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# Is the C*-Algebraic Approach to Quantum Mechanics an Alternative Formulation to the Dominant One?

**Antonino Drago**

Department of Physical Science, Naples University "Federico II", Naples, Italy
Email: drago@unina

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## Abstract

Since 1947 a foundation of Quantum Mechanics (QM) on functional analysis was suggested by Segal. By defining the C*-algebra of the observables, then the Gelfand-Naimark-Segal theorem faithfully represents this algebra into Hilbert space. In the 70's Emch has reiterated this formulation and improved it. Recently Strocchi improved it even more. First, he suggested an axiomatization of the paradigmatic Dirac-von Neumann's formulation of QM to which he addresses two basic criticisms, i.e. a weak linkage with the experimental basis of theoretical physics and the obscurity about the separation mark between classical mechanics and QM. Afterwards, through an analysis of the experimental basis of a physical theory he suggests an explanation of Segal's restriction of the operators to be bounded. Eventually, he represents this algebra into Hilbert space and at last, by means of Weyl algebra he obtains the symmetries of the dynamics of a particle theory. In fact, several characteristic features of this formulation correspond to those determined by the two choices which are the alternative ones to the choices of the dominant formulation. It is a problem-based theory, since it starts rather from than axioms a problem (i.e. the indeterminacy); then, it argues through both doubly negated propositions and an *ad absurdum* proof. Moreover, its theoretical development is similar to that of an alternative classical theory since it put, before the geometry, the algebra; the bounded operators are represented by a polynomial algebra; which pertains to constructive mathematics. Eventually, he obtains the symmetries of the theory. The problems to be overcome in order to accurately re-construct his formulation according to the two alternative choices which are listed. It is concluded that rather an alternative role, it plays a complementary role to the paradigmatic formulation.

## Keywords

Quantum Mechanics, C*-Algebra Approach, Strocchi's Formulation, Two

Dichotomies, Constructive Mathematics, Non-Classical Logic

## 1. Introduction

In previous papers I have characterized the foundations of Theoretical physics as constituted by two dichotomies; one concerns the two kinds of infinity—either the actual infinity (AI), or the potential infinity (PI)—, or, in formal terms, the two kinds of Mathematics—either the classical Mathematics making use of the idealistic axioms (e.g. Zermelo's), or the constructive Mathematics—; the other dichotomy concerns the two kinds of organization of a theory—either the axiomatic-deductive one (AO) or the problem-based organization aimed at solving a basic problem through the invention of a new scientific method (PO)—; or, in formal terms, the two kinds of Logic—either the classical Logic or the intuitionist one (Drago, 1996).

I have characterized the basic choices of each of the main classical theories, both the dominant ones and those based on the alternative choices. Through the basic choices intended as interpretative categories, I have characterized 1) the birth of Quantum Mechanics (QM) through Albert Einstein's paper on quanta (Drago, 2013), 2) the entire history of the theory (Drago, 2002), 3) and its kind of logic (Drago & Venezia, 2002).

The dominant formulation of QM, the Dirac-von Neumann's one (DvNQM) is clearly based on the choices AI (since the Hilbert space of all square summable functions pertains to classical Mathematics) and AO (since its mathematical framework is applied to the reality as an a priori structure).

In the aim at discovering an alternative formulation, i.e. a formulation of QM which is based on the choices PI&PO, in a first time I have studied Weyl's formulation of QM, because this author wanted to base it upon an elementary mathematics and moreover he formulated it through the symmetries (which constitute the characteristic mathematical tool of the classical theory of Lazare Carnot's mechanics, whose choices are the alternative ones, PI & PO; Drago, 2004). Yet, in order to obtain as most as possible results, Weyl has changed its starting Mathematics in the classical one (AI), when he has taken the limit to continuous groups; and moreover he has assumed Schroedinger's equation as an a priori hypothesis (AO) (Drago, 2000).

I then have classed all the formulations of QM (24) according to rough evaluations of their basic choices (Drago, 2014). Three formulations appeared as based on the choices PI&PO. Yet, under a further, detailed analysis they resulted either incomplete (e.g., Heisenberg's) or inadequate. I have then followed the program of reiterating through the alternative formulations of the classical theories the historical process which led to the birth of QM, in such a way to generate it according to the alternative choices. It resulted in a too difficult task (Drago, 2016).

Then, I have focused the attention on those formulations which, although not based by the author on the choices PI&PO, may be easily reformulated according to them. The most appealing one resulted Franco Strocchi's formulation (SQM) (Strocchi, 2008), which pertains to a theoretical tradition started by the seminal paper (Segal, 1947), suggesting as basic a particular algebra (C*-algebra) in place to Hilbert space. Given the ancient tradition of the algebraic approach—to be based on the choice PO (problems instead of axioms) and PI (finitist, or at most constructive methods, Monna, 1973: pp. 147-148)—, this approach is the most promising for discovering an alternative formulation of QM. In addition, Strocchi suggests to the original approach many improvements which are useful for formulating an alternative QM. However, some unresolved questions are recognized and listed. It is concluded that at present time rather an alternative role, this formulation plays a complementary role to the paradigmatic formulation.

## 2. Segal's Seminal Paper on C*-Algebra in Quantum Mechanics

Let us come back to the historical origin of this algebraic approach which exploits functional analysis. Roughly speaking, functional analysis applies the usual notions of calculus employed inside the space of real numbers points, to the space of functions on real or complex numbers. Imagine the space of all the functions representing the possible evolutions of a physical system. An operator transforming one function in another or evenly a function of such functions is a functional. A physical magnitude is represented as a functional, i.e. as an operator on each of these functions giving the value of the physical system at a given state.

In 1947 Irving E. Segal has assumed as mathematical basis of QM a particular polynomial algebra of operators,[1] a C*-algebra.[2] The point is very important. After having met the problem to know an entirely new world, as it is the microscopic world, wisely physicists have analyzed the structure which is under their control, i.e. all the macroscopic instruments which can be applied to measure observables; or better, the mathematical structure of them in order to exploit all its potentialities.

This approach opposes to Hilbert's one, which has dominated past studies on QM. In reaction to the novelty of the microscopic world, as first step Hilbert has put the space of all the (candidate) useful functions for representing all possible

[1]The set of polynomial functions may be compared with the set of the analytical functions, which represent almost all the functions of theoretical physics. Each of the latter ones, being spanned in an infinite series of powers of the variables, is unboundedly approximated by polynomials. In particular, a constructive function may be considered as a sequence of approximating polynomials (Pour-El, 1975).

[2]I recall that a norm is essentially a bound; more precisely, it is a function which assigns a strictly positive length or size to each vector in a vector space. may be basically viewed as the application of linear algebra to spaces of functions. A Banach algebra is a linear associative algebra over the field C of the complex numbers with a norm | |. A C*-algebra is a Banach algebra on a complex field, together an involution * with the property $|A * A| = |A^2|$.

mathematical results of measurement apparatuses (the Hilbert space), in order to a priori assure to the physicists the full capability of the mathematical calculations (even at the cost, as both Segal (1947: p. 930) and Strocchi (2012, p. 3) remark, of having ill-defined states and operators); that means that the more powerful as possible mathematics is put as the basis of conceiving reality—that is the same prejudice of the Mathematical physics with respect to the entire physical world. In such a way Hilbert has skipped the first step of a natural development of a theory, which put first the basic notions and the physical principles; among the latter ones, the principle about the relationship between mathematics and physics; not before this step, the theory starts the formally mathematical development. It is not a chance that this approach was suggested by a mathematician, Hilbert, who was the founder of the formalist school and moreover a hard advocate of the classical mathematics against the suggestions of less idealistic kinds of mathematics by both Brouwer and Weyl (as a fact, Hilbert space does not directly deal with the discrete mathematics of quanta).

In the following, I quote the main points of Segal's presentation of his alternative approach: The first one is the most important and it will be maintained by his followers.

> We present in this paper a set of postulates for a physical system and deduce from these the main general features of the quantum theory of stationary states. [In opposition to the BvNQM)] Our theory is strictly operational in the sense that only the observables of the physical system are involved in the postulates. [Indeed,] The collection of all bounded self-adjoint[3] operators on a Hilbert space,[4] which has previously been used [by me] as a mathematical model for the observables in quantum mechanics, satisfy the [previously stated] postulates, as do a variety of considerably more general mathematical structures (Segal, 1947: p. 930).

Then, he illustrates this novelty with respect to the paradigmatic approach:

> Inasmuch as Hilbert space plays no role in our theory, our proofs are necessarily of a different character from the proofs of these results for the case of the system of all bounded self-adjoint operators. Actually, Hilbert space appears to be somewhat inadequate as a state space even for the latter system, in that there exist pure states of the system which cannot be represented in the usual way by rays of the Hilbert space.
>
> The [above] postulates are partly algebraic and partly metric. The algebraic postulates require essentially that an observable can be multiplied by real numbers and rose to integral powers, and that any two observables can be added. It is assumed that the usual algebraic laws are satisfied so that 1) the observables can be treated like the elements of a linear space, and 2) the

[3]An operator A is adjoint if there is A* such that (Ax, y) = (x, A * y), where * is the involution. It is self-adjoint if A = A*. It is Hermtian or symmetric if (Ax, y) = (x, Ay).

[4]A Hilbert space is a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product.

usual rules for dealing with polynomials in one variable with real coefficients remain valid when the variable is replaced by an observable. It is not assumed that two observables have a product.[5] The metric postulates require that for each observable there be defined a kind of maximum numerical value, which plays the part of a norm, and has various properties in accord with its physical significance. While this norm is quite essential to the development of the theory, an interesting consequence of the theory is that the norm can be (uniquely) defined in a purely algebraic fashion. This shows that the objective features of a physical system,—the spectral values and probability distributions of the observables, and the pure states,—are completely determined by the algebra of observables, i.e., by the rules for addition, scalar multiplication, and powers, of observables (Segal, 1947: p. 930).

Segal's mathematics may seem less powerful than the usual one, allowing whatsoever mathematical functions. However, more general functions may be obtained, so that he recovers Hilbert space.

On the other hand, it is interesting to note that if our algebraic postulates are strengthened sufficiently, then it can be shown that the collection of observables is isomorphic, (algebraically and metrically) with all self-adjoint operators in an algebra of bounded operators on Hilbert space (the norm corresponding to the operator bound) (p. 931).

Indeed, he improves a previous result:

Our result is formally similar to a result of Gelfand-Neumark [sic]. Giving a representation for a certain kind of complex Banach algebra…. our proof is concerned with showing that, in the present case, such quotient algebras are actually isomorphic to the real field (p. 394).

This result constitutes the so-called Gelfand-Naimark-Segal theorem (GNS).

## 3. Emch's Presentation of the C*-Algebraic Approach to QM

In order to motivate the introduction of a C*-algebra Gerard G. Emch (1984) stated:

As the years, however, quantum statistical mechanics and relativistic quantum field theory were grudgingly recognized to lie somewhere beyond the reach of this formalism [of Hilbert space]. Moreover, the somewhat *ad hoc*, or a priori, introduction of a Hilbert space on which to build the theory was leaving room for a conceptually tighter approach.…. [; indeed,] a closer adherence to empirically verifiable structural relations between the fundamental objects of the theory—the observables and the states—suggests that

[5]The multiplications of the operators are avoided by Segal because it concerns their commutativity. In p. 391 he offers an interesting comparison between his approach to the algebraic interpretation of QM and previous von Neumann's (von Neumann, 1936).

the observables should be constructed from [solely] the self-adjoint elements of a C*-algebra, characteristic of the system considered, and that the states should be identified as the elements of [a specific] convex set··· [Then,] The Gelfand-Naimark-Segal theorem [GNS] allows··· to reconstruct a Hilbert space representation appropriate to each given physical situation, i.e. to a class of compatible preparations of the system··· having thus established known things on firmer foundations, the main point is that the algebraic formulation is genuinely more general, and precisely so where more generality is needed. Drastically different (in technical terms "disjoint") representations [i.e. commutative vs. non-commutative] do occur in cases of interest both to microphysics (leading to renewed hopes that one might handle elementary particle interactions in a mathematically consistent manner) and to macrophysics (Emch, 1984: pp. 361-362).

Afterwards, Emch suggests its basic assumption:

*The C*-Axiomatic Postulate: A physical system is characterized by a triple* $\{S^*, A, \langle .\,;.\rangle\}$ *where:* $A$, *the set of its observables* (*or measurable attributes*), *is the collection of all the self-adjoint elements A of a C*-algebra B with identity I satisfying* $|\, I \,| = 1$, $S^*$, *the set of its states* (*or modes of preparation*), *is the collection of all real-valued, positive linear functional* $\Phi$ *on* $A$, *normalized by the condition* $\langle \Phi; I\rangle = 1$; *and the prediction rule* $\langle .\,, .\rangle$ *which attributes, to every pair* $\{\Phi, A\} \in S^* x A$, *the value* $\langle \Phi; A \rangle$ *of* $\Phi$ *at* $A$, *interpreted as the expectation of the observable A when the system is in the state* $\Phi$ (Emch, 1984: p. 362).

In such a way

… a truly fundamental postulate… cover[s] simultaneously the situations encountered in quantum mechanics, in classical mechanics, and in the intermediate case of a theory with superselection rules[6] (Emch, 1984: p. 369).

From that he draws the following remarks:

*Firstly*, there should be no [negative] argument on the fact that the postulate is both concise and mathematically legible. *Secondly*, it is obtained by induction from the von Neumann synthesis of the quantum theories of Heisenberg, Schroedinger and Born… which the C*-algebraic postulate therefore encompasses. *Thirdly*, the universe of discourse of the C*-algebraic postulate is genuinely more general than allowed by the framework delineated by the von Neumann postulate: the algebras of observables which we can now consider are more general than B (H); and the states which naturally appear in the theory are more general than the density-matrices of ordinary quantum mechanics. Moreover, the objects described by the

[6]The superselection rules are caused by the operators, as electric charge, which commute with all observables. Owing to theme not all projections are observables and it is impossible to measure *coherent* superpositions of states belonging to superselections sectors.

> C*-algebraic postulate appear in all the physical situations so far encountered not only in ordinary quantum mechanics, but also in the quantum theories with superselection rules··· and even in classical mechanics. This is a synthesis which, therefore, goes beyond that achieved by the von Neumann postulate. *Fourthly*, the GNS construction allows to bring back into the formalism the technical powers of Hilbert space representations, on which depended many of the successes of the quantum mechanics described by the von Neumann postulate. Hence the generality added to the von Neumann framework··· is not crippling; quite to the contrary, in fact, since the Hilbert space representations, with which we are now prepared to work, correspond precisely to the physical situation in which one takes into account the modes of preparation of the systems considered··· [*Fifthly*,] Our fifth point has to do with the epistemological question of the empirical foundations of the C*-algebraic postulate··· [; a] similarly remote role [to von Neumann' assumption of Hilbert space although lacking of a clear empirical basis] is played, in the C*-algebraic postulate, by the C'-algebra B of which only the self-adjoint elements are identified as observables. It is, therefore, a legitimate question to ask whether one can decide empirically when "the observables of a physical system can be identified with the self-adjoint part of a C"-algebra. This is a hard question··· (Emch, 1984: pp. 378-379).

To which he devotes a long analysis, obtaining "a chain of operational [necessary] axioms"; which however are not "compelling". Indeed, he concludes:

> The formal presentation there is intrinsic and explicit enough so that one empirically decides whether each of the axioms is a reasonable [sic!] idealized description of a given physical system (Emch, 1984: p. 383).

Moreover, he supposes that the boundedness of the operators of a C*-algebra represents a mere first stage of historical development of these algebras, whose further stages will include the unbounded operators (Emch, 1984: p. 383).

## 4. Strocchi's Axiomatic of the Dominant Formulation of QM

Before illustrating the basic point of his formulation, a last paper by Strocchi (2012) addresses some radical criticisms to DvNQM. He points out the lack of a priori physical motivation, as acknowledged by Dirac himself ("The justification of the whole scheme depends on the agreement of the formal results with the experiments"). He adds that "··· in most textbooks··· no attempt is made of improving the presentation of the axioms from the experimental/operational point of view" (ivi, p. 1). He then wants to make apparent the theoretical structure of the dominant DvNQM by offering a five axioms axiomatic.

> The basic idea [of DvNQM] is Dirac realization of the linear structure of the quantum states, the so-called *superposition principle.* It codifies the distinctive feature of Schroedinger wave mechanics with respect to standard

classical [i.e. Hamilton's] mechanics... in the quantum superposition the coefficient $c_\alpha$ of the state gives the probability amplitude for the outcome corresponding to the state $\alpha$, i.e. the probability $p_\alpha$ for the outcome, i.e. $|c_a|^2$.

Hence, "*Axiom I. States*". The states are represented by rays (or matrices) in a Hilbert space, *H.* This axiom

> ··· formalizes the superposition principle by realizing the underlying structure of vector space spanned by the states. The physical basis of such a principle is taken from the analysis of photon polarization experiments, which however do not provide a clear cut distinction with respect to the classical wave picture··· in Dirac's words "[the axiom] cannot be explained in terms of familiar physical concepts" (ivi, p. 1).

The following Strocchi's suggestion for an axiom of the DvNQM departs from the standard presentation of the observables because it imposes (as Segal and Emch do) their boundedness (on this subject we will come back in the following):

> *Axiom II. Observables.* The observables of a quantum mechanical systems, i.e. the quantities which can be measured, are described by the set of bounded self-adjoined operators in a Hilbert space H.
>
> Some remarks follow:
>
> In Dirac formulation the operators describing observables are not required to be bounded and no distinction was made between hermiticity and self-adjointness. However, for an unbounded operator hermiticity is not enough for defining its spectrum[7] and continuous functions of it; hence, its physical interpretation is not well-defined. Moreover, the sum of two unbounded self-adjoint operators does not define a self-adjoint operator… and therefore without the condition of boundedness the whole linear structure of the observables is in question (ivi, p. 2). In Dirac presentation the physical motivations for the description of the observables by self-adjoint operators look rather weak. Dirac arguments in support of axiom II are somewhat interlaced with implicit assumption about the spectrum of observables and its relation with the outcomes of measurements.
> In the standard presentation of the principles of QM such an axiom appears as a distinctive feature of QM with respect to classical mechanics... [yet,] no a priori physical motivation is given (ivi, p. 2).
>
> He continues the presentation of DvNQM by adding:
>
> As a consequence of Axioms I and II, the states··· define *positive linear functionals* on the algebra of observables··· The following axiom relates such functionals to the experimental expectations···

[7]The spectrum of an operator A, $\sigma(A)$ may be defined, inside a C*-algebra also in an algebraic way; it is the set of all complex numbers $\lambda$ such that $|A - \lambda I|$ does not have a (two-sided) inverse in A.

*Axiom III. Expectations* [of an experiment applying an operator to a state ω] are given by the Hilbert space matrix element $<A_\omega> = (\Psi_\omega, A\,\Psi_\omega)$ (ivi, p. 2).

Strocchi remarks that:

The assertion that the experimental expectations have a Hilbert space realization may look a very strong assumption with no classical counterpart.
Then, a crucial axiom is introduced.
*Axiom IV. Dirac canonical quantization.* The operators which describe the canonical coordinates $q_i$ and moment $p_i$, $i = 1, \cdots, s$ of a quantum system of $2s$ degrees of freedom obey the canonical commutation relations.
$\left[q_i, q_j\right] = 0 = \left[p_i, p_j\right].\left[p_i, q_j\right] = ih/2\pi\delta_{ij}$. (1.1)
This axiom reflects the commutation relations of the infinite matrices for the position and momentum proposed by Heisenberg... and later related to the uncertainty principle. The attempts to justify such an axiom by Heisenberg and Dirac [are questionable. Moreover, the above] Equations imply that the canonical variables cannot be described by bounded operators [an explanation will be presented later] and therefore are not observables according to Axiom II.
The following axiom provides the bridge between Heisenberg and Schroedinger formulations of QM, a deep open problem at the birth of QM. The compatibility of the two descriptions has been the subject of philosophical debates; the recognition that a quantum particle has multiple properties which look contradictory and mutually exclusive has led Bohr... to the formulation of his *complementarity principle* as the basic feature of quantum physics. Bohr's statement is not mathematically precise and it is not sharp enough to lead to a unique interpretation... This is probably the origin of the still lasting philosophical debates on its meaning.
The following axiom provides the mathematical formulation of the coexistence of the particle and the wave picture and, together with axiom IV, may be regarded as the substitute of Bohr principle.
*Axiom V. Schroedinger representation*. The commutation relations (1.1) are represented by the following operators in the Hilbert space $H = L^2(\boldsymbol{R}^s, dx)$:
$q_i\psi(x) = x_i\psi(x); p_j\psi(x) = ih/2\pi\psi/x_j(x)$ (ivi, p. 3)

Strocchi's general motivation for a new formulation with respect to DvNQM is given by the following criticism he addresses to it:

The Dirac-von Neumann axioms provide a neat mathematical foundation of quantum mechanics, but their a priori justification is not very compelling, their main support, as stressed by Dirac, being the a posteriori success of the theory they lead to. The dramatic departure from the general philosophy and ideas of classical physics may explain the many attempts of obtaining quantum mechanics by a deformation of classical mechanics or by

> the so-called geometric quantization. Thus, a more argued motivation on the basis of physical considerations is desirable (ivi, p. 3).

And this is the purpose of a specific section. Then, Strocchi declares in the following terms his strategy for constructing a new theory:

> The discussion of the principles of QM gets greatly simplified, from a conceptual point of view, if one first clarifies what are the [physical] objects of the [subsequent] mathematical formulation (ivi, p. 3).

These objects are essentially the physical apparatuses, which both Segal and later Emch ignored. Hence, as a first step, Strocchi wants to make a clear distinction between the physical content and the mathematical framework in order to suggest a clearly operative support to Segal's alternative approach.

After the previous presentation of the Axiom II, he had stated:

> A trivial consequence of the Axiom II is that, through their linear combinations and products the operators generate an algebra A over the complex numbers··· which coincides with the whole set *B*(*H*) of the bounded operators of *H*[8] (ivi, p. 2).

He adds

> In this section we argue that the structure of C*-algebra of observables and states is the suitable language for the mathematical description of a physical system in general (including the atomic systems), with no reference to classical mechanics and its standard paradigms (Strocchi, 2010: p. 16).

In this aim, he tries to justify this algebra through an accurate analysis of the experimental basis of physics. He next constructs the C*-algebra such that it is "experimentally motivated", because it originates from the previous basic considerations on the observables.

However, it presents some difficulties. At the end of his analysis Strocchi honestly admits that his work for operationally justifying the wanted C*-algebra is only partially successful. Hence the boundedness of all operators, although qualified by Strocchi as "natural", has to be considered as an assumption.

As an important consequence, the ambiguity about hermiticity and self-adjointness of the operators in DvNQM is cancelled because for bounded operators hermiticity implies self-adjointness.

## 5. Strocchi's Formulation of Quantum Mechanics: II. The C*-Algebra and His Axiom A

We thus skip to his Axiom

[8]"Owing to the superselection rules [caused by the operators, as electric charge, which commute with all observables].... not all projections are observables and it is impossible to measure *coherent* superpositions of states belonging to superselections sectors. Hence, [for this case] Axiom II must become: The observables of a quantum mechanical system are described by real vectors space generated by a set of bounded self-adjoint operators on *H*" (ivi, p. 2).

> ··· the following Axiom··· partly goes beyond the implications of the operational analysis discussed so far; however, in our opinion, it represents a more physically motivated alternative to Dirac-von Neumann axiom II. All the preceding discussions and arguments are meant to provide [no more than] a physical justification of such an axiom and are completely summarized and superseded by it. An indirect justification of it as a property of the description of a general physical systems is that it is satisfied by *both CM and QM*.[9]

Hence his operative suggestions are not enough to obtain the final "Axiom A", which supersedes them. Of his elaboration, he:

> *Axiom A*. The observables generate a [polynomial] C*-algebra A, with identity···; the *states* which by eq. (2.1) define positive linear functionals on the Algebras $A_A$ A, for any observable A, separate such algebras in the sense of eq. (2.6) and extend to positive linear functional on A (ivi, p. 6).[10]

In conclusion, from the above considerations it follows that the right language for the mathematical description of quantum systems is the theory of (non-abelian) C*-algebras and as such the mathematical structure of quantum mechanics can be viewed as a chapter of that theory (Strocchi, 2010: p. 42).

## 6. Strocchi's Formulation of Quantum Mechanics: III. Relationship of C*-Algebra with Hilbert Space

Then Strocchi exploits some mathematical advancements obtained by GNS for recovering the Hilbert space and hence all the mathematical description of a physical system.

> It is important to mention that quite generally, by the Gelfand-Naimark representation theorem, an (abstract) abelian C*-algebra A (with identity) is isometrically isomorphic to the algebra of complex continuous functions C(X) on a compact Hausdorff topological space X,[11] where X is intrinsically defined as the Gelfand spectrum of A (Strocchi, 2010: p 15).

As a very important consequence; its theoretical approach is independent

[9]Notice the similar conclusion written in his book: "The arguments discussed in this section do not pretend to prove as a mathematical theorem that the general physical requirements on the set of observables necessarily lead to a C*-algebraic structure, but they should provide sufficient motivations in favor of it. In any case, the above mathematical structure is by far more general than the concrete structure discussed in Sect. 2 [classical Hamiltonian systems] for classical systems" (Strocchi, 2010: p. 24).

[10]Notice the similar words in his book: "For these reasons we adopt the following mathematical framework: 1. A physical system is *defined* by its *C*-algebra A of observables* (with identity). 2. The states of the given physical system are identified by the measurements of the observables, i.e. a state is *a normalized positive linear functional on A*. The set S of physical states separates the observables, technically one says that S is full, and conversely the observables separate the states" (Strocchi, 2010: p. 24).

[11]In a Hausdorff space any singleton set $\{x\} \subset X$ is equal to the intersection of all closed neighborhoods of $x$.

from the space-time variables or any other geometrical representation, as instead a Hilbert space is.

> From the point of view of general philosophy, the picture emerging from the Gelfand theory of abelian C*-algebras has far reaching consequences and it leads to a rather drastic change of perspective [in theoretical physics]. In the standard description of a physical system the geometry comes first: one first specifies the coordinate space (more generally a manifold or a Hausdorff topological space), which yields the geometrical description of the system, and then one considers the abelian algebra of continuous functions on that space. By the Gelfand theory [instead] the relation can be completely reversed: one may start from [an algebra, i.e. [the abstract abelian C*-algebra, which in the physical applications may be the abstract characterization of the observables, in the sense that it encodes the relations between the physical quantities of the system, and then one reconstructs the Hausdorff space such that the given C*-algebra [with identity] can be seen as the C*-algebra of continuous functions on it. In this perspective, one may say that the algebra comes first, the geometry comes later··· (Strocchi, 2010: p. 15).

He adds:

> The recognition of the··· mathematical structure at the basis of the standard description of the classical systems suggests an abstract characterization of a classical (Hamiltonian) system with no a priori reference to the explicit realization in terms of canonical variables, phase space, continuous functions in the phase space, etc. In this perspective since a physical system is described in terms of measurements of its observables, one may take the point of view that a classical system is *defined* by [only] its physical properties, i.e. by the algebraic structure of the set of its measurable quantities or observables, which [can be translated in mathematical terms in order to] generate an abstract abelian C*-algebra A with identity. The states of the system being fully characterized by the expectations of the observables are described by normalized positive linear functionals on A··· (Strocchi, 2010: p. 15).

The GNS construction is very important from a general mathematical [and also theoretical physics'] point of view, since it reduces the existence of Hilbert space representations of a C*-algebra to the existence of states, which is guaranteed by Proposition 1.6.4 of Appendix C··· Thus, the basis of the mathematical description of quantum mechanical systems [i.e. Hilbert space] need not to be postulated, as in the Dirac-Von Neumann axiomatic setting of quantum mechanics, but it is merely a consequence of the C*-algebra structure of the observables argued in Sect. 1.3 and of the fact that, by its operational definition, a state defines a positive linear functional on them··· (Strocchi, 2010: p. 45).

## 7. Strocchi's Formulation of Quantum Mechanics: IV. The Principle of Indeterminacy and Its Representation

So far, Strocchi did not mention quantum systems as being different from classical ones. An important clarification is his sharp answer to the following question: What characterize QM with respect to classical theories? At the beginning of the above-mentioned paper he had remarked:

> The lack of a clear distinction between the role of the two sets of axioms, I, II, III and IV, V, is at the origin of the widespread point of view, adopted by many textbooks, by which all of them are characteristic of quantum systems. The distinction between classical and quantum systems is [read: ought to be] rather given by the mathematical structure of A and it will have different realizations depending on the particular [either classical or not] class of systems (Strocchi, 2012: p. 3).

He underlines that the first three axioms of the above list represent also a classical system; the quantum characterization enters through the Axiom IV, concerning the non-commutativity of the two conjugate observable defining a states. As a consequence, Classical mechanics results in a Hilbertian description which is equivalent to one in terms of an algebra of functions, whereas this kind of algebra is impossible when the observables do not commute, since two mutually interfering variables cannot be governed by the notion of a function.

This quantum/classical distinction was blurred for a long time because the status of the principle of indetermination was unclear to most physicians. In 1947 Segal had still to write that he had:

> To confute the view that the indeterminacy principle is a reflection of an unduly complex formulation of Quantum mechanics and to [strength] the view that the principle is quite intrinsic in physics, or in an empirical science based on quantitative measurement (Segal, 1947: p. 931).

Strocchi remarks in addition that the usual mathematical relations of non-commutation are not valid for finitely measurable operators, essentially because a sharp measurement of one observable ($\Delta p = 0$ exacly) ought to have in correspondence an infinite value of the other observable; yet, this value cannot be operationally obtained (ivi, p. 8). Hence, he evaluates as insufficient Born's and Heisenberg's experimental justifications of these relations which in the standard mathematical representation link these relations with Hilbert's operators. Rather, he advances reasons of experimental methodology for suggesting a new mathematical version of them (called by him "complementarity relations"),

$$\Delta_\omega(A) + \Delta_\omega(B) \geq C > 0 \text{ for all } \omega$$

where $\Delta$ is the mean square deviation. Notice that this relation is not the mere logarithm of the previous one because they may differ at the infinity points.

About this result he adds the following comment.

> This provides a precise *operational and mathematical formulation* of

> complementarity with the advantage, w.r.t. the Heisenberg uncertainty relations, of being meaningful and therefore testable for operationally defined observables, necessarily represented by bounded operators.... (ivi, p. 8).

In particular, he proves that his version is more effective than Heisenberg's in the case of the two components $s_1$ and $s_3$ of momenta of spin ½ (ivi, p. 9).

In sum, through the technique of the representations of C*-algebras he has obtained a complete formulation of QM.

## 8. Strocchi's Formulation of Quantum Mechanics: V. Symmetries

After the complementary relations Dirac canonical quantization is re-formulated according to an algebraic comprehensive approach of Classical mechanics and QM. By starting from a free C*-algebra[12] he nicely obtains two cases of quantization, Z = 0 and Z = $i$h/2π, which correspond respectively to classical mechanics and QM. That moreover proves that no other case is possible.

Axiom V of DvNQM gives the Schroedinger representation inside Hilbert space. In SQM

In SQM Schroedinger QM follows from the von Neumann uniqueness theorem through the canonical commutators relations. His treatment includes the symmetries too, as it is shown in the case of the dynamics in a one-parameter group of *-automorphisms of A. At this aim, in order to take in account the unboundedness of the operators, he defines the Weyl algebra of the two variables, $p$ and $q$, defining the state of the particle (rather than the Heisenberg algebra).

> For finite degrees of freedom, the Weyl algebra codifies the experimental limitations on the measurements of position and momentum (Heisenberg uncertainty relations) and Schroedinger QM follows from the von Neumann uniqueness theorem (Strocchi, 2008: p. 4th of the cover).

And also the symmetries follow.

At last, he summarizes his formulation through the following features:

> In conclusion, the operational definition of states and observables motivates the physical principle or axiom that, quite generally the observables of a physical (not necessarily quantum mechanical) system generate a C*-algebra. The Hilbert space realization of states and observables (Dirac-von Neumann Axioms I-III) is then [obtained as] a mathematical result. The existence of observables which satisfy the operationally defined complementarity relations implies that the algebra of observables is not Abelian and it marks the difference between CM and QM. Thus, for a quan-

[12] A free algebra is the noncommutative analogue of a polynomial ring since its elements may be described as "polynomials" with non-commuting variables.

tum mechanical system the Poisson algebra generated by the canonical variables [i.e. the algebraic-differential relationships between the variables] cannot be represented by commuting operators [owing to the indetermination relationships] and actually canonical quantization (Axiom IV) follows from [different,] general geometrical structures. The Schroedinger representation (Axiom V) is selected by the general properties of irreducibility and regularity. The general setting discussed so far may then provide a more economical and physically motivated alternative to the Dirac-von Neumann axioms for the foundation of quantum mechanics (Strocchi, 2012: p, 12) (ivi, p. 12).[13]

## 9. Strocchi's Formulation as a PI Theory. The Lacking Characteristic Features

Hilbert space clearly represents the AI attitude, Segal's tradition which is based out from it, according to an algebraic approach, whose tradition relies on constructive mathematical tools promises an entirely new foundation of QM. Moreover SQM introduces as a fundamental mathematical technique the symmetries, which are the theoretical techniques of PI & PO theories. As a fact, according to Segal QM is formulated. In the literature on the QM that I know, I have found no one formulation presenting these merits; only Weyl formulation presents symmetries yet based in an approximative way.

Segal's tradition assumes the boundedness of each physical variable. This assumption is necessary in order to obtain a C*-algebra of the observables; it assures both the hermiticity of all operators and moreover the solutions of all relevant, differential equations (Pour-El & Richards, 1989). Strocchi tries to justify this thesis of boundedness through an operational analysis of experimental physics. In my opinion this thesis remains as questionable on an epistemological basis.[14]

[13]For a similar conclusion: "… the mathematical setting of quantum mechanics can be derived with a very strict logic solely from the C*-algebraic structure of the observables and the operational information of non-commutativity codified by the Heisenberg uncertainty relations (Section 2.1). In this way one has a (in our opinion better motivated) alternative to the Dirac-Von Neumann axiomatic setting, which [however] can actually be derived [from the previous framework] through the GNS theorem 2.2.4 [about the representation of a C*-algebra into Hilbert space], the Gelfand-Naimark theorem 2.3.1 [about the faithful representation on Hilbert space in the case of Abelian algebra] and Von Neumann theorem 3.2.2 [about the unitarily equivalence of all regular irreducible representations of Weyl algebras]" (Strocchi, 2008: p. 23).

[14]Surely, each apparatus is bounded in its result of the measurement processes, but the set of all apparatuses defining an observable may be infinite in number and hence this set may produce a result beyond a whatsoever bound. This point was debated by Bridgman in his discussion of the operative dependence of each physical observable on the apparatuses of measurements (Bridgman, 1927, chp. 1). This point may also be discussed by considering which numbers result from measurements. Each measurement process gives a number having a finite number of digits, hence a rational number. Yet, each result of a measurement may be improved beyond whatsoever bound by means of new apparatuses; hence, it is a privilege of a theoretical physicist to idealize an experimental number as a real number, with possibly an infinite number of digit, as e.g. π, to which the results of all the equivalent processes of measurements presumably converge. Hence he overcomes the finitist bound in the aim at easily operating with real numbers (which however may be constructive or not). Likewise, the unboundedness of the physical variables represents an idealization of its range of values.

However, the same Strocchi admits that his "preliminary basic consideration" is not enough to conclude his Axiom A (ivi, p. 6), which is the actual point of departure of his formal development. Hence, one can consider SQM as no more than relying upon the mathematical content of Axiom A, i.e. the polynomial C*-algebras of the observables, given as an a priori. But, the previous objection to his thesis challenges not only Strocchi's criticisms to the dominant formulation, but also the very basis of some theorems (e.g. his result about the indetermination relationships). Hence, the thesis of a bounded experimental basis as suitable for theoretical physics in general, and in particular for QM rather seems a reduction of the very mathematical basis of DvNQM.

However, one may suppose that Segal's tradition represents a unaware and incomplete attempt by many scholars to achieve a formulation of QM which is based on constructive mathematics. SQM looks as a good basis for obtaining a constructive (PI) formulation of QM. In view of improving it as an entirely constructive formulation one has to solve the problems of discovering the constructive counter-parts of the following steps of this theory:

1) The mathematical definition of a C*-algebra. There exists, if one accepts the apartness definition (see Bishop & Bridges, 1985: chp. 7, p. 157; Takamura, 2005: p. 81).

2) GN theorem. Yes in the case of Abelian algebras; its constructive counter-part was obtained by (Bridges, 1979: sect. 6.7; Takamura, 2005: p. 289). Through a slightly different notion of norm instead, in the case of a non-Abelian algebra, that necessary for QM, to find a solution seems hopeless.[15]

3) The proof of the *ad absurdum* proof (AAP) in next sect requires to derive from a polynomial C*-algebra a C*-algebra of general functions. Open problem.

4) In the case of a finite number of observables the introduction of both Weyl's and Heisenberg' algebras and groups. Open problem.

5) Von Neumann theorem (all regular irreducible representation of Weyl C-*algebras are unitarily equivalent). Open problem.

The already obtained resolutions of the first problems in the above list are comfortable; they mean that the above problems are relevant. However, the difficulties presented by the unsolved ones are formidable.

## 10. Strocchi's Formulation as a PO Theory: The Lacking Characteristic Features

It is not possible to represent the C*-algebra of the complementary relations, based only on algebraic reasoning, through functions, which of course imply classical logic through their equality symbols. This fact leads to suspect that SQM may be a PO theory. An accurate inspection of SQM shows that in fact Strocchi presents most of his theory according to some characteristic features of a PO, illustrated by (Drago, 2012).

First, he lucidly bases his theory on a *problem*. SQM is based on the funda-

[15]Bridges D.: Personal communication, 20/12/2017.

mental problem of how our knowledge can overcome the unavoidable uncertainty of the measurements of two conjugate observables. In particular, he put the problem of which experimental reasons justify the non-commutation relations.

> The main problem is the precise interpretation of the principle [of non commutativity of conjugate variables] in terms of unambiguous experimental operations and its precise mathematical formulation (ivi, p. 15).

Second. He argues by means of the intuitionist logic inside which the law of double negation fails. Indeed, he makes use of doubly negated propositions whose corresponding affirmative propositions lack of evidence or are false (DNPs). In the following, I will list the DNPs occurring in (Strocchi, 2012):[16]

1) It is impossible to measure coherent superpositions of states belonging to different superselection sectors. [≠one measures coherent superpositions of states inside a single sector] (ivi, p. 2).

2) Thus, if two states defined by two apparently different preparation procedures yield the same results of measurements for all observables, i.e. expectations, from an experimental point of view they cannot be considered as physically different [≠ they are the same]··· [to be cont.ed].

3). ... since there is no measurement which distinguishes them [≠the results of all measurements are equal] (ivi, p. 3).

4) Similarly... there is no available operational way to distinguish them [≠all operations give the same result] (ivi, p. 3).

5) … the in-evitable limitations in the preparation of states and measurements of *A* in general preclude the possibility of obtaining sharp values of A, i.e. $\Delta_\omega(A) = 0$··· [≠ the freedom of preparations··· gives... sharp values of···] (ivi, p. 8).

6) *Experimental principle*··· For any given observable A, one can correspondingly prepare states for which a sharp value may be approximated as well as one likes [Here the nature of DNP is given by the point underlined words; they are equivalent to "beyond any bound"; ≠ at the infinity] (ivi, p. 8).

7) This means that it is impossible to have a direct [non mediated] experimental check of the uncertainty relations [≠one has a mediated experimental check of the uncertainty relations]··· [to be cont.ed].

8) ···since one only [=not otherwise ≠ surely] measures bounded functions of the position and the momentum (ivi, p. 8).

A last proposition of this kind is presented by Strocchi when he introduces a crucial notion. Consistently with the PO model of a theory, he looks for the mathematical version of these uncertainty relations by proceeding in a heuristic way. In addition, his main result (the proposition 2.8) is a DNP as it will be proved in the following. In a first time he suggests the new definition of complementarity through a negative word:

[16]In the following I will underline the negative words inside a DNP in order to make apparent its logical nature. Notice that the modal words are equivalent to a DNP (e.g. may: "it not false that it is the case that…"). They will be point underlined.

> *Definition* 2.7. Two observable A, B are called *complementary* if the following bound holds
> $\Delta(A)+\Delta(B)>0$ (ivi, p. 8).
>
> Then he states the DNP 9:
>
> *Proposition* 2.8. If the above experimental principle holds, given a representation $\pi$ of $A$, the existence of two observables $\pi(A)$, $\pi(B)$ which are *complementary*, implies that the C*-algebra $A(A,B)$ generated by $\pi(A)$, $\pi(B)$ cannot be commutative [≠ two observable with $\Delta(A) + \Delta(B) = 0$ commute] (Strocchi, 2012: p. 9) (ivi, p. 9).[17]

The given problem is not considered as solved without showing the relation between the old and the new notions. First, he relaxes the previous limitation of the observables to be represented by polynomial functions.

> The relation between complementarity and non-commutativity is easily displayed if one realizes that in each irreducible representation pi(A) of the algebra of observables one may enlarge the notion of observables by considering as observables the weak limits of any Abelian C*-subalgebra B pi(A). Technically, this amounts to consider the von Neumann algebra $B^w$ generated by B; one may show that the former contains all the spectral projections of the elements of B. In the Gelfand representation of the Abelian C*-algebra B by the set of continuous functions on the spectrum of B, such weak limits correspond to the pointwise limits of the continuous functions. They are operationally defined by instruments whose outcomes yield the pointwise limits of the functions defined by the measurements of the elements of B.
> This means that one recognizes as observables not only the polynomial functions of elements B belongs to *B* and therefore by norm closure the continuous functions of *B*, but also their pointwise limits (ivi, p. 9).

Then the relationship between the two above relations is stated by means of an AAP, exactly the way of reasoning of a PO theory. The argument can be summarized in the following way. By calling "complementarity of A,B" *Cp* and their "commutativity" *Cm*, he wants to prove that when *Cp* holds true then ¬*Cm* follows. He starts by negating the thesis, ¬ ¬*Cm*, which describes a situation where both π(A) and π(B) (according to a von Neumann's theorem) can be written as functions of C, i.e. in this case the C*-algebra is an algebra of functions. Hence, in this algebra the classical logic holds true, and thus ¬ ¬*Cm* → *Cm*. His arguing obtains that *Cm* → ¬*Cp*, i.e. the negation of the starting hypothesis, an absurd. Hence, it is not possible that *Cm* → *Cp*, or, ¬ (*Cp* → ¬*Cm*), i.e. the new notion *Cp* surely grasps more content than the old notion *Cm*.[18]

[17]Notice that the second negative proposition is not a mere explanation of the first negative proportion, because they are different, physical the former one and mathematical the latter one.

[18]Incidentally, in classical logic the proved formula *Cp* → ¬*Cm* is classically equivalent to ¬*Cp* v ¬*Cm* = ¬*Cm* v ¬*Cp* = *Cm* →¬*Cp*.

Yet, the above AAP concerns the relationships of the experimental basis of QM with DvNQM, not the conclusion of the theoretical development of SQM as it occurs in the model of a PO theory. However, the previous development of SQM may be organized anew for fitting the model of a PO theory at the cost to change some its parts. The following are the moves to be performed:

1) To make use of more DNPs than those used by Strocchi.

2) To invent a chain of new AAPs concerning the resolution of the previous problem, more specifically the problem of a faithful representation of a C*-algebra of the operators into Hilbert space; in addition one may include the previous AAP of SQM, i.e. to find out the correct representation of the C*-algebra of the operators into the Hilbert space.

3) At last, one has to apply to the conclusion of the final AAP the principle of sufficient reason for translating this conclusion in an affirmative proposition; from which one has to obtain the symmetries and the results of the measurements.

In sum, apart the stating the basic problem, the entire development of the theory has to invented. The tasks are hard, but a priori not impossible.

## 11. Conclusion

A merit of Strocchi's work is to have suggested two clever and sharp criticisms to respectively BvNQM and the usual mathematical representation of the commutation relation. In addition, from both Segal's and Strocchi's works we have obtained an at all new look at QM: 1) It sharply characterizes its mathematical contents. 2) It constructs a C*-algebra of bounded observables; i.e., it before put the algebra and later the geometry, as Heisenberg's formulation did. 3) It stresses the experimental characteristic feature of Heisenberg's principle, which is represented according to a new mathematical formula. 4) This principle is recognized as constituting the sharp separation mark between Classical mechanics and QM. 5) With respect to the expectations of the measurements his approach deals with the operators, rather that the states, as Hilbert space does. 6) Its theoretical development obtains the symmetries through Weyl's algebra.

Yet, SQM is not the alternative formulation to DvnQM for the following reasons. 1) Its mathematics is only partially the constructive one; in particular, its description of the operative basis of theoretical physics is not enough to fully justify the boundedness of the physical operators, i.e. the postulate of a C*-algebra. 2) Its organization is only partially the problem-based one. 3) In a more specific way, its theoretical development is aimed to recover Hilbert space through a suitable representation, although this space represents the choices AI&AO. 4) Rather than the Haniltonian, one could base the theoretical development of QM on the basic phenomenon of the bodies impact (where moreover the boundedness of all operators is fully justified), rather than the continuous motions of a particle; more in general on discrete phenomena, rather than continuous motions; in terms of formulations, on the physical principle of the Lazare Carnot's mechanics.

All in all, although its starting point is an alternative one, the resulting SQM plays rather an alternative role to BvNQM a parallel role to it.

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