The Proof of Goldbach's Conjecture on Prime Numbers

Silviu Guiasu

Department of Mathematics and Statistics, York University, Toronto, Canada

Correspondence to: Silviu Guiasu, guiasus@pascal.math.yorku.ca
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ABSTRACT

Goldbach's Conjecture ("Every even positive integer strictly larger than 4 is the sum of two primes") has remained unproven since 1742. This paper contains the proof that every positive composite integer n strictly larger than 3, is located at the middle of the distance between two primes, which implicitly proves Goldbach's Conjecture for $2n$ as well.

1. INTRODUCTION

The primes (positive integers divisible only by 1 and themselves) are the fundamental atoms of number theory. According to the fundamental theorem of arithmetic, every positive integer larger than 1 is the unique product of primes. The prime numbers are randomly distributed in the set of positive integers but, there is a hidden regularity and stability in the way they are generated ([1]-[3]).

A conjecture is a plausible statement which has not been proven to be true or false. Number theory is full of conjectures mainly about prime numbers due to their mysterious behavior. The oldest conjecture in number theory is considered to be that formulated long ago by Christian Goldbach. As shown in [4], in his letter to Leonhard Euler, dated 7 June 1742, Christian Golbach formulated several conjectures. Among them, every integer greater than 5 can be written as the sum of three primes. Euler replied in a letter dated 30 June 1742 reminding Goldbach of an earlier conversation they had in which Goldbach remarked that his original conjecture was that every even integer greater than 2 can be written as the sum of two primes. In this letter, Euler stated: "That… every even integer is the sum of two primes, I regard as a completely certain theorem, although I cannot prove it." As Goldbach considered 1 to be a prime, today his conjecture is known as stating that: "Every even integer larger than 4 is the sum of two primes." Along centuries, Goldbach's Conjecture has been verified for larger and larger even integers. Today, there is no shortage of means for verifying Goldbach's conjecture for big even integers. Going to https://www.dcode.fr/goldbach-conjecture and clicking on "Goldbach Conjecture Calculator-Tester-Online Tool-dCode" we get Goldbach solutions for even integers up to 10^7. Recently, Olivera e Silva *et al.* ([5]) claimed that they have empirically verified the conjecture up to 4×10^{18}. Thus, Goldbach's Conjecture has become more plausible but still remained unproven.

If Goldbach's Conjecture is true then to every even positive integer $n > 4$ there correspond two primes p and q such that: $n = p + q$. Assuming $p \le q$, this implies that:

$$n/2 - p = q - n/2,$$

showing that either: $n/2 = p = q$, or, if $p < q$, that $n/2$ is located at the middle of the distance between the two primes.

Conversely, if n is a prime then $2n = n + n$ and Goldbach's Conjecture is true. If n is a positive composite integer strictly larger than 3, located at the middle of the distance between two primes $p < q$, then:

$$n - p = q - n,$$

implying that:

$$2n = p + q.$$

Therefore, if n is a prime, or a positive composite integer $n > 3$, and every such an integer is located at the middle distance between two primes, then $2n$ is the sum of two primes and Goldbach's Conjecture is true for every $2n \ge 6$, and therefore, for every $2n > 4$.

The objective of this paper is to prove that every positive composite integer n, strictly larger than 3, is located at the middle of the distance between two primes and, implicitly, that Goldbach's Conjecture is true for $2n$ as well. The existence and the number of pairs of primes symmetric with respect to the positive integer n and, therefore the existence and the number of Goldbach solutions for $2n$, essentially depend on the symmetric pairs, with respect to n, of relative prime odd composite multiples of primes from the interval $\left(2,(n-1)/3\right]$ (if n is an even composite positive integer) or from the interva $\left(2,(n-2)/3\right]$ (if n is an odd composite positive integer) that are not factors of n.

2. MATCHING SYMMETRIC ODD POSITIVE INTEGERS

2.1. The Symmetry with Respect to an Odd Composite Positive Integer

Assume that n is an odd composite positive integer strictly larger than 3.

Frame I for n odd composite positive integer:

Group 2:	$2n$-3	$2n$-5	⋯	n+4	n+2
Group 1:	3	5	⋯	n-4	n-2

The odd integers from the interval $[0,2n]$, symmetric with respect to n, are put in two groups, as shown in Frame I. Each column contains two odd integers, symmetric with respect to n, called matching integers, whose sum is equal to $2n$. Except n, which is the center of symmetry, the even numbers from the interval $[0,2n]$ are not included in Frame I. Also, the odd integer $2n-1$ is also missing from Frame I because it matches the integer 1 which is not relevant in this context. The smallest integer from group 1 is matched with the largest integer from group 2 but the difference between the matching integers gradually decreases by a constant rate equal to 4.

In each group of Frame I, there are primes or odd composite positive integers. Let us notice that if p_i is a prime factor of n and an integer s_1 from any group is a multiple of p_i, then its matching integer s_2 is also a multiple of p_i because:

$$s_1 + s_2 = 2n$$

As a consequence, if an odd integer s_1 from a group is relative prime with n then its matching integer s_2 is also relative prime with n. An odd prime which is not a factor of n can match with an odd prime or with an odd composite integer relative prime with n but not with an odd composite positive integer divisible by an odd prime factor of n. An odd composite integer can match with a prime or with another odd composite integer but the two odd composite integers must be either relative prime with n or divisible by the same od prime factor of n.

In Frame I, the number of odd integers from group 1 is:

$$N_I = \left[\left((n-2) - 2 \right) / 2 \right] + 1 = \left[(n-4)/2 \right] + 1, \tag{1}$$

and in group 2 the number is the same:

$$N_I = \left[\left((2n-3) - (n+1) \right) / 2 \right] + 1 = \left[(n-4)/2 \right] + 1,$$

where $[x]$ is the integer part of the real number x.

The number of primes in group 1 is: $\pi(n-2)-1$, where $\pi(x)$ is the number of primes not exceeding the real number x. The number of primes in group 2 is: $\pi(2n-3)-\pi(n+1)$. According to Bertrand's theorem ([6]), if $n > 3$, there is at least one prime q such that:

$$n < q < 2n-2 \quad \text{or} \quad n < q \le 2n-3.$$

As n is an odd composite integer, let:

$$n = p_1^{k_1} \cdots p_r^{k_r}$$

be the prime factorization of n, where $p_1, \cdots, p_r$ are odd primes.

The number of odd multiples of p_i in group 1 is:

$$a_I(p_i) = \left(\left(n/(p_i) \right) - 1 \right) / 2, \tag{2}$$

where −1 is needed because n is not in group 1 and cannot be counted amongst the multiples of p_i in group 1. The number of odd multiples of the product $p_i p_j$ in group 1 is:

$$a_I(p_i, p_j) = \left(\left(n/(p_i p_j) \right) - 1 \right) / 2, \tag{3}$$

where −1 is needed because n itself is not in group 1 and cannot be counted amongst the multiples of the product $p_i p_j$ in group 1. The number of odd composite multiples of the product $p_i p_j p_k$ in group 1 is:

$$a_I(p_i, p_j, p_k) = \left(\left(n/(p_i p_j p_k) \right) - 1 \right) / 2, \tag{4}$$

where −1 is needed because n itself is not in group 1 and cannot be counted. amongst the multiples of the product $p_i p_j p_k$ in group 1. And so on, for the other products of prime factors of n.

The total number of odd multiples of prime factors of n in group 1 is:

$$a_I = \sum_i a_I(p_i) - \sum_{i<j} a_I(p_i, p_j) + \sum_{i<j<k} a_I(p_i, p_j, p_k) - \cdots \tag{5}$$

We denote by A_I the set of multiples of prime factors of n in group 1. The number of its elements is a_I.

If an integer s_1 from group 1 is a multiple of an odd prime factor p_i of n, its natching integer s_2 from group 2 is also a multiple of the same prime factor p_i of n because:

$$s_1 + s_2 = 2n.$$

Therefore, counting the multiples of the prime factors of n in group 2 is the same as counting the multiples of the prime factors of n in group 1 with the following essential difference, namely, that in group 2 all odd multiples of the prime factors of n are composite integers. For instance, if p_i is a prime factor of n in group 1, its matching integer from group 2 is a composite multiple of p_i. Consequently, the number of odd multiples of the prime factor p_i of n in group 2 is:

$$b_I(p_i) = a_I(p_i), \tag{6}$$

the number of odd multiples of the product $p_i p_j$ of prime factors of n in group 2 is:

$$b_I\left(p_i,p_j\right)=a_I\left(p_i,p_j\right), \tag{7}$$

the number of odd multiples of the product $p_i p_j p_k$ of prime factors of n in group 2 is:

$$b_I\left(p_i,p_j,p_k\right)=a_I\left(p_i,p_j,p_k\right), \tag{8}$$

and so on for the other products of prime factors of n.

The total number of odd multiples of prime factors of n in group 2 is:

$$b_I=\sum_i b_I\left(p_i\right)-\sum_{i<j} b_I\left(p_i,p_j\right)+\sum_{i<j<k} b_I\left(p_i,p_j,p_k\right)-\cdots=a_I$$

Denote by B_I the set of odd multiples of prime factors of n in group 2. The number of its elements is b_I.

The primes from Frame I are grouped into three disjoint classes:

Class 1 is the set of primes belonging to the interval $\left(2,(n-2)/3\right]$. Let $P_{I,11}$ be the set of primes from Class 1, in group 1, that are not factors of n. Their number is $\pi_{I,11}$.

Class 2 is the set of primes belonging to the interval $\left((n-2)/3,(2n-3)/3\right]$. Let $P_{I,21}$ be the set of primes from Class 2, in group 1, that are not factors of n. Their number is $\pi_{I,21}$.

Class 3 is the set of primes belonging to the interval $\left((2n-3)/3,2n-3\right]$. Let $P_{I,31}$ be the set of primes from Class 3, in group 1, that are not factors of n, and $P_{I,32}$ be the set of primes from Class 3, in group 2, that are not factors of n. The number of elements of $P_{I,31}$ is $\pi_{I,31}$ and the number of elements of $P_{I,32}$ is $\pi_{I,32}$.

The primes from Class 1 have odd composite multiples in both groups. The primes from Class 2 have odd composite multiples only in group 2. The primes from Class 3 have no odd composite multiples in the corresponding Frame I.

Let $M_{I,1}$ be the set of odd composite multiples, in group 1, of primes that are not factors of n, and are relative prime with respect to n. Let $m_{I,1}$ be the number of the elements of the set $M_{I,1}$. Let $M_{I,2}$ be the set of odd composite multiples, in group 2, of primes that are not factors of n, and are relative prime with respect to n. Let $m_{I,2}$ be the number of the elements of the set $M_{I,2}$. Finally, let m_I be the number of integers from $M_{I,1}$ matching integers from $M_{I,2}$. If s_1 and s_2 are two integers from $M_{I,1}$ and $M_{I,2}$, respectively, they are relative prime with respect to n but if they match, namely if $s_1+s_2=2n$, then they must be relative prime with respect to each other, otherwise a common prime p of s_1 and s_2 would be a factor of n, which is absurd. Let m_I be the number of such pairs of matching multiples from $M_{I,1}$ and $M_{I,2}$.

Taking into account the structure of the two groups of Frame I, we have:

$$N_I=\pi_{I,11}+\pi_{I,21}+\pi_{I,31}+m_{I,1}+a_I=\pi_{I,32}+m_{I,2}+b_I. \tag{10}$$

As $a_I=b_I$, from (10) we get:

$$\pi_{I,11}+\pi_{I,21}+\pi_{I,31}+m_{I,1}=\pi_{I,32}+m_{I,2}, \tag{11}$$

Implying:

$$\pi_{I,11}+\pi_{I,21}+\pi_{I,31}+m_{I,1}-m_I=\pi_{I,32}+m_{I,2}-m_I. \tag{12}$$

As $m_{I,1}-m_I$ is the number of multiples from the set $M_{I,1}$ in group 1 matching primes from group 2, and $m_{I,2}-m_I$ is the number of multiples from the set $M_{I,2}$ in group 2 matvhing primes from group 1, the number of pairs of primes matching in Frame I, which is the number of Goldbach solutions for $2n$ is:

$$\pi_{I,11}+\pi_{I,21}+\pi_{I,31}-\left(m_{I,2}-m_I\right)=\pi_{I,32}-\left(m_{I,1}-m_I\right). \tag{13}$$

2.2. The Symmetry with Respect to an Even Composite Positive Integer with Odd Prime Factors

Assume that n is an even composite positive integer strictly larger than 3. Its Frame is:

Frame II for n even composite positive integer:

Group 2:	$2n$-3	$2n$-5	$\cdots$	n+3	n+1
Group 1:	3	5	$\cdots$	n-3	n-1

All the comments made about Frame I remain valid for Frame II as well. Briefly, in Frame II, the number of odd integers from group 1 is:

$$N_{II} = \left[\left((n-1)-2\right)/2\right]+1 = \left[(n-3)/2\right]+1, \tag{14}$$

and in group 2 the number of odd integers is the same:

$$N_{II} = \left[\left((2n-3)-n\right)/2\right]+1 = \left[(n-3)/2\right]+1.$$

The number of primes in group 1 is: $\pi(n)-1$, where -1 is needed because the prime 2 does not belong to group 1 and cannot be counted. The number of primes in group 2 is: $\pi(2n-3)-\pi(n)$.

According to Bertrand's theorem ([6]), if $n>3$, there is at least one prime q such that:

$$n<q<2n-2 \quad \text{or} \quad n<q\le 2n-3.$$

As n is an even composite integer with at least one odd prime factor:

$$n = 2^k p_1^{k_1}\cdots p_r^{k_r},\ (r\ge 1),$$

is the prime factorization of n, where $p_1,\cdots,p_r$ are odd primes.

The number of odd multiples of p_i in group 1 is:

$$a_{II}(p_i) = \left[\left(\left[n/(p_i)\right]-1\right)/2\right]+1, \tag{15}$$

where $[x]$ is the integer part of the real number x. In the above equality, −1 is needed because n is not in group 1 and cannot be counted amongst the composite multiples of p_i in group 1.

The number of odd multiples of the product $p_i p_j$ in group 1 is:

$$a_{II}(p_i,p_j) = \left[\left(\left[n/(p_i p_j)\right]-1\right)/2\right]+1, \tag{16}$$

where −1 is needed because n itself is not in group 1 and cannot be counted amongst the multiples of the product $p_i p_j$ in group 1.

The number of odd multiples of the product $p_i p_j p_k$ in group 1 is:

$$a_{II}(p_i,p_j,p_k) = \left[\left(\left[n/(p_i p_j p_k)\right]-1\right)/2\right]+1, \tag{17}$$

where −1 is needed because n itself is not in group 1 and cannot be counted. amongst the multiples of the product $p_i p_j p_k$ in group 1. And so on, for the other products of prime factors of n.

The total number of odd multiples of prime factors of n in group 1 is:

$$a_{II} = \sum_i a_{II}(p_i) - \sum_{i<j} a_{II}(p_i,p_j) + \sum_{i<j<k} a_{II}(p_i,p_j,p_k) - \cdots \tag{18}$$

We denote by A_{II} the set of multiples of prime factors of n in group 1. The number of its elements is a_{II}.

If an integer s_1 from group 1 is a multiple of an odd prime factor p_i of n, its natching integer s_2 from group 2 is also a multiple of the same prime factor p_i of n because:

$$s_1 + s_2 = 2n.$$

Therefore, counting the multiples of the prime factors of n in group 2 is the same as counting the multiples of the prime factors of n in group 1 with the following essential difference, namely, that in group 2, all odd multiples of the prime factors of n are composite integers. For instance, if p_i is a prime factor of n in group 1, its matching integer from group 2 is a composite multiple of p_i. Consequently, the number of odd composite multiples of the prime factor p_i of n in group 2 is:

$$b_{II}\left(p_i\right)=a_{II}\left(p_i\right) \tag{19}$$

whereas the number of odd multiples of the product $p_i p_j$ of prime factors of n in group 2 is:

$$b_{II}\left(p_i,p_j\right)=a_{II}\left(p_i,p_j\right), \tag{20}$$

the number of odd multiples of the product $p_i p_j p_k$ of prime factors of n in group 2 is:

$$b_{II}\left(p_i,p_j,p_k\right)=a_{II}\left(p_i,p_j,p_k\right), \tag{21}$$

and so on for the other products of prime factors of n.

The total number of odd multiples of prime factors of n in group 2 is:

$$b_{II}=\sum_i b_{II}\left(p_i\right)-\sum_{i<j} b_{II}\left(p_i,p_j\right)+\sum_{i<j<k} b_{II}\left(p_i,p_j,p_k\right)-\cdots=a_{II} \tag{22}$$

Denote by B_{II} the set of odd multiples of prime factors of n in group 2. The number of its elements is b_{II}.

The primes from Frame II are grouped in three disjoint classes:

Class 1 is the set of primes belonging to the interval $\left(2,(n-1)/3\right]$. Let $P_{II,11}$ be the set of primes from Class 1, in group 1, that are not factors of n. Their number is $\pi_{II,11}$.

Class 2 is the set of primes belonging to the interval $\left((n-1)/3,(2n-3)/3\right]$. Let $P_{II,21}$ be the set of primes from Class 2, in group 1, that are not factors of n. Their number is $\pi_{II,21}$.

Class 3 is the set of primes belonging to the interval $\left((2n-3)/3,2n-3\right]$. Let $P_{II,31}$ be the set of primes from Class 3, in group 1, that are not factors of n, and $P_{II,32}$ be the set of primes from Class 3, in group 2, that are not factors of n. The number of elements of $P_{II,31}$ is $\pi_{II,31}$ and the number of elements of $P_{II,32}$ is $\pi_{II,32}$.

The primes from Class 1 have odd composite multiples in both groups. The primes from Class 2 have odd composite multiples only in group 2. The primes from Class 3 have no odd composite multiples in the corresponding Frame II.

Let $M_{II,1}$ be the set of odd composite multiples, in group 1, of primes that are not factors of n, and are relative prime with respect to n. Let $m_{II,1}$ be the number of the elements of the set $M_{II,1}$. Let $M_{II,2}$ be the set of odd composite multiples, in group 2, of primes that are not factors of n, and are relative prime with respect to n. Let $m_{II,2}$ be the number of the elements of the set $M_{II,2}$. Finally, let m_{II} be the number of integers from $M_{II,1}$ matching integers from $M_{II,2}$. If s_1 and s_2 are two integers from $M_{II,1}$ and $M_{II,2}$, respectively, they are relative prime with respect to n but if they match, namely if $s_1+s_2=2n$, then they must be relative prime with respect to each other, otherwise a common prime p of s_1 and s_2 would be a factor of n, which is absurd. Let m_{II} be the number of such pairs of matching multiples from $M_{II,1}$ and $M_{II,2}$.

Taking into account the structure of the two groups of Frame II, we have:

$$N_{II}=\pi_{II,11}+\pi_{II,21}+\pi_{II,31}+m_{II,1}+a_{II}=\pi_{II,32}+m_{II,2}+b_{II}. \tag{23}$$

As $a_{II}=b_{II}$, from (23) we get:

$$\pi_{II,11}+\pi_{II,21}+\pi_{II,31}+m_{II,1}=\pi_{II,32}+m_{II,2}, \tag{24}$$

implying:

$$\pi_{II,11}+\pi_{II,21}+\pi_{II,31}+m_{II,1}-m_{II}=\pi_{II,32}+m_{II,2}-m_{II}. \tag{25}$$

As $m_{II,1}-m_{II}$ is the number of multiples from the set $M_{II,1}$ in group 1 matching primes from group 2, and $m_{II,2}-m_{II}$ is the number of multiples from the set $M_{II,2}$ in group 2 matvhing primes from group 1, the number of pairs of primes matching in Frame I, which is the number of Goldbach solutions for $2n$ is:

$$\pi_{II,11}+\pi_{II,21}+\pi_{II,31}-\left(m_{II,2}-m_{II}\right)=\pi_{II,32}-\left(m_{II,1}-m_{II}\right). \tag{26}$$

2.3. The Symmetry with Respect to a Positive Integer Power of 2

In this case, $n=2^k$ and there are no odd prime factors of *n.* The formalism given for Frame II can still be applied but there are some basic differences. As there are no odd prime factors of *n*, we have $a_{II}=b_{II}=0$. If s_1 is an integer from group 1 and s_2 a matching integer from group 2, we have:

$$s_1+s_2=2^k.$$

If s_1 is a prime, then s_2 cannot be a multiple of s_1 because this would make s_1 to be a prine factor of 2^k, which is absurd. Similarly, if s_2 is a prime, then s_1 cannot be a multiple of s_2. Also, s_1 and s_2 must be relative prime because if they have a common prime factor *p*, it would be a factor of 2^k as well, which is absurd. The sets $M_{II,1}$ and $M_{II,2}$ consist of the odd multiples of the primes from Class 1 in group 1. Two such matching multiples must be relative prime. Therefore, the number m_{II}, from (26), is the number of pairs of matching relative prime odd multiples of some of the $\pi_{II,11}$ primes from the interval $\left(2,(n-1)/3\right]$.

3. NUMERICAL EXAMPLES

In this Section we take successively the numerical values *n* = 21, 30, 32, 33, and 50. The problem is not to get the corresponding Goldbach solutions of the corresponding $2n$ as a sum of two primes, because Goldbach solutions for the corresponding even integers $2n$ = 42, 60, 64, 66, and 100, may be obtained online, by typing on GOOGLE: "Goldbach Conjecture Calculator-Tester-Online Tool-dCode" and running it, one by one, for the values of $2n$ given above. These numerical examples are given here just to illustrate how the formulas from Section 2 may be used for justifying the existence and the number of Goldbach solutions in these cases. In all these examples we use formulas (1)-(13) for Frame I and (14)-(26) for Frame II.

Example 1. Let $n=21=3\times 7$. The corresponding Frame I is:

Group 2:	39	37	35	33	31	29	27	25	23
Group 1:	3	5	7	9	11	13	15	17	19

Applying the formulas (1)-(13), we get: $N_I=9$,

$A_I=\{3,7,9,15\}$, $a_I=4$, $B_I=\{39,35,33,27\}$, $b_I=4$,

$\text{Class }1=\{3,5,7\}$, $P_{I,11}=\{5\}$, $\pi_{I,11}=1$,

$\text{Class }2=\{11,13\}$, $P_{I,21}=\{11,13\}$, $\pi_{I,21}=2$,

$\text{Class }3=\{17,19,23,29,31,37\}$, $P_{I,31}=\{17,19\}$, $\pi_{I,31}=2$,

$P_{I,32}=\{23,29,31,37\}$, $\pi_{I,32}=4$,

$M_{I,1}=\varnothing$, $m_{I,1}=0$, $M_{I,2}=\{25\}$, $m_{I,2}=1$, $m_I=0$.

The number of matching primes, or the number of Goldbach solutions for 42, is:

$$\pi_{I,11}+\pi_{I,21}+\pi_{I,31}-\left(m_{I,2}-m_I\right)=4,$$

or, equivalently,

$$\pi_{I,32}-\left(m_{I,1}-m_I\right)=4.$$

Indeed, they are: (5, 37), (11, 31), (13, 29), (19, 23).

Example 2. Let $n=30=2\times 3\times 5$. The corresponding Frame II is:

Group 2:	57	55	53	51	49	47	45	43	41	39	37	35	33	31
Group 1:	3	5	7	9	11	13	15	17	19	21	23	25	27	29

Applying the formulas (14)-(26), we get: $N_{II}=14$,

$A_{II}=\{3,5,9,15,21,25,27\}$, $a_{II}=7$,

$B_{II}=\{57,55,51,45,39,35,33\}$, $b_{II}=7$,

$\text{Class }1=\{3,5,7\}$, $P_{II,11}=\{7\}$, $\pi_{II,11}=1$,

$\text{Class }2=\{11,13,17,19\}$, $P_{II,21}=\{11,13,17,19\}$, $\pi_{II,21}=4$,

$\text{Class }3=\{23,29,31,37,41,43,47,53\}$, $P_{II,31}=\{23,29\}$, $\pi_{II,31}=2$,

$P_{II,32}=\{31,37,41,43,47,53\}$, $\pi_{II,32}=6$,

$M_{II,1}=\varnothing$, $m_{II,1}=0$, $M_{II,2}=\{49\}$, $m_{II,2}=1$, $m_{II}=0$.

The number of matching primes, or the number of Goldbach solutions for 60, is:

$$\pi_{II,11}+\pi_{II,21}+\pi_{II,31}-\left(m_{II,2}-m_{II}\right)=6,$$

or, equivalently,

$$\pi_{II,32}-\left(m_{II,1}-m_{II}\right)=6.$$

Indeed, they are: (7, 53), (13, 47), (17, 43), (19, 41), (23, 37), (29, 31).

Example 3. Let $n=32=2^5$. The corresponding Frame II is:

Group 2:	61	59	57	55	53	51	49	47	45	43	41	39	37	35	33
Group 1:	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31

There are no odd prime factors of *n*. Applying the formulas (14) and (23)-(26), we get:

$N_{II}=15$,

$\text{Class }1=\{3,5,7\}$, $P_{II,11}=\{3,5,7\}$, $\pi_{II,11}=3$,

$\text{Class }2=\{11,13,17,19\}$, $P_{II,21}=\{11,13,17,19\}$, $\pi_{II,21}=4$,

$\text{Class }3=\{23,29,31,37,41,43,47,53,59,61\}$, $P_{II,31}=\{23,29,31\}$, $\pi_{II,31}=3$,

$P_{II,32}=\{37,41,43,47,53,59,61\}$, $\pi_{II,32}=7$,

$M_{II,1} = \{9,15,21,25,27\}$, $m_{II,1} = 5$,

$M_{II,2} = \{57,55,51,49,45,39,35,33\}$, $m_{II,2} = 8$, $m_{II} = 3$.

The number of matching primes, or the number of Goldbach solutions for 64, is:

$$\pi_{II,11} + \pi_{II,21} + \pi_{II,31} - (m_{II,2} - m_{II}) = 5,$$

or, equivalently,

$$\pi_{II,32} - (m_{II,1} - m_{II}) = 5.$$

Indeed, they are: (3, 61), (5, 59), (11, 53), (17, 47), (23, 41). Let us notice that in the three pairs of matching odd composite multiples, namely,

$$\begin{matrix} 55 & 49 & 39 \\ 9 & 15 & 25 \end{matrix}$$

the components are relative prime multiples of prime numbers from Class 1.

Example 4: Let $n = 33 = 3 \times 11$. The corresponding Frame I is:

Group 2:	63	61	59	57	55	53	51	49	47	45	43	41	39	37	35
Group 1:	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31

Applying the formulas (1)-(13), we get: $N_I = 15$,

$A_I = \{3,9,111,15,21,27\}$, $a_I = 6$,

$B_I = \{63,57,55,51,45,39\}$, $b_I = 6$,

$\text{Class } 1 = \{3,5,7\}$, $P_{I,11} = \{5,7\}$, $\pi_{I,11} = 2$,

$\text{Class } 2 = \{11,13,17,19\}$, $P_{I,21} = \{13,17,19\}$, $\pi_{I,21} = 3$,

$\text{Class } 3 = \{23,29,31,37,41,43,47,53,59,61\}$, $P_{I,31} = \{23,29,31\}$, $\pi_{I,31} = 3$,

$P_{I,32} = \{37,41,43,47,53,59,61\}$, $\pi_{I,32} = 7$,

$M_{I,1} = \{25\}$, $m_{I,1} = 1$, $M_{I,2} = \{49,35\}$, $m_{I,2} = 2$, $m_I = 0$.

The number of matching primes, or the number of Goldbach solutions for 66, is:

$$\pi_{I,11} + \pi_{I,21} + \pi_{I,31} - (m_{I,2} - m_I) = 6$$

or, equivalently,

$$\pi_{I,32} - (m_{I,1} - m_I) = 6.$$

Indeed, they are: (5, 61), (7, 59), (13, 53), (19, 47), (23, 43), (29, 37).

Example 5. Let $n = 50 = 2 \times 5^2$.

Using the formulas (14)-(26), we get: $N_{II} = 24$,

$A_{II} = \{5,15,25,35,45\}$, $a_{II} = 5$, $B_{II} = \{95,85,75,65,55\}$, $b_{II} = 5$,

$\text{Class } 1 = \{3,5,7,11,13\}$, $P_{II,11} = \{3,7,11,13\}$, $\pi_{II,11} = 4$,

$\text{Class } 2 = \{17,19,23,29,31\}$, $P_{II,21} = \{17,19,23,29,31\}$, $\pi_{II,21} = 5$,

$\text{Class } 3 = \{37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}$,

$P_{II,31} = \{37, 41, 43, 47\}$, $\pi_{II,31} = 4$,

$P_{II,32} = \{53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}$, $\pi_{II,32} = 10$,

$M_{II,1} = \{9, 21, 27, 33, 39, 49\}$, $m_{II,1} = 6$,

$M_{II,2} = \{93, 91, 87, 81, 77, 69, 63, 57, 51\}$, $m_{II,2} = 9$, $m_{II} = 2$.

The number of matching primes, or the number of Goldbach solutions for 100, is:

$$\pi_{II,11} + \pi_{II,21} + \pi_{II,31} - (m_{II,2} - m_{II}) = 6$$

or, equivalently,

$$\pi_{II,32} - (m_{II,1} - m_{II}) = 6.$$

Indeed, they are: (3, 97), (11, 89), (17, 83), (29, 71), (41, 59), (47, 53).

Let us notice that in the two pairs of matching odd composite multiples, namely,

$$\begin{matrix} 91 & 51 \\ 9 & 49 \end{matrix}$$

the components are relative prime multiples of primes from Class 1 that are not factors of n.

4. CONCLUSION

Goldbach's Conjecture is the oldest conjecture in number theory. It states that every even integer strictly greater than 4 is the sum of two primes. There have been many empirical verifications of it, up to astronomic numbers, but it has remained unproven since 1742. The objective of this paper is to formulate an equivalent property about a hidden symmetry of primes stating that for every positive composite number n, strictly larger than 3, there are two primes symmetric with respect to n. The paper contains a proof of this prime symmetry property and, implicitly, of Goldbach's conjecture for $2n$ as well. To prove that every positive composite integer n, strictly larger than 3, is located at the middle of the distance between two primes is equivalent to prove that Goldbach's Conjecture is true for $2n$ as well. The existence and the number of pairs of primes symmetric with respect to the positive integer n and, therefore the existence and number of Goldbach solutions for $2n$, essentially depend on the symmetric pairs with respect to n of relative prime odd composite multiples of primes from the interval $(2, (n-1)/3]$ (if n is an even composite positive integer) or from the interval $(2, (n-2)/3]$ (if n is an odd composite positive integer) that are not factors of n.

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CONFLICTS OF INTEREST

The author declares no conflicts of interest regarding the publication of this paper.

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