \sum_{j=0}^n \alpha_j z^j$ be a polynomial of degree n such that $\alpha_j = a_j + ib_j$ where a_j and b_j , $j = 0, 1, \dots, n$ are real numbers. If $t_1 > t_2 \geq 0$ can be found such that for a certain integer λ , $1 < \lambda \leq n - 1$

$$a_r t_1 t_2 + a_{r-1} (t_1 - t_2) - a_{r-2} \begin{cases} \geq 0, r = 2, 3, \dots, \lambda + 1 \\ \leq 0, r = \lambda + 2, \lambda + 3, \dots, n + 1 \end{cases}$$

and

$$b_r t_1 t_2 + b_{r-1} (t_1 - t_2) - b_{r-2} \begin{cases} \geq 0, r = 2, 3, \dots, \lambda + 1 \\ \leq 0, r = \lambda + 2, \lambda + 3, \dots, n + 1 \end{cases}$$

$$(a_{-1} = a_{-2} = a_{n+1} = 0 = b_{-1} = b_{-2} = b_{n+1}),$$

then all the zeros of $P(z)$ lie in

$$\left| z + \frac{\alpha_{n-1} - (t_1 - t_2)}{\alpha_n} \right| \leq R_1, \tag{2}$$

where

$$R_1 = \frac{t_1}{|\alpha_n|} \left\{ \left(\frac{2a_\lambda}{t_1^{n-\lambda}} - \frac{a_{n-1}}{t_1} \right) + \frac{1}{t_1^n} (|a_0| - a_0) \right\} + \frac{t_2}{|\alpha_n|} \left\{ \left(\frac{2a_{\lambda+1}}{t_1^{n-\lambda-1}} - a_n \right) + \frac{1}{t_1^n} (|a_0| - a_0) \right\} + \frac{t_1}{|\alpha_n|} \left\{ \left(\frac{2b_\lambda}{t_1^{n-\lambda}} - \frac{b_{n-1}}{t_1} \right) + \frac{1}{t_1^n} (|b_0| - b_0) \right\} + \frac{t_2}{|\alpha_n|} \left\{ \left(\frac{2b_{\lambda+1}}{t_1^{n-\lambda-1}} - b_n \right) + \frac{1}{t_1^n} (|b_0| - b_0) \right\}$$

Remark 3. Theorem 4 of [4] immediately follows from Theorem 3 when $t_1 = 1$, $t_2 = 0$ and the coefficients α_j , $j = 0, 1, \dots, n$ are real.

On combining Theorem 2 and Theorem 3 the following more interesting result is immediate.

Corollary 5. Let $P(z) := \sum_{j=0}^n a_j z^j$ be a polynomial of degree n such that $\alpha_j = a_j + ib_j$ where a_j and b_j , $j = 0, 1, \dots, n$ are real. If $t_1 > t_2 \geq 0$ can be found such that for a certain integer λ , $\lambda \leq n - 1$

$$a_r t_1 t_2 + a_{r-1} (t_1 - t_2) - a_{r-2} \begin{cases} \geq 0, r = 2, 3, \dots, \lambda + 1 \\ \leq 0, r = \lambda + 2, \lambda + 3, \dots, n + 1 \end{cases}$$

$$b_r t_1 t_2 + b_{r-1} (t_1 - t_2) - b_{r-2} \begin{cases} \geq 0, r = 2, 3, \dots, \lambda + 1 \\ \leq 0, r = \lambda + 2, \lambda + 3, \dots, n + 1 \end{cases} \quad a_{n+1} = 0 = b_{n+1},$$

then all the zeros of $P(z)$ lie in the intersection of the two circles given by (1) and (2).

If we take $\lambda = n - 1$ and the coefficients $\alpha_j, j = 0, 1, \dots, n$ are real in Theorem 3, we get the following result.

Corollary 6. Let $P(z) := \sum_{j=0}^n a_j z^j$ be a polynomial

$$\left| z + \frac{a_{n-1}}{a_n} - (t_1 - t_2) \right| \leq \frac{1}{|a_n|} \left[t_2 a_n + a_{n-1} + \frac{1}{t_1^n} (|a_0| - a_0) (t_1 + t_2) \right].$$

The following result also follows from Theorem 3, when $\lambda = n - 1$, the coefficients $\alpha_j, j = 0, 1, 2, \dots, n$ are real and $t_2 = 0$.

Corollary 7. Let $P(z) := \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with real coefficients. If for some $t > 0$,

$$a_n t^n \geq a_{n-1} t^{n-1} \geq a_{n-2} t^{n-2} \geq \dots \geq a_1 t \geq a_0,$$

then $P(z)$ has all the zeros in

$$\left| z + \frac{a_{n-1}}{a_n} - t \right| \leq \frac{1}{|a_n|} \left\{ a_{n-1} + \frac{(|a_0| - a_0)}{t^{n-1}} \right\}.$$

2. Lemmas

For proving the above theorems, we require the following lemmas. The first Lemma which we need is due to Rahman and Schmeisser [11].

Lemma 1. If $f(z)$ is analytic in $|z| \leq 1, f(0) = a$, where $|a| < 1, f'(0) = b, |f(z)| \leq 1$ on $|z| \leq 1$, then for $|z| \leq 1$,

$$\begin{aligned} F(z) &= (t_2 + z)(t_1 - z)P(z) = -\alpha_n z^{n+2} + (\alpha_n(t_1 - t_2) - \alpha_{n-1})z^{n+1} + (\alpha_n t_1 t_2 + \alpha_{n-1}(t_1 - t_2) - \alpha_{n-2})z^n + \dots \\ &\quad + (\alpha_2 t_1 t_2 + \alpha_1(t_1 - t_2) - \alpha_0)z^2 + (\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2))z + \alpha_0 t_1 t_2 \\ &= -a_n z^{n+2} + (a_n(t_1 - t_2) - a_{n-1})z^{n+1} + \dots + (a_1 t_1 t_2 + a_0(t_1 - t_2))z + a_0 t_1 t_2 \\ &\quad + i[-b_n z^{n+2} + (b_n(t_1 - t_2) - b_{n-1})z^{n+1} + \dots + (b_1 t_1 t_2 + b_0(t_1 - t_2))z + b_0 t_1 t_2]. \end{aligned} \tag{4}$$

Further, let

$$\begin{aligned} G(z) &= z^{n+2} F\left(\frac{1}{z}\right) = -\alpha_n + (\alpha_n(t_1 - t_2) - \alpha_{n-1})z + (\alpha_n t_1 t_2 + \alpha_{n-1}(t_1 - t_2) - \alpha_{n-2})z^2 + \dots \\ &\quad + (\alpha_2 t_1 t_2 + \alpha_1(t_1 - t_2) - \alpha_0)z^n + (\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2))z^{n+1} + \alpha_0 t_1 t_2 z^{n+2} \\ &= -a_n + (a_n(t_1 - t_2) - a_{n-1})z + (a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2})z^2 + \dots + (a_2 t_1 t_2 + a_1(t_1 - t_2) - a_0)z^n \\ &\quad + (a_1 t_1 t_2 + a_0(t_1 - t_2))z^{n+1} + a_0 t_1 t_2 z^{n+2} + i[-b_n + (b_n(t_1 - t_2) - b_{n-1})z + (b_n t_1 t_2 + b_{n-1}(t_1 - t_2) - b_{n-2})z^2 + \dots \\ &\quad + (b_2 t_1 t_2 + b_1(t_1 - t_2) - b_0)z^n + (b_1 t_1 t_2 + b_0(t_1 - t_2))z^{n+1} + b_0 t_1 t_2 z^{n+2}] = -\alpha_n + \psi(z) \end{aligned} \tag{5}$$

of degree n with real coefficients. If $t_1 > t_2 \geq 0$ can be found such that

$$a_r t_1 t_2 + a_{r-1} (t_1 - t_2) - a_{r-2} \begin{cases} \geq 0, r = 2, 3, \dots, n \\ \leq 0, r = n + 1 \end{cases}$$

then all the zeros of $P(z)$ lie in

$$|f(z)| \leq \frac{(1 - |a|)|z|^2 + |b||z| + |a|(1 - |a|)}{|a|(1 - |a|)|z|^2 + |b||z| + (1 - |a|)}.$$

From Lemma 1, one can easily deduce the following :

Lemma 2. If $f(z)$ is analytic in $|z| \leq R, f(0) = 0, f'(0) = b$ and $|f(z)| \leq M$ for $|z| = R$, then

$$|f(z)| \leq \frac{M|z|}{R^2} \frac{M|z| + R^2|b|}{M + |b||z|}, \quad |z| \leq R.$$

The next Lemma is due to Aziz and Mohammad [2].

Lemma 3. Let $f(z) := a_n z^n + \sum_{j=0}^p a_j z^j, 0 \leq p \leq n - 1$ be a polynomial of degree n with complex coefficients. Then for every positive real number r , all the zeros of $f(z)$ lie in the disk

$$|z| \leq \max \left\{ r, \sum_{k=0}^p \frac{|a_k|}{|a_n| r^{n-k-1}} \right\}. \tag{3}$$

3. Proofs of the Theorems

Proof of Theorem 1. Consider the polynomial

where

$$\begin{aligned} \psi(z) = & (a_n(t_1 - t_2) - a_{n-1})z + (a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2})z^2 + \dots \\ & + (a_2 t_1 t_2 + a_1(t_1 - t_2) - a_0)z^n + (a_1 t_1 t_2 + a_0(t_1 - t_2))z^{n+1} + a_0 t_1 t_2 z^{n+2} \\ & + i \left[(b_n(t_1 - t_2) - b_{n-1})z + (b_n t_1 t_2 + b_{n-1}(t_1 - t_2) - b_{n-2})z^2 + \dots \right. \\ & \left. + (b_2 t_1 t_2 + b_1(t_1 - t_2) - b_0)z^n + (b_1 t_1 t_2 + b_0(t_1 - t_2))z^{n+1} + b_0 t_1 t_2 z^{n+2} \right]. \end{aligned}$$

Now

$$\begin{aligned} |\psi(z)| \leq & |(a_n(t_1 - t_2) - a_{n-1})z| + |(a_n t_1 t_2 + a_{n-1}(t_1 - t_2) - a_{n-2})z^2| + \dots + |(a_2 t_1 t_2 + a_1(t_1 - t_2) - a_0)z^n| \\ & + |(a_1 t_1 t_2 + a_0(t_1 - t_2))z^{n+1}| + |a_0 t_1 t_2 z^{n+2}| + |(b_n(t_1 - t_2) - b_{n-1})z| + |(b_n t_1 t_2 + b_{n-1}(t_1 - t_2) - b_{n-2})z^2| + \dots \\ & + |(b_2 t_1 t_2 + b_1(t_1 - t_2) - b_0)z^n| + |(b_1 t_1 t_2 + b_0(t_1 - t_2))z^{n+1}| + |b_0 t_1 t_2 z^{n+2}| \end{aligned}$$

This gives after using hypothesis, for $|z| = 1/t_1$

$$|\psi(z)| \leq (a_n + b_n) + (|a_0| - a_0) \frac{t_2}{t_1^{n+1}} + (|b_0| - b_0) \frac{t_2}{t_1^{n+1}} = K_1 \text{ (Say),}$$

Clearly, $\psi(0) = 0, \psi'(0) = \alpha_n(t_1 - t_2) - \alpha_{n-1}$ and

$$|\psi(z)| \leq K_1 \text{ for } |z| = 1/t_1$$

Thus, it follows by Lemma 2 that

$$|\psi(z)| \leq \frac{K_1 |z|}{\frac{1}{t_1^2}} \left[\frac{K_1 |z| + \frac{1}{t_1^2} |\alpha_n(t_1 - t_2) - \alpha_{n-1}|}{K_1 + |\alpha_n(t_1 - t_2) - \alpha_{n-1}| |z|} \right], |z| \leq \frac{1}{t_1}.$$

From (5), we get

$$\begin{aligned} |G(z)| & \geq |\alpha_n| - \frac{K_1 |z|}{\frac{1}{t_1^2}} \left[\frac{K_1 |z| + \frac{1}{t_1^2} |\alpha_n(t_1 - t_2) - \alpha_{n-1}|}{K_1 + |\alpha_n(t_1 - t_2) - \alpha_{n-1}| |z|} \right] \\ & = \frac{K_1 |\alpha_n| + |\alpha_n| |\alpha_n(t_1 - t_2) - \alpha_{n-1}| |z| - K_1^2 t_1^2 |z|^2 - K_1 |z| |\alpha_n(t_1 - t_2) - \alpha_{n-1}|}{K_1 + |\alpha_n(t_1 - t_2) - \alpha_{n-1}| |z|} \\ & = \frac{-K_1^2 t_1^2 |z|^2 + (|\alpha_n| - K_1) |\alpha_n(t_1 - t_2) - \alpha_{n-1}| |z| + K_1 |\alpha_n|}{K_1 + |\alpha_n(t_1 - t_2) - \alpha_{n-1}| |z|} > 0, \end{aligned}$$

if

$$K_1^2 t_1^2 |z|^2 - (|\alpha_n| - K_1) |\alpha_n(t_1 - t_2) - \alpha_{n-1}| |z| - K_1 |\alpha_n| < 0.$$

This gives $|G(z)| > 0$, if

$$|z| < \frac{-(|\alpha_n| - K_1) |\alpha_n(t_1 - t_2) - \alpha_{n-1}| + \left\{ (|\alpha_n| - K_1)^2 |\alpha_n(t_1 - t_2) - \alpha_{n-1}|^2 + 4K_1^3 t_1^2 |\alpha_n| \right\}^{\frac{1}{2}}}{2K_1^2 t_1^2} = \frac{1}{r_1}.$$

Consequently, all the zeros of $G(z)$ lie in

$$|z| \geq \min \left(\frac{1}{r_1}, \frac{1}{t_1} \right).$$

of $F(z)$ and hence all the zeros of $P(z)$ lie in

$$|z| \leq \max(r_1, t_1). \tag{6}$$

Again from (4)

Since $F(z) = z^{n+2}G(1/z)$, it follows that all the zeros

$$|F(z)| \geq |\alpha_0| t_1 t_2 - |\phi(z)|, \tag{7}$$

where

$$|\phi(z)| \leq |a_n| |z|^{n+2} + |(a_n(t_1 - t_2) - a_{n-1})| |z|^{n+1} + \dots + |(a_1 t_1 t_2 + a_0(t_1 - t_2))| |z| + |b_n| |z|^{n+2} \\ + |(b_n(t_1 - t_2) - b_{n-1})| |z|^{n+1} + \dots + |(b_1 t_1 t_2 + b_0(t_1 - t_2))| |z|.$$

Therefore, for $|z| = t_1$, we have by using the hypothesis

$$|\phi(z)| \leq |a_n| |t_1|^{n+2} + |(a_n(t_1 - t_2) - a_{n-1})| |t_1|^{n+1} + \dots \\ + |(a_1 t_1 t_2 + a_0(t_1 - t_2))| |t_1| + |b_n| |t_1|^{n+2} + |(b_n(t_1 - t_2) - b_{n-1})| |t_1|^{n+1} + \dots + |(b_1 t_1 t_2 + b_0(t_1 - t_2))| |t_1| \\ = (a_n + b_n) t_1^{n+2} + (|a_n| + |b_n|) t_1^{n+2} - (a_0 + b_0) t_1 t_2 = K_2 \text{ (say) for } |z| = t_1$$

Therefore, it follows again by Lemma 2 that

$$|\phi(z)| \leq \frac{K_2 |z|}{t_1^2} \left[\frac{K_2 |z| + t_1^2 |\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)|}{K_2 + |\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)| |z|} \right].$$

Using this result in (7), we get

$$|F(z)| \geq |\alpha_0| t_1 t_2 - \frac{K_2 |z|}{t_1^2} \left[\frac{K_2 |z| + t_1^2 |\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)|}{K_2 + |\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)| |z|} \right] \\ = \frac{|\alpha_0| t_1^3 t_2 [K_2 + |\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)| |z|] - K_2^2 |z|^2 - K_2 t_1^2 |\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)| |z|}{t_1^2 (K_2 + |\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)| |z|)} \\ = \frac{-K_2^2 |z|^2 - [K_2 - |\alpha_0| t_1 t_2] |\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)| t_1^2 |z| + K_2 |\alpha_0| t_1^3 t_2}{t_1^2 (K_2 + |\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)| |z|)} > 0,$$

if

$$K_2^2 |z|^2 + [|\alpha_0| t_1 t_2 - K_2] |\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)| t_1^2 |z| - K_2 |\alpha_0| t_1^3 t_2 < 0.$$

Thus $|F(z)| > 0$, if

$$|z| < \frac{-[|\alpha_0| t_1 t_2 - K_2] |\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)| t_1^2 + \left\{ [|\alpha_0| t_1 t_2 - K_2]^2 |\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)|^2 t_1^4 + 4K_2^3 |\alpha_0| t_1^3 t_2 \right\}^{\frac{1}{2}}}{2K_2^2} = r_2.$$

This shows that all the zeros of $F(z)$ and hence of the polynomial $P(z)$ lie in

$$|z| \geq \min \left(r_2, \frac{1}{t_1} \right). \quad (8)$$

Combining (6) and (8), we get the desired result.

Proof of Theorem 2. Consider the polynomial

$$f(z) = (t_2 + z)(t_1 - z)P(z) \\ = -\alpha_n z^{n+2} + \sum_{r=0}^{n+1} (\alpha_r t_1 t_2 + \alpha_{r-1}(t_1 - t_2) - \alpha_{r-2}) z^r.$$

Since $f(z)$ is a polynomial of degree $n + 2$, it follows by applying Lemma 3 to $f(z)$ with $p = n + 1$ and $r = t_1$, that all the zeros of $f(z)$ lie in

$$|z| \leq \max \left\{ t_1, \sum_{r=0}^{n+1} \frac{|\alpha_r t_1 t_2 + \alpha_{r-1}(t_1 - t_2) - \alpha_{r-2}|}{|\alpha_n| t_1^{n-r+1}} \right\}. \quad (9)$$

Now

$$\begin{aligned}
 t_1 &= \left| \frac{\sum_{r=0}^{n+1} (\alpha_r t_1 t_2 + \alpha_{r-1} (t_1 - t_2) - \alpha_{r-2})}{|\alpha_n| t_1^{n-r+1}} \right| = \left| \frac{\sum_{r=0}^{n+1} (a_r t_1 t_2 + a_{r-1} (t_1 - t_2) - \alpha_{r-2} + i(b_r t_1 t_2 + b_{r-1} (t_1 - t_2) - b_{r-2}))}{|\alpha_n| t_1^{n-r+1}} \right| \\
 &\leq \sum_{r=0}^{n+1} \frac{|(a_r t_1 t_2 + a_{r-1} (t_1 - t_2) - \alpha_{r-2})|}{|\alpha_n| t_1^{n-r+1}} + \sum_{r=0}^{n+1} \frac{|(b_r t_1 t_2 + b_{r-1} (t_1 - t_2) - b_{r-2})|}{|\alpha_n| t_1^{n-r+1}} \\
 &= \frac{|a_0| t_1 t_2}{|\alpha_n| t_1^{n+1}} + \frac{|a_1 t_1 t_2 + a_0 (t_1 - t_2)|}{|\alpha_n| t_1^{n+1}} + \sum_{r=2}^{\lambda+1} \frac{|(a_r t_1 t_2 + a_{r-1} (t_1 - t_2) - \alpha_{r-2})|}{|\alpha_n| t_1^{n-r+1}} + \sum_{r=\lambda+2}^{n+1} \frac{|(a_r t_1 t_2 + a_{r-1} (t_1 - t_2) - \alpha_{r-2})|}{|\alpha_n| t_1^{n-r+1}} \\
 &\quad + \left[\frac{|b_0| t_1 t_2}{|\alpha_n| t_1^{n+1}} + \frac{|b_1 t_1 t_2 + b_0 (t_1 - t_2)|}{|\alpha_n| t_1^{n+1}} + \sum_{r=2}^{\lambda+1} \frac{|(b_r t_1 t_2 + b_{r-1} (t_1 - t_2) - b_{r-2})|}{|\alpha_n| t_1^{n-r+1}} + \sum_{r=\lambda+2}^{n+1} \frac{|(b_r t_1 t_2 + b_{r-1} (t_1 - t_2) - b_{r-2})|}{|\alpha_n| t_1^{n-r+1}} \right] \\
 &\leq \frac{|a_0| t_2}{|\alpha_n| t_1^n} + \frac{|a_1 t_1 - a_0| t_2}{|\alpha_n| t_1^n} + \frac{|a_0|}{|\alpha_n| t_1^{n-1}} + \sum_{r=2}^{\lambda+1} \frac{|(a_r t_1 t_2 + a_{r-1} (t_1 - t_2) - \alpha_{r-2})|}{|\alpha_n| t_1^{n-r+1}} + \sum_{r=\lambda+2}^{n+1} \frac{|(a_r t_1 t_2 + a_{r-1} (t_1 - t_2) - \alpha_{r-2})|}{|\alpha_n| t_1^{n-r+1}} \\
 &\quad + \left[\frac{|b_0| t_2}{|\alpha_n| t_1^n} + \frac{|b_1 t_1 - b_0| t_2}{|\alpha_n| t_1^n} + \frac{|b_0|}{|\alpha_n| t_1^{n-1}} + \sum_{r=2}^{\lambda+1} \frac{|(b_r t_1 t_2 + b_{r-1} (t_1 - t_2) - b_{r-2})|}{|\alpha_n| t_1^{n-r+1}} + \sum_{r=\lambda+2}^{n+1} \frac{|(b_r t_1 t_2 + b_{r-1} (t_1 - t_2) - b_{r-2})|}{|\alpha_n| t_1^{n-r+1}} \right].
 \end{aligned}$$

Using the hypothesis, we get

$$\begin{aligned}
 t_1 &\leq \frac{|a_0| t_2}{|\alpha_n| t_1^n} + \frac{a_1 t_2}{|\alpha_n| t_1^{n-1}} - \frac{a_0 t_2}{|\alpha_n| t_1^n} + \frac{|a_0|}{|\alpha_n| t_1^{n-1}} + \sum_{r=2}^{\lambda+1} \frac{(a_r t_1 t_2 + a_{r-1} (t_1 - t_2) - \alpha_{r-2})}{|\alpha_n| t_1^{n-r+1}} + \sum_{r=\lambda+2}^{n+1} \frac{(a_r t_1 t_2 + a_{r-1} (t_1 - t_2) - \alpha_{r-2})}{|\alpha_n| t_1^{n-r+1}} \\
 &\quad + \left[\frac{|b_0| t_2}{|\alpha_n| t_1^n} + \frac{b_1 t_2}{|\alpha_n| t_1^{n-1}} - \frac{b_0 t_2}{|\alpha_n| t_1^n} + \frac{|b_0|}{|\alpha_n| t_1^{n-1}} + \sum_{r=2}^{\lambda+1} \frac{(b_r t_1 t_2 + b_{r-1} (t_1 - t_2) - b_{r-2})}{|\alpha_n| t_1^{n-r+1}} + \sum_{r=\lambda+2}^{n+1} \frac{(b_r t_1 t_2 + b_{r-1} (t_1 - t_2) - b_{r-2})}{|\alpha_n| t_1^{n-r+1}} \right] \\
 &= \frac{t_1}{|\alpha_n|} \left\{ \left(\frac{2a_{\lambda}}{t_1^{n-\lambda}} - a_n \right) + \frac{1}{t_1^n} (|a_0| - a_0) \right\} + \frac{t_2}{|\alpha_n|} \left\{ \frac{2a_{\lambda+1}}{t_1^{n-\lambda-1}} + \frac{1}{t_1^n} (|a_0| - a_0) \right\} \\
 &\quad + \frac{t_1}{|\alpha_n|} \left\{ \left(\frac{2b_{\lambda}}{t_1^{n-\lambda}} - b_n \right) + \frac{1}{t_1^n} (|b_0| - b_0) \right\} + \frac{t_2}{|\alpha_n|} \left\{ \frac{2b_{\lambda+1}}{t_1^{n-\lambda-1}} + \frac{1}{t_1^n} (|b_0| - b_0) \right\}.
 \end{aligned}$$

Hence by (9) all the zeros of $f(z)$ lie in the circle $|z| \leq R$,

where

$$\begin{aligned}
 R &= \frac{t_1}{|\alpha_n|} \left\{ \left(\frac{2a_{\lambda}}{t_1^{n-\lambda}} - a_n \right) + \frac{1}{t_1^n} (|a_0| - a_0) \right\} + \frac{t_2}{|\alpha_n|} \left\{ \frac{2a_{\lambda+1}}{t_1^{n-\lambda-1}} + \frac{1}{t_1^n} (|a_0| - a_0) \right\} + \frac{t_1}{|\alpha_n|} \left\{ \left(\frac{2b_{\lambda}}{t_1^{n-\lambda}} - b_n \right) + \frac{1}{t_1^n} (|b_0| - b_0) \right\} \\
 &\quad + \frac{t_2}{|\alpha_n|} \left\{ \frac{2b_{\lambda+1}}{t_1^{n-\lambda-1}} + \frac{1}{t_1^n} (|b_0| - b_0) \right\}.
 \end{aligned}$$

Since every zero of $P(z)$ is also a zero of $f(z)$, the theorem is proved completely.

$$\begin{aligned}
 f(z) &= (t_2 + z)(t_1 - z)P(z) = -\alpha_n z^{n+2} + (\alpha_n (t_1 - t_2) - \alpha_{n-1}) z^{n+1} + (\alpha_n t_1 t_2 + \alpha_{n-1} (t_1 - t_2) - \alpha_{n-2}) z^n + \dots \\
 &\quad + (\alpha_2 t_1 t_2 + \alpha_1 (t_1 - t_2) - \alpha_0) z^2 + (\alpha_1 t_1 t_2 + \alpha_0 (t_1 - t_2)) z + \alpha_0 t_1 t_2 \\
 &= -\alpha_n z^{n+2} + (\alpha_n (t_1 - t_2) - \alpha_{n-1}) z^{n+1} + \sum_{r=0}^n \{a_r t_1 t_2 + a_{r-1} (t_1 - t_2) - a_{r-2}\} z^r + i \sum_{r=0}^n \{b_r t_1 t_2 + b_{r-1} (t_1 - t_2) - b_{r-2}\} z^r.
 \end{aligned}$$

This gives

$$\begin{aligned}
|f(z)| &\geq \left| -\alpha_n z^{n+2} + (\alpha_n(t_1 - t_2) - \alpha_{n-1})z^{n+1} - \left[\sum_{r=0}^n \{a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2}\} z^r \right] \right. \\
&\quad \left. - \left[\sum_{r=0}^n \{b_r t_1 t_2 + b_{r-1}(t_1 - t_2) - b_{r-2}\} z^r \right] \right| \\
&\geq |z|^{n+1} \left| \alpha_n z + \alpha_{n-1} - \alpha_n(t_1 - t_2) - \sum_{r=0}^n |a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2}| |z|^r \right. \\
&\quad \left. - \sum_{r=0}^n |b_r t_1 t_2 + b_{r-1}(t_1 - t_2) - b_{r-2}| |z|^r \right|.
\end{aligned}$$

Let $|z| > t_1$, we get by using the hypothesis

$$\begin{aligned}
|f(z)| &\geq |z|^{n+1} \left[\left| \alpha_n z + \alpha_{n-1} - \alpha_n(t_1 - t_2) - \sum_{r=0}^n |a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2}| \frac{1}{t_1^{n-r+1}} - \sum_{r=0}^n |b_r t_1 t_2 + b_{r-1}(t_1 - t_2) - b_{r-2}| \frac{1}{t_1^{n-r+1}} \right| \right. \\
&= |z|^{n+1} \left[\left| \alpha_n z + \alpha_{n-1} - \alpha_n(t_1 - t_2) - \left\{ \frac{|a_0|t_2}{t_1^n} + |(a_1 t_1 - a_0)t_2 + a_0 t_1| \frac{1}{t_1^n} \right\} - \left\{ \frac{|b_0|t_2}{t_1^n} + |(b_1 t_1 - b_0)t_2 + b_0 t_1| \frac{1}{t_1^n} \right\} \right. \right. \\
&\quad \left. - \sum_{r=2}^{\lambda+1} |a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2}| \frac{1}{t_1^{n-r+1}} - \sum_{r=\lambda+2}^n |a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2}| \frac{1}{t_1^{n-r+1}} \right. \\
&\quad \left. - \sum_{r=2}^{\lambda+1} |b_r t_1 t_2 + b_{r-1}(t_1 - t_2) - b_{r-2}| \frac{1}{t_1^{n-r+1}} - \sum_{r=\lambda+2}^n |b_r t_1 t_2 + b_{r-1}(t_1 - t_2) - b_{r-2}| \frac{1}{t_1^{n-r+1}} \right] \\
&\geq |z|^{n+1} \left[\left| \alpha_n z + \alpha_{n-1} - \alpha_n(t_1 - t_2) - \left\{ \frac{|a_0|t_2}{t_1^n} + \frac{|a_1 t_1 - a_0|t_2}{t_1^n} + \frac{|a_0|t_1}{t_1^n} \right\} \right. \right. \\
&\quad \left. - \left\{ \frac{|b_0|t_2}{t_1^n} + \frac{|b_1 t_1 - b_0|t_2}{t_1^n} + \frac{|b_0|t_1}{t_1^n} \right\} - \sum_{r=2}^{\lambda+1} |a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2}| \frac{1}{t_1^{n-r+1}} \right. \\
&\quad \left. - \sum_{r=\lambda+2}^n |a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2}| \frac{1}{t_1^{n-r+1}} - \sum_{r=2}^{\lambda+1} |b_r t_1 t_2 + b_{r-1}(t_1 - t_2) - b_{r-2}| \frac{1}{t_1^{n-r+1}} \right. \\
&\quad \left. - \sum_{r=\lambda+2}^n |b_r t_1 t_2 + b_{r-1}(t_1 - t_2) - b_{r-2}| \frac{1}{t_1^{n-r+1}} \right] \\
&= |z|^{n+1} \left[\left| \alpha_n z + \alpha_{n-1} - \alpha_n(t_1 - t_2) - \frac{|a_0|t_2}{t_1^n} - \frac{(a_1 t_1 - a_0)t_2}{t_1^n} - \frac{|a_0|t_1}{t_1^n} + \left\{ -\frac{|b_0|t_2}{t_1^n} - \frac{(b_1 t_1 - b_0)t_2}{t_1^n} - \frac{|b_0|t_1}{t_1^n} \right\} \right. \right. \\
&\quad \left. - \sum_{r=2}^{\lambda+1} (a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2}) \frac{1}{t_1^{n-r+1}} - \sum_{r=\lambda+2}^n (a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2}) \frac{1}{t_1^{n-r+1}} \right. \\
&\quad \left. - \sum_{r=2}^{\lambda+1} (b_r t_1 t_2 + b_{r-1}(t_1 - t_2) - b_{r-2}) \frac{1}{t_1^{n-r+1}} - \sum_{r=\lambda+2}^n (b_r t_1 t_2 + b_{r-1}(t_1 - t_2) - b_{r-2}) \frac{1}{t_1^{n-r+1}} \right] \\
&= |z|^{n+1} \left[\left| \alpha_n z + \alpha_{n-1} - \alpha_n(t_1 - t_2) - \frac{|a_0|t_2}{t_1^n} - \frac{a_1 t_2}{t_1^{n-1}} + \frac{a_0 t_2}{t_1^n} - \frac{|a_0|t_1}{t_1^n} \right. \right. \\
&\quad \left. + \left\{ -\frac{|b_0|t_2}{t_1^n} - \frac{b_1 t_2}{t_1^{n-1}} + \frac{b_0 t_2}{t_1^n} - \frac{|b_0|t_1}{t_1^n} \right\} - \left\{ -\left(\frac{a_1 t_2}{t_1^{n-1}} + \frac{a_0}{t_1^{n-1}} \right) + \left(\frac{a_{\lambda+1} t_2 + a_\lambda}{t_1^{n-\lambda-1}} \right) \right\} + \left\{ -\left(\frac{a_{\lambda+1} t_2 + a_\lambda}{t_1^{n-\lambda-1}} \right) + a_n t_2 + a_{n-1} \right\} \right. \\
&\quad \left. - \left\{ -\left(\frac{b_1 t_2}{t_1^{n-1}} + \frac{b_0}{t_1^{n-1}} \right) + \left(\frac{b_{\lambda+1} t_2 + b_\lambda}{t_1^{n-\lambda-1}} \right) \right\} + \left\{ -\left(\frac{b_{\lambda+1} t_2 + b_\lambda}{t_1^{n-\lambda-1}} \right) + b_n t_2 + b_{n-1} \right\} \right] \\
&= |z|^{n+1} \left[\left| \alpha_n z + \alpha_{n-1} - \alpha_n(t_1 - t_2) - \frac{|a_0|t_2}{t_1^n} + \frac{a_0 t_2}{t_1^n} - \frac{|a_0|t_1}{t_1^n} + \frac{a_0}{t_1^{n-1}} - 2 \left(\frac{a_{\lambda+1} t_2 + a_\lambda}{t_1^{n-\lambda-1}} \right) + a_n t_2 + a_{n-1} \right. \right. \\
&\quad \left. + \left\{ -\frac{|b_0|t_2}{t_1^n} + \frac{b_0 t_2}{t_1^n} - \frac{|b_0|t_1}{t_1^n} + \frac{b_0}{t_1^{n-1}} - 2 \left(\frac{b_{\lambda+1} t_2 + b_\lambda}{t_1^{n-\lambda-1}} \right) + b_n t_2 + b_{n-1} \right\} \right] > 0,
\end{aligned}$$

if

$$\begin{aligned} |\alpha_n z + \alpha_{n-1} - \alpha_n (t_1 - t_2)| &> 2 \left(\frac{a_{\lambda+1} t_2 + a_\lambda}{t_1^{n-\lambda-1}} \right) - a_n t_2 - a_{n-1} + \frac{|a_0| t_2 - a_0 t_1}{t_1^n} + \frac{|a_0| t_1 - a_0 t_1}{t_1^n} \\ &+ \left\{ 2 \left(\frac{b_{\lambda+1} t_2 + b_\lambda}{t_1^{n-\lambda-1}} \right) - b_n t_2 - b_{n-1} + \frac{|b_0| t_2 - b_0 t_1}{t_1^n} + \frac{|b_0| t_1 - b_0 t_1}{t_1^n} \right\}. \end{aligned}$$

Thus $|z| > 0$, if

$$\begin{aligned} \left| z + \frac{\alpha_{n-1}}{\alpha_n} - (t_1 - t_2) \right| &> \frac{t_1}{|\alpha_n|} \left\{ \left(\frac{2a_\lambda}{t_1^{n-\lambda}} - \frac{a_{n-1}}{t_1} \right) + \frac{1}{t_1^n} (|a_0| - a_0) \right\} + \frac{t_2}{|\alpha_n|} \left\{ \left(\frac{2a_{\lambda+1}}{t_1^{n-\lambda-1}} - a_n \right) + \frac{1}{t_1^n} (|a_0| - a_0) \right\} \\ &+ \left[\frac{t_1}{|\alpha_n|} \left\{ \left(\frac{2b_\lambda}{t_1^{n-\lambda}} - \frac{b_{n-1}}{t_1} \right) + \frac{1}{t_1^n} (|b_0| - b_0) \right\} + \frac{t_2}{|\alpha_n|} \left\{ \left(\frac{2b_{\lambda+1}}{t_1^{n-\lambda-1}} - b_n \right) + \frac{1}{t_1^n} (|b_0| - b_0) \right\} \right]. \end{aligned}$$

T

his shows that those zeros of $f(z)$ whose modulus is greater than t_1 , lie in the circle

$$\begin{aligned} \left| z + \frac{\alpha_{n-1}}{\alpha_n} - (t_1 - t_2) \right| &\leq \frac{t_1}{|\alpha_n|} \left\{ \left(\frac{2a_\lambda}{t_1^{n-\lambda}} - \frac{a_{n-1}}{t_1} \right) + \frac{1}{t_1^n} (|a_0| - a_0) \right\} + \frac{t_2}{|\alpha_n|} \left\{ \left(\frac{2a_{\lambda+1}}{t_1^{n-\lambda-1}} - a_n \right) + \frac{1}{t_1^n} (|a_0| - a_0) \right\} \\ &+ \frac{t_1}{|\alpha_n|} \left\{ \left(\frac{2b_\lambda}{t_1^{n-\lambda}} - \frac{b_{n-1}}{t_1} \right) + \frac{1}{t_1^n} (|b_0| - b_0) \right\} + \frac{t_2}{|\alpha_n|} \left\{ \left(\frac{2b_{\lambda+1}}{t_1^{n-\lambda-1}} - b_n \right) + \frac{1}{t_1^n} (|b_0| - b_0) \right\}. \end{aligned}$$

It can be easily verified that those zeros of $f(z)$ whose modulus is less than t_1 , lie in the circle as well.

Therefore, we conclude that all zeros of $f(z)$ and hence $P(z)$ lie in

$$\begin{aligned} \left| z + \frac{\alpha_{n-1}}{\alpha_n} - (t_1 - t_2) \right| &\leq \frac{t_1}{|\alpha_n|} \left\{ \left(\frac{2a_\lambda}{t_1^{n-\lambda}} - \frac{a_{n-1}}{t_1} \right) + \frac{1}{t_1^n} (|a_0| - a_0) \right\} + \frac{t_2}{|\alpha_n|} \left\{ \left(\frac{2a_{\lambda+1}}{t_1^{n-\lambda-1}} - a_n \right) + \frac{1}{t_1^n} (|a_0| - a_0) \right\} \\ &+ \frac{t_1}{|\alpha_n|} \left\{ \left(\frac{2b_\lambda}{t_1^{n-\lambda}} - \frac{b_{n-1}}{t_1} \right) + \frac{1}{t_1^n} (|b_0| - b_0) \right\} + \frac{t_2}{|\alpha_n|} \left\{ \left(\frac{2b_{\lambda+1}}{t_1^{n-\lambda-1}} - b_n \right) + \frac{1}{t_1^n} (|b_0| - b_0) \right\}. \end{aligned}$$

This completes the proof of the theorem.

4. References

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