# Exponential Stability Analysis of a System Comprised of a Robot and its Associated Safety Mechanism

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**Abstract:** The system comprised of a robot and its associated safety mechanism is studied in the paper. By the method of strong continuous semi-group, the paper analyzes the restriction of essential spectral growth bound of the system operator. The essential spectral radius of the system operator is also discussed before and after perturbation. The results show that the dynamic solution of the system is exponential stability and tends to the steady solution of the system.

**Key words:** robot; strictly dominant eigenvalue; essential spectrum; disturbance; exponential stability.

## 1 Introduction

With rapid development of science and technology, the application of robots has increased at an impressive rate in the area of industrial sector, construction, fire prevention, underwater exploration, outer space exploration, and medicine. Undoubtedly, robot will be a household good as normal the nearest future. Hence, robot safety and reliability has become an important factor. The authors established the mathematic models of the system consisting of a Robot associated safety mechanism (see system (1)), and studied the steady solution of the system by the method of Laplace-transform[1]. By using the linear operator theory in Banach space, the authors proved that the solution of the system is asymptotic stability and its steady solution is the eigenvector which is corresponding to the eigenvalue 0 of the system operator[2]. In this paper, under more normal assumptions, we will prove the existence of rigorous dominant eigenvalue, and analyze the restriction of essential spectral growth bound of the system operator and the change of the essential spectral radius after perturbation. The results show that the dynamic solution of the system is exponential stability and tends to the steady solution of the system.

## 2 The Basic Assumptions and Mathematic Model of the System

### 2.1 The Basic Assumptions

The following assumptions are associated with the analysis presented in this article:

1) The system is composed of two items: a robot and its associated safety mechanism system.

2) Failures are statistically independent.

3) Times to failure other than that of common-cause failures are exponentially distributed.

4) System fails when the robot fails.

5) The repaired robot or its associated safety mechanism is as good as new.

6) The failed system repair times are arbitrarily distributed.

7) The partially failed system repair times are exponentially distributed.

8) The application of the device of stages method may involve some approximation.

9) Common-cause failure times are gamma distributed.

The following symbols are associated with Figure 1 or its related analyses:

$t$ time

$i$ $i$th state of the system:

$i$=0 (Robot and its associated, safety mechanism working normally),

$i$=l (Robot working normally, safety mechanism failed),

$i$=2 (Robot failed with an incident),

$i$=3 (Robot failed safely),

$i$=4 (Robot failed, safety system operating normally),

$i$=5 (Robot failed due to a common-cause failure),

$i$ =5$a$ (dummy state).

$\lambda_1$ Constant failure rate of the safety mechanism / system.

$\lambda_2$ Constant failure rate of the robot failing with an incident.

$\lambda_3$ Constant failure rate of the robot failing safely.

$\lambda_4$ Constant failure rate of the robot.

$\lambda_{5a}, \lambda_5$ Parameters associated with common-cause failures.

$\mu_i$ Constant repair rate from state $i$; $i$ = 1, 2, 3, 4, 5.

$\mu_i(x)$ Time-dependent repair rate when the failed

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system is in state $i$ and has an elapsed repair time of $x$; for $i = 2,3,4,5$.

$P_i(t)$ Probability that the robot is in state $i$ at time t; $i = 0, 1, 2, 3, 4, 5$.

$p_i(x,t)\Delta x$ The probability that at time $t$, the failed system is in state $i$ and the elapsed repair time lies in the interval $[x, x+\Delta x]$; for $i = 2,3,4,5$.

The state-space diagram of the system is shown in Figure 1.

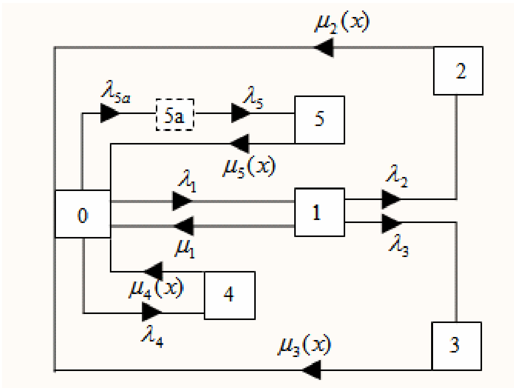


**Figure 1. State space diagram of the system containing a robot and its associated safety mechanism.**

## 2.2 The Mathematical Model

In the assumptions as before, by supplementary variables, we can get the following integral differential equation group which describes the system.

$$\begin{cases} \frac{dP_0(t)}{dt} = -(\lambda + \lambda_4 + \lambda_{5a})P_0(t) + \mu_1 P_1(t) + \sum_{i=2}^{5} \int_0^\infty P_i(x,t)\mu_i(x)dx \\ \frac{dP_1(t)}{dt} = \lambda_1 P_0(t) - (\mu_1 + \lambda_2 + \lambda_3)P_1(t) \\ \frac{dP_{5a}(t)}{dt} = -\lambda_5 P_{5a}(t) + \lambda_{5a} P_0(t) \\ \frac{\partial P_i(x,t)}{\partial x} + \frac{\partial P_i(x,t)}{\partial t} = -\mu_i(x)P_i(x,t), i = 2,3,4,5 \\ P_j(0,t) = \lambda_2 P_j(t) \quad j = 2,3 \\ P_4(0,t) = \lambda_4 P_0(t) \\ P_5(0,t) = \lambda_5 P_{5a}(t) \\ P_0(t) = 1, P_1(0) = P_{5a}(0) = P_i(x,0) = 0, i = 2,3,4,5 \end{cases} \tag{1}$$

The repair function $\mu_i(x)$ is assumed to be bounded[2]. So, when we discuss the system, some practical actual meaning and some good properties will be lost. In fact, in the practical application, $\mu_i(x)$ is unbounded generally. In order to perfect the system (1), we assume

$$\int_0^\infty e^{-\int_0^x \mu_i(\xi)d\xi} dx < +\infty \tag{2}$$

$$0 < c_i < \inf_{x\in R^+} \mu_i(x) < \sup_{x\in R^+} \mu_i(x) = +\infty; \quad i = 2,3,4,5 \tag{3}$$

Assume a state space as following

$$X = C^2 \times (L^1(\mathrm{R}^+))^4 \times C .$$

For $y = (y_0, y_1, y_2(x), y_3(x), y_4(x), y_5(x), y_{5a}(x)) \in X$, the norm of $y$ is defined

$$\|y\| = |y_0| + |y_1| + \|y_2\|_{L^1} + \|y_3\|_{L^1} + \|y_4\|_{L^1} + \|y_5\|_{L^1} + |y_{5a}| .$$

Obviously, ( $X$ , $\|\cdot\|$ ) is a Banach space. Denote

$$a_0 = \lambda_1 + \lambda_4 + \lambda_{5a}, a_1 = \mu_1 + \lambda_2 + \lambda_3 .$$

In $X$ , we define operator

$$A = diag(-a_0, -a_1, -\frac{d}{dx} - \mu_2(x), -\frac{d}{dx} - \mu_3(x), -\frac{d}{dx} - \mu_4(x), -\frac{d}{dx} - \mu_5(x), \lambda_5) \cdot$$

Let the defined region of $A$ be

$$D(A) = \{P \in X \left| \frac{dP_i(x)}{dx} \in L^1(R^+), i = 3,4,5\ P_2(0) = \lambda_2 P_1,\ P_3(0) = \lambda_3 P_1, \right.$$
$$P_4(0) = \lambda_4 P_0, P_5(0) = \lambda_5 P_{5a}\}$$

Define operator $B : X \to X$ , and $B = \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix}$.

$$B_1 = (\mu_1, \int_0^\infty \mu_2(x)dx, \int_0^\infty \mu_3(x)dx, \int_0^\infty \mu_4(x)dx, \int_0^\infty \mu_5(x)dx, 0);$$
$$B_2 = (\lambda_1, 0,0,0,0, \lambda_{5a})^T$$

The system (1) can be described as an abstract Cauchy problem:

$$\begin{cases} \frac{dP(t)}{dt} = (A+B)P(t) \\ P(0) = (1,0,0,0,0,0,0) \\ P(t) = (P_0(t), P_1(t), P_2(x,t), P_3(x,t), P_4(x,t), P_5(x,t), P_{5a}(t)) \end{cases} \tag{4}$$

# 3 The Steady Solution of the System

Firstly, we study the nonzero solution's existence about $(\gamma I - A - B)P = 0$, where

$$P = (P_0, P_1, P_2(x), P_3(x), P_4(x), P_5(x), P_{5a}) .$$

The equation can be expressed by

$$\begin{cases} (\gamma + a_0)P_0 - \mu_1 - \sum_{i=2}^{5} \int_0^\infty P_i(x,t)\mu_i(x)dx = 0 \\ \lambda_1 P_0 - (\gamma + a_1)P_1 = 0 \\ (\gamma + \lambda_5)P_{5a} - \lambda_{5a} = 0 \\ \frac{dP_j(x)}{dx} + (\gamma + \mu_i(x))P_i(x) = 0, i = 2,3,4,5 \\ P_i(0) = \lambda_i P_1 \quad i = 2,3 \\ P_4(0) = \lambda_4 P_0 \\ P_5(0) = \lambda_5 P_{5a} \end{cases} \tag{5}$$

By solving (5), we can get

$$P_i(x) = P_i(0)e^{-\int_0^x (\gamma + \mu_i(\xi))d\xi} \quad i = 2,3,4,5 . \tag{6}$$

So the first equation of (5) become

$$(\gamma+a_0)P_0-\mu_1P_1-\sum_{i=2}^{5}P_i(0)\int_0^\infty\mu_i(x)e^{-\int_0^x(\gamma+\mu_i(\xi))d\xi}dx=0 \quad (7)$$

Let $\mu_{i,\gamma}=\int_0^\infty\mu(x)e^{-\int_0^x(\gamma+\mu_i(\xi))d\xi}dx, i=3,4,5$, then we have a equation group as following

$$\begin{cases}(\gamma+a_0)P_0-\mu_1P_1-\sum_{i=2}^{5}P_i(0)\mu_{i,\gamma}=0\\ \lambda_1P_0-(\gamma+a_1)P_1=0\\ \lambda_{5a}P_0-(\gamma+\lambda_5)P_{5a}=0\\ \lambda_2P_1-P_2(0)=0\\ \lambda_3P_1-P_3(0)=0\\ \lambda_4P_1-P_4(0)=0\\ \lambda_5P_{5a}-P_5(0)=0\end{cases} \quad (8)$$

Denote the coefficient determinant of (8) by $D(\gamma)$, and then we have

$$D(\gamma)=\begin{vmatrix}\gamma+a_0 & -\mu_1 & \mu_{2,\gamma} & \mu_{3,\gamma} & \mu_{4,\gamma} & \mu_{5a} & 0\\ \lambda_1 & -(\gamma+a_1) & 0 & 0 & 0 & 0 & 0\\ \lambda_{5a} & 0 & 0 & 0 & 0 & 0 & -(\gamma+\lambda_5)\\ 0 & \lambda_2 & -1 & 0 & 0 & 0 & 0\\ 0 & \lambda_3 & 0 & -1 & 0 & 0 & 0\\ \lambda & 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & \lambda_5\end{vmatrix}$$

When $\gamma\in C$ and $\gamma$ is an eigenvalue of $A+B$, then $D(\gamma)=0$. In turn, if $\gamma\in C$ and satisfies $D(\gamma)=0$, then the equation group (8) has a nonzero solution $(P_0,P_1,P_2(x),P_3(x),P_4(x),P_5(x),P_{5a})$, hence

$$(P_0,P_1,P_2(0)e^{-\int_0^x(\gamma+\mu_2(\xi))d\xi},P_3(0)e^{-\int_0^x(\gamma+\mu_3(\xi))d\xi},$$
$$P_4(0)e^{-\int_0^x(\gamma+\mu_4(\xi))d\xi},P_5(0)e^{-\int_0^x(\gamma+\mu_5(\xi))d\xi},P_{5a})$$
$$\in D(A+B)$$

and it is a solution of (5). Specially, when $\gamma=0$,

$$\mu_{i,\gamma}=\int_0^\infty\mu_i(x)e^{-\int_0^x(\gamma+\mu_i(\xi))d\xi}dx=1, i=2,3,4,5,$$

hence $D(\gamma)=0$. So, $\gamma=0$ is an eigenvalue of the operator $A+B$, and its latent vector $P$ has the following components:

$$\begin{cases}P_1=\dfrac{\lambda}{a_1}P_0\\ P_i=\dfrac{\lambda_1\lambda_i}{a_1}P_0e^{-\int_0^x\mu_i(\xi)d\xi}, i=2,3\\ P_4(x)=\lambda_4P_0e^{-\int_0^x\mu_4(\xi)d\xi}\\ P_5(x)=\lambda_5P_0e^{-\int_0^x\mu_5(\xi)d\xi}\\ P_{5a}=\dfrac{\lambda_{5a}}{\lambda_5}P_0\end{cases} \quad (9)$$

Let $Q=(1,1,1,1,1,1,1)$, then we have

$$(P,Q)=P_0+P_1+\int_0^\infty P_2(x)dx+$$
$$\int_0^\infty P_3(x)dx+\int_0^\infty P_4(x)dx+\int_0^\infty P_5(x)dx+P_{5a}>0.$$

For any $P\in D(A+B)$, we have $<(A+B)P,Q>=0$, i.e. $(A+B)^*Q=0$, hence $0$ is an simple eigenvalue of the operator $A+B$.

Hence, we can get the steady positive solution of the system

$$\hat{P}_0=(P_0,P_1,P_2(x),P_3(x),P_4(x),P_5(x),P_{5a}) \quad (10)$$

where $p_1,p_i(x),P_{5a}$ $(i=2,3,4,5)$ are expressed as in (9).

## 4. The Exponential Stability of the Solution of System

The authors have proved that the solution of system (1) is the model is progressively steady, and $\gamma=0$ is a simple eigenvalue of the operator of the system[2]. In this section, we will illuminate that with the more strongly conditions, the system has more well stability.

**Theorem 4.1**[2] 1) $\gamma=0$ is a simple eigenvalue of the operator $A+B$; 2) $\{\gamma\in C:\mathrm{Re}\gamma\ge 0,\gamma\ne 0\}\subset\rho(A)$.

**Theorem 4.2** Suppose that $A$ is defined as before and there is a positive number $c$ satisfied

$$c=\min\{c_i,\lambda_1,\lambda_i,\lambda_{5a}\} \quad (i=2,3,4,5).$$

Then, when $\mathrm{Re}\gamma>-c$, we have $\gamma\in\rho(A)$, and

$$\left\|(\gamma I-A)^{-1}\right\|\le\frac{1}{\mathrm{Re}\gamma+c}.$$

**Proof** When $\mathrm{Re}\gamma>-\eta$, for any given

$$y=(y_0,y_1,y_2(x),y_3(x),y_4(x),y_5(x),y_{5a})\in X,$$

we consider the resolvent equation $(\gamma I-A)P=y$, where $P=(P_0,P_1,P_2(x),P_3(x),P_4(x),P_5(x),P_{5a})$.

The analytic expression of the resolvent equation is as following

$$\begin{cases}(\gamma+a_0)P_0=y_0\\(\gamma+a_1)P_1=y_1\\(\gamma+\lambda_5)P_{5a}=y_{5a}\\\dfrac{dP_i(x)}{dx}+(\gamma+\mu_i(x))P_i(x)=y_i(x),\ i=2,3,4,5\\P_i(0)=\lambda_iP_1,\ i=2,3\\P_4(0)=\lambda_4P_0\\P_5(0)=\lambda_5P_{5a}\end{cases}\quad(11)$$

When $\operatorname{Re}\gamma>-c$, we have $\gamma\neq -a_i,(i=0,1)$. By solving (11), we have

$$\begin{cases}P_i=\dfrac{y_i}{\gamma+a_i}\quad i=0,1\\P_{5a}=\dfrac{y_{5a}}{\gamma+\lambda_{5a}}\\P_j(x)=\dfrac{\lambda_jy_1}{\gamma+a_1}e^{-\gamma x-\int_0^x\mu_j(\xi)d\xi}\\\qquad+\int_0^x e^{-\gamma(x-\tau)-\int_\tau^x\mu_j(\xi)d\xi}y_j(\tau)d\tau,\ j=2,3\\P_4(x)=\dfrac{\lambda_4y_0}{\gamma+a_0}e^{-\gamma x-\int_0^x\mu_4(\xi)d\xi}\\\qquad+\int_0^x e^{-\gamma(x-\tau)-\int_\tau^x\mu_4(\xi)d\xi}|y_4(\tau)|d\tau\\P_5(x)=\dfrac{\lambda_5y_{5a}}{\gamma+\lambda_{5a}}e^{-\gamma x-\int_0^x\mu_5(\xi)d\xi}\\\qquad+\int_0^x e^{-\gamma(x-\tau)-\int_\tau^x\mu_5(\xi)d\xi}|y_5(\tau)|d\tau\end{cases}\quad(12)$$

Since, according to the inference [3], we have

$$\|P\|=\sum_{i=o}^{1}|p_i|+\sum_{j=2}^{5}\|P_j\|+|P_{5a}|$$
$$\le\frac{1}{\operatorname{Re}\gamma+c}(|P_0|+|P_1|+\sum_{i=2}^{5}\|P_i\|+|P_{5a}|)=\frac{1}{\operatorname{Re}\gamma+c}\|y\|.$$

It illuminates that when $\operatorname{Re}\gamma+c>0,(\gamma I-A)^{-1}:X\to X$ is bounded, hence $\gamma\in\rho(A)$ and $\|(\gamma I-A)^{-1}\|\le\dfrac{1}{\operatorname{Re}\lambda+c}$.

According to the Lumer-Philips theorem[4], we can conclude the following inference.

**Inference 4.1** Suppose that $A$ and $c$ is defined as before, then the compression semi-group $S(t)$ spanned by operator $A$ is exponentially digressive, namely for any $c>\omega>0$, then $\|S(t)\|\le e^{-\omega t},t\ge 0$.

Since the operator $B$ is a finite rank operator, and then it is a compact operator. According to the operator semi-group theorem and compact perturbation of semi-group, we have the following conclusion.

**Theorem 4.3** Suppose that $A$ and $c$ is defined as before, then the compression semi-group $T(t)$ which is spanned by $A+B$ possesses the properties as follow.

1).When $\gamma\in C$, $\operatorname{Re}\gamma+c>0$, we have

$$\gamma\in\sigma(A)\Leftrightarrow D(\gamma)=0;$$

2).Suppose that $\gamma_0=0$, for any $\gamma_k\in\sigma(A)\cap\{\gamma\in C|\operatorname{Re}\gamma\ge -c,D(\gamma)=0\},\gamma_k\neq\gamma_0$, where $\operatorname{Re}\gamma_{k+1}\le\operatorname{Re}\gamma_k$, $k=1,2,3\ldots N$, then we have $\gamma_0=0$ is a rigorous dominant eigenvalue.

3). Suppose that $\hat{P}_0=(P_1,P_2(x),P_3(x),P_4(x),P_5(x),P_{5a})$ is the steady solution of the system and $\operatorname{Re}\gamma_1<\omega<\gamma_0$, then for any $P\in X$, $Q=(1,1,1,1,1,1,1)$ ,there exists a constant $M(\omega)>0$ such that

$$\|T(t)P-(P,Q)\hat{P}_0\|\le e^{-\omega t}\|P\|,\ \text{where } t\ge 0.$$

**Proof** 1) When $\operatorname{Re}\gamma>-c$, according to the theorem 4.2, we have $\gamma I-A-B=(\gamma-A)(I-R(\gamma,A)B)$.

Since $B$ is a third order rank operator, then $R(\gamma,A)B$ is a compact operator, hence, $\gamma\in\rho(A+B)$ if and only if 1 is not an eigenvalue of $R(\gamma,A)B$. So when $\operatorname{Re}\gamma+c>0$, we have $\gamma\in\sigma(A)\Leftrightarrow D(\gamma)=0$.

2) Since $D(\gamma)$ is an analytic function, then at most there are finite zero points and there is no accumulation point in finite region.

Suppose that $\gamma_0=0$, hence we have

$$\gamma_k\in\sigma(A)\cap\{\gamma\in C|\operatorname{Re}\gamma\ge -c,D(\gamma)=0\},\gamma_k\neq\gamma_0,$$

Where $\operatorname{Re}\gamma_{k+1}\le\operatorname{Re}\gamma_k$, $k=1,2,3\ldots N$. According to the discreteness of eigenvalue and theorem 4.1, we have $\operatorname{Re}\gamma_k\le\gamma_0$. Since the eigenfunction that corresponding to $\gamma_0$ is positive, then we have $\gamma_0=0$ is a rigorous dominant eigenvalue.

3) Finally, by the perturbation theorem of a semi-group, compact perturbation does not change the essential spectrum bound of the semi-group, hence the semi-group $T(t)$ spanned by $A+B$ and semi-group $S(t)$ spanned by $A$ have the same the essential spectrum bound[5-6]. Hence, for the essential spectrum bound of $T(t)$, we have $\omega(A+B)\le\omega_0(A)$.

Suppose that $\hat{P}_0$ is defined as in (10), and $\operatorname{Re}\gamma_1<\omega<\gamma_0$, according to the finite expansion theorem of semi-group, we have the following conclusion. For any $P \in X$, $Q = (1,1,1,1,1,1,1)$ ,there exists a constant $M(\omega) > 0$ such that

$$\left\| T(t)P - (P,Q)\hat{P}_0 \right\| \le M(\omega)e^{-\omega t} \text{, where } t \ge 0 .$$

The above conclusions show that under some definite conditions the dynamic solution of the system is exponential stability and tends to the steady solution of the system.

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