

Tensor-Centric Warfare IV: Kähler Dynamics of Battlefields

Vladimir Ivancevic¹, Darryn Reid¹, Peyam Pourbeik²

¹Joint and Operations Analysis Division, Defence Science & Technology Group, Adelaide, Australia

²Cyber and Electronic Warfare Division, Defence Science & Technology Group, Adelaide, Australia

Email: Vladimir.Ivancevic@dsto.defence.gov.au, Darryn.Reid@dsto.defence.gov.au, Peyam.Pourbeik@dsto.defence.gov.au

How to cite this paper: Ivancevic, V., Reid, D. and Pourbeik, P. (2018) Tensor-Centric Warfare IV: Kähler Dynamics of Battlefields. *Intelligent Control and Automation*, 9, 123-146.

<https://doi.org/10.4236/ica.2018.94010>

Received: September 28, 2018

Accepted: November 12, 2018

Published: November 15, 2018

Copyright © 2018 by authors and Scientific Research Publishing Inc.

This work is licensed under the Creative Commons Attribution International License (CC BY 4.0).

<http://creativecommons.org/licenses/by/4.0/>



Open Access

Abstract

This paper presents the complex dynamics synthesis of the combat dynamics series called *tensor-centric warfare* (TCW; for the first three parts of the series, see [1] [2] [3]), which includes tensor generalization of classical Lanchester-type combat equations, entropic Lie-dragging and commutators for modeling warfare uncertainty and symmetry, and various delta-strikes and missiles (both deterministic and random). The present paper gives a unique synthesis of the Red vs. Blue vectorfields into a single complex battle-vectorfield, using dynamics on Kähler manifolds as a rigorous framework for extending the TCW concept. The global Kähler dynamics framework, with its rigorous underpinning called the Kähler-Ricci flow, provides not only a new insight into the “geometry of warfare”, but also into the “physics of warfare”, in terms of Lagrangian and Hamiltonian structures of the battlefields. It also provides a convenient and efficient computational framework for entropic wargaming.

Keywords

Tensor-Centric Warfare, Kähler Geometry, Complex Battle-Vectorfield, Lagrangian and Hamiltonian Battlefields, Kähler-Ricci Flow, Entropic Computational Wargaming

1. Introduction

In the series of papers called the *tensor-centric warfare* (TCW; see [1] [2] [3]) we have developed a tensor architecture for general Red-Blue combat dynamics. The TCW framework starts by providing tensorial generalization of the *Lanchester*-type combat equations [1], and then includes *entropic Lie-dragging* (for modeling warfare uncertainty) and *commutators* (for analysis of warfare

symmetry) [2], as well as various kinds of *delta-strikes* and missiles [3]. The whole TCW architecture is defined by the following pair of Red-and-Blue tensorial dynamical systems (formally, the pair of Red-Blue vectorfields):

$$\begin{aligned} \text{Red: } \overbrace{\partial_t R^a}^{\text{Red.vecfield}} &= \overbrace{kA_b^a B^b}^{\text{lin.Lanchester}} + \overbrace{k_b F_{cd}^{ab} R^c B^d}^{\text{quad.Lanchester}} + \overbrace{R^b \mathcal{L}_R \mathcal{N}_b^a}^{\text{Lie.dragging}} + \overbrace{\left[R^a, B^a \right]}^{\text{war.symmetry}} + \overbrace{\delta R^a}^{\text{delta.strikes}} (\text{H-L}), \\ \text{Blue: } \overbrace{\partial_t B^a}^{\text{Blue.vecfield}} &= \overbrace{\kappa C_b^a R^b}^{\text{lin.Lanchester}} + \overbrace{\kappa_b G_{cd}^{ab} R^c B^d}^{\text{quad.Lanchester}} + \overbrace{B^b \mathcal{L}_B \mathcal{N}_b^a}^{\text{Lie.dragging}} + \overbrace{\left[B^a, R^a \right]}^{\text{war.symmetry}} + \overbrace{\delta B^a}^{\text{delta.strikes}} (\text{H-L}). \end{aligned} \quad (1)$$

In Equations (1) the Red and Blue Hamilton-Langevin delta strikes, δR^a (H-L) and δB^a (H-L), partially derived from the Ising-type *battle Hamiltonian*: $H = -J_{ab} R^a B^b$, with the connection tensor:

$J_{ab} = A_b^a C_a^b \eta_{ab}$ (weighted by the random noise; see [3] for details), read:

$$\begin{aligned} \delta R^a (\text{H-L}) &= \overbrace{\sum_{j=1}^N \alpha_j^a \delta(t - \tau_j^R)}^{\text{disc.spectrum}} + \overbrace{\int_{t_0}^{t_1} \alpha^a(t) \delta(t - \tau_R) dt}^{\text{cont.spectrum}} + \overbrace{\alpha^a \sum_{j=1}^N \delta(t - \tau_j^R) (\pm \rho^R)_j}^{\text{bidirect.rnd}} \\ &\quad + \overbrace{\left[R^a, B^a \right] \frac{\partial H}{\partial B^b} R^b}^{\text{RedHam.vecfield}} - \overbrace{\gamma_b^a R^b}^{\text{self.dissipat}} - \overbrace{\gamma^{ab} \frac{\partial H}{\partial B^b}}^{\text{oppon.dissipat}} + \overbrace{f_{\text{rnd}}^a(t)}^{\text{rnd.force}}, \\ \delta B^a (\text{H-L}) &= \overbrace{\sum_{j=1}^M \beta_j^a \delta(t - \tau_j^B)}^{\text{disc.spectrum}} + \overbrace{\int_{t_0}^{t_1} \beta^a(t) \delta(t - \tau_B) dt}^{\text{cont.spectrum}} + \overbrace{\beta^a \sum_{j=1}^M \delta(t - \tau_j^B) (\pm \rho^B)_j}^{\text{bidirect.rnd}} \\ &\quad + \overbrace{\left[B^a, R^a \right] \frac{\partial H}{\partial R^b} B^b}^{\text{BlueHam.vecfield}} - \overbrace{\chi_b^a B^b}^{\text{self.dissipat}} - \overbrace{\chi^{ab} \frac{\partial H}{\partial R^b}}^{\text{oppon.dissipat}} + \overbrace{g_{\text{rnd}}^a(t)}^{\text{rnd.force}}. \end{aligned} \quad (2)$$

In the Red-Blue Equations (1)-(2), $\partial_t \equiv \partial/\partial t$ and the Red and Blue forces are defined as vectors $R^a = R^a(x, t) \in M_{\text{Red}}$ and $B^a = B^a(q, t) \in M_{\text{Blue}}$, defined on their respective configuration n -manifolds M_{Red} (with local coordinates $\{x^a\}$, for $a = 1, \dots, n$) and M_{Blue} (with local coordinates $\{q^a\}$). The Red and Blue vectorfields, $\partial_t R^a$ and $\partial_t B^a$, include the following terms (placed on the right-hand side of Equations (1)):

- Linear Lanchester-type terms, $kA_b^a B^b \in M_{\text{Red}}$ and $\kappa C_b^a R^b \in M_{\text{Blue}}$, with combat tensors A_b^a and C_b^a defined via bipartite and tripartite adjacency matrices, respectively defining Red and Blue aircraft formations (according to the aircraft-combat scenario from [4] and [1]);
- Quadratic Lanchester-type terms, $k_b F_{cd}^{ab} B^c R^d \in M_{\text{Red}}$ and $\kappa_b G_{cd}^{ab} R^c B^d \in M_{\text{Blue}}$, with the 4th-order tensors F_{cd}^{ab} and G_{cd}^{ab} representing strategic, tactical and operational capabilities of the Red and Blue forces (see [1]);
- Entropic Lie-dragging of the opposite side terms, $R^b \mathcal{L}_R \mathcal{N}_b^a \in M_{\text{Red}}$ and $B^b \mathcal{L}_B \mathcal{N}_b^a \in M_{\text{Blue}}$, where $\mathcal{N}_b^a = C_b^a + G_{bc}^{ca}$ and $\mathcal{N}_b^a = A_b^a + F_{bc}^{ca}$. In case of *resistance*, the Lie derivatives are positive, $|\mathcal{L}_R \mathcal{N}_b^a| > 0$ and $|\mathcal{L}_B \mathcal{N}_b^a| > 0$, so that the non-equilibrium battlefield *entropy* grows, $\partial_t S > 0$; in case of non-resistance, the Lie derivatives vanish, $|\mathcal{L}_R \mathcal{N}_b^a| = 0$ and $|\mathcal{L}_B \mathcal{N}_b^a| = 0$, so that the battlefield entropy is conserved, $\partial_t S = 0$ (see [2]);
- Entropic Red-Blue commutators, $\left[\left[R^a, B^a \right] \right] \geq 0 \in M_{\text{Red}}$ and

$\left[\left[B^a, R^a \right] \right] \geq 0 \in M_{\text{Blue}}$, for modeling warfare *symmetry* (see [2]);

- Hamilton-Langevin delta strikes, δR^a (H-L) and δB^a (H-L), on both sides, including discrete striking spectra (slow-fire missiles) and continuous striking spectra (rapid-fire missiles), as well as bidirectional random strikes, Hamiltonian vectorfields, self-dissipation, opponent-caused dissipation and non-delta random forces (see [3]).

The first three models of the TCW series have been developed on the Red and Blue configuration manifolds, M_{Red} and M_{Blue} , intentionally without specifying any geometric structures on these manifolds. In the present paper, we use the most sophisticated geometric structure of Kähler manifolds, which allows development of both Lagrangian and Hamiltonian formalisms on it. Here we summarize and reformulate the two Red-and-Blue dynamical systems (1)-(2) in the form of a unique *Kähler dynamical system*, together with its specific geometrical underpinning called the Kähler-Ricci flow. This sophisticated geometric framework gives a new insight into deep mathematical and physical structures of battlefields and also provides a convenient computational wargaming framework.

2. Kähler Dynamics of Battlefields

The concept of *Kähler dynamics*, or tensor dynamics on Kähler manifolds (see **Appendix 1** for a technical exposition), has been formally developed in [5], based on our previous work on self-organization entropy [6], controllable complexity [7] and autonomy of cyber-physical-cognitive systems [8].

Briefly, the *Kähler dynamics* is defined by the complex-valued vectorfield:

$$\partial_t V^a(z^a, t) = \partial_t R^a(x^a, t) + i \partial_t B^a(q^a, t),$$

which flows along the *Kähler battle-manifold* \mathcal{K} defined by the complexified sum (*i.e.*, the sum with the imaginary unit: $i^2 = -1$):

$$\widehat{\mathcal{K}} = \underbrace{\partial_t V^a(z^a, t)}_{\partial_t R^a(x^a, t)} + i \underbrace{\partial_t B^a(q^a, t)}_{\partial_t B^a(q^a, t)},$$

where TM_{Red} is the tangent bundle (or, velocity phase-space) of the Red forces with Riemannian structure g_R and natural Lagrangian dynamics (derived from the Lagrangian energy function L), and T^*M_{Blue} is the cotangent bundle (or, momentum phase-space) of the Blue forces with symplectic structure ω_S and natural Hamiltonian dynamics (derived from the Hamiltonian energy function H).

More specifically, a *Kähler manifold*, $\mathcal{K} \equiv (\mathcal{K}, g) \equiv (\mathcal{K}, \omega)$, represents a Hermitian manifold of complex dimension $2n$ (or, real dimension $4n$), defined by the Hermitian metric form: $g = g_R + i\omega_S$, where $g_R = g_{ab} dx^a dx^b \in TM_{\text{Red}}$ is the Riemannian metric on the TM_{Red} tangent bundle and $\omega_S = dp_a \wedge dq^a \in T^*M_{\text{Blue}}$ is the symplectic form on the T^*M_{Blue} cotangent bundle. In our case of the two-party battlefield with the Red forces defined by the (real) configuration n -manifold M_{Red} (with its tangent bundle $TM_{\text{Red}} = \sqcup_{x \in M_{\text{Red}}} T_x M_{\text{Red}}$ which is

the Riemannian $2n$ -manifold) and the Blue forces defined by the configuration n -manifold M_{Blue} (with its cotangent bundle $T^*M_{\text{Blue}} = \sqcup_{q \in M_{\text{Blue}}} T_q^*M_{\text{Blue}}$ which is the symplectic $2n$ -manifold),¹ our Kähler battle-manifold \mathcal{K} is defined as the complexified sum:

$(\mathcal{K}, g) = TM_{\text{Red}} + iT^*M_{\text{Blue}}$ with the Hermitian metric form g :
 $g = g_R + i\omega_S$, where $g_R = g_{ab} dx^a dx^b \in TM_{\text{Red}}$, $\omega_S = dp_a \wedge dq^a \in T^*M_{\text{Blue}}$,
 with the local coordinates on the component bundles, $(x^a, \dot{x}^a) \in TM_{\text{Red}}$ and $(q^a, p_a) \in T^*M_{\text{Blue}}$. For further technical details on Kähler manifolds, see **Appendix 1**.

The unique battlefield dynamics defined by the *battle-vectorfield*, $\partial_t V^a(z^a, t)$, has several advantages over the real-valued Red-Blue Equations (1)-(2):

- $\partial_t V^a(z^a, t)$ is mathematically more consistent than the pair $[\partial_t R^a(x^a, t), \partial_t B^a(q^a, t)]$, since dynamics in the complex plane \mathbb{C} includes dynamics in the real plane \mathbb{R}^2 but reveals much richer structure (including polar form, Euler relation, conjugation, etc.);
- $\partial_t V^a(z^a, t)$ has a rigorous geometric underpinning called the Kähler-Ricci flow;
- $\partial_t V^a(z^a, t)$ gives a new insight into the physics of warfare in terms of its natural/embedded Lagrangian and Hamiltonian dynamics, and
- $\partial_t V^a(z^a, t)$ has a straightforward implementation in the computational wargame called the *Entropy Battle*.

Deep mathematical and physical aspects of this new concept are briefly defined in the next two subsections, based on the rigorous technical exposition given in **Appendix 1**.

2.1. Geometry of Warfare: Kähler Battle-Vectorfield

Our Kähler dynamics of the battlefield is defined as a complex-valued nD vectorfield $\partial_t V^a(z^a, t)$, called the *Kähler battle-vectorfield*, flowing along the Kähler battle-manifold \mathcal{K} and defined in the following two steps. Firstly, the above two real-valued Red-and-Blue vectorfields (1) can be rewritten in terms of the real and imaginary components of a single complex-valued vectorfield, defined on \mathcal{K} as:

$$\begin{aligned} \text{Red} : \partial_t \text{Re}(V)^a &= kA_b^a \text{Im}(V)^b + k_b F_{cd}^{ab} \text{Im}(V)^c \text{Re}(V)^d + \text{Re}(V)^b \mathcal{L}_{\text{Re}(V)} \mathcal{N}_b^a \\ &\quad + [\text{Re}(V)^a, \text{Im}(V)^a] + \text{Re}(\delta V^a \text{ (H-L)}) \\ \text{Blue} : \partial_t \text{Im}(V)^a &= \kappa C_b^a \text{Re}(V)^b + \kappa_b G_{cd}^{ab} \text{Im}(V)^c \text{Re}(V)^d + \text{Im}(V)^b \mathcal{L}_{\text{Im}(V)} \mathcal{N}_b^a \\ &\quad + [\text{Im}(V)^a, \text{Re}(V)^a] + \text{Im}(\delta V^a \text{ (H-L)}) \end{aligned} \tag{3}$$

Secondly, from the split real-Red and imaginary-Blue vectorfields, $\partial_t \text{Re}(V)^a$ and $\partial_t \text{Im}(V)^a$ in (3), we can directly compose the following single

¹It is clear that we can reverse this formal representation of the Red and Blue forces, so that the Blue manifold M_{Blue} admits the tangent bundle TM_{Blue} and the Red manifold admits the cotangent bundle T^*M_{Red} .

complex-valued vectorfield, $\partial_t V^a(z^a, t): \mathcal{K} \rightarrow \mathbb{C}$, as a unique description of the battlefield dynamics:

$$\partial_t V^a = k \mathcal{A}_b^a V^b + k_b \mathcal{F}_{cd}^{ab} V^c V^d + V^b \mathcal{L}_V \mathcal{N}_b^a + [\text{R-I}(V)^a, \text{I-R}(V)^a] + \delta V^a \text{ (H-L)}, \quad (4)$$

where $V^a(z^a, t) = R^a(x^a, t) + iB^a(q^a, t) \in \mathcal{K}^a$ is the unique complex vector. Its time derivative, $\partial_t V^a(z^a, t)$, is our main actor, the *battle-vectorfield*, defined as the mapping from the Kähler battle-manifold \mathcal{K} to the complex plane \mathbb{C} :

$$\partial_t V^a(z^a, t) = \partial_t R^a(x^a, t) + i\partial_t B^a(q^a, t): \mathcal{K} \rightarrow \mathbb{C}.$$

The battle-vectorfield $\partial_t V^a(z^a, t)$, defined by the complex-valued system of tensor differential Equations (4), represents a dynamical game played on the Kähler battle-manifold \mathcal{K} , in which the actors are the following complex tensors:

- $\mathcal{A}_b^a = A_b^a + iC_b^a \in \mathcal{K}$,
- $\mathcal{F}_{cd}^{ab} = F_{cd}^{ab} + iG_{cd}^{ab} \in \mathcal{K}$,
- $\mathcal{L}_V \mathcal{N}_b^a = \mathcal{L}_{\text{Re}(V)} \mathcal{N}_b^a + i\mathcal{L}_{\text{Im}(V)} \mathfrak{N}_b^a \in \mathcal{K}$,
- $[\text{R-I}(V)^a, \text{I-R}(V)^a] = [\text{Re}(V)^a, \text{Im}(V)^a] + i[\text{Im}(V)^a, \text{Re}(V)^a] \in \mathcal{K}$, and
- $\delta V^a \text{ (H-L)} = \text{Re}[\delta V^a \text{ (H-L)}] + i\text{Im}[\delta V^a \text{ (H-L)}] \in \mathcal{K}$.

The promised rigorous geometric underpinning of the battle-vectorfield $\partial_t V^a(z^a, t) \in \mathcal{K}$, defined by (4), is provided by the *Kähler-Ricci (KR) flow*, a geometric-dynamics structure defined on the Kähler battle-manifold $\mathcal{K} \equiv (\mathcal{K}, g) \equiv (\mathcal{K}, \omega)$ in the following four equivalent ways:

1) Globally, in terms of the Kähler form $\omega = \omega(t)$ and the Ricci curvature form $\text{Ric}[\omega(t)]$, as:

$$\partial_t \omega(t) = -\text{Ric}[\omega(t)], \quad \omega(0) = \omega_0.$$

2) Locally, in terms of the Kähler metric tensor $g_{\bar{j}j} = g_{\bar{j}j}(t)$ and the Ricci curvature tensor $R_{\bar{j}j} = R_{\bar{j}j}(t)$, as:

$$\partial_t g_{\bar{j}j}(t) = g_{\bar{j}j}(t) - R_{\bar{j}j}(t), \quad g_{\bar{j}j}(0) = g_0.$$

3) In terms of the Kähler potential $\varphi = \varphi(t)$ and volume forms $(\omega^n, \omega_\varphi^n)$ as:

$$\partial_t \varphi(t) = \varphi(t) + \log(\omega_\varphi^n / \omega^n) - g(t), \quad \varphi(0) = \varphi_0.$$

4) In the form of the *Monge-Ampère equation* (with Dolbeault's $(\partial, \bar{\partial})$ -operators and the Kähler class condition, $\omega_0 + i\partial\bar{\partial}\varphi > 0$):

$$\partial_t \varphi(t) = \log\left[\left(\omega_0 + i\partial\bar{\partial}\varphi\right)^n / \omega^n\right], \quad \varphi(0) = \varphi_0.$$

The solutions of these four KR equations are called the *KR solitons*. They uniquely exist in the case of Kähler-Einstein metric: $R_{\bar{j}j} = \lambda g$, for some real constant λ . KR solitons can be threefold: *shrinking* (if $\lambda > 0$), *steady* (if $\lambda = 0$), or *expanding* (if $\lambda < 0$). For more technical details on the Kähler-Ricci

flow, see **Appendix 1.2**.

In summary, the proposed Red-Blue combat dynamics model:

$$\partial_t V^a(z^a, t) = \partial_t R^a(x^a, t) + i \partial_t B^a(q^a, t)$$

is defined on the Kähler battle-manifold \mathcal{K} by a single battle-vectorfield:

$$\partial_t V^a = k \mathcal{A}_b^a V^b + k_b \mathcal{F}_{cd}^{ab} V^c V^d + V^b \mathcal{L}_V \mathcal{N}_b^a + \left[\mathbf{R}\text{-I}(V)^a, \mathbf{I}\text{-R}(V)^a \right] + \delta V^a \text{ (H-L)},$$

which is underpinned by the geometric Kähler-Ricci flow on \mathcal{K} :

$$\partial_t \omega(t) = -\text{Ric}[\omega(t)].$$

Since the Kähler-Ricci flow $\partial_t \omega(t)$ has threefold solitary solutions: shrinking, steady and expanding (depending on the parameters), by analogy, we conjecture that the battle-vectorfield $\partial_t V^a(z^a, t)$ also has solitary solutions of shrinking, steady and expanding nature. Therefore, the *battle dynamics can be shrinking, steady, or expanding*—as expected from the classical warfare analysis.

2.2. Physics of Warfare: Lagrangian and Hamiltonian Structures of the Red and Blue Forces

Now we give a physical interpretation of warfare, using geometric insights from the Kähler dynamics provided above (and in the **Appendix 1**). The Kähler battle-manifold (\mathcal{K}, g) , with the fundamental complex structure defined by its Hermitian metric $g = g_R + i \omega_S$, includes the Riemannian structure g_R (for the Red force) and the symplectic structure ω_S (for the Blue force)—or vice versa. The Riemannian structure, $g_R = g_{ab} dx^a dx^b \in TM_{\text{Red}}$, naturally admits Lagrangian dynamics for the Red force, derived from the Lagrangian energy function, $L(x, \dot{x}): TM_{\text{Red}} \rightarrow \mathbb{R}$; the symplectic structure, $\omega_S = dp_a \wedge dq^a \in T^*M_{\text{Blue}}$, naturally admits Hamiltonian dynamics for the Blue force, derived from the Hamiltonian energy function, $H(q, p): T^*M_{\text{Blue}} \rightarrow \mathbb{R}$ —or vice versa.

Next, we recall that general forced-and-dissipative mechanics (see, e.g. [5] and the references therein) in Lagrangian form reads:

$$\dot{L}_{\dot{x}^a} + \Phi_{\dot{x}^a} = L_{x^a} + F_a, \tag{5}$$

and in Hamiltonian form reads:

$$\dot{q}^a = H_{p_a} - \Phi_{p_a}, \quad \dot{p}_a = F_a - H_{q^a} + \Phi_{q^a}, \tag{6}$$

where new (x, p, q) -indices denote partial derivatives (which is common with PDEs), F_a represents the covector of generalized external forces and the scalar function Φ , given by the mappings $\Phi(\dot{x}): TM_{\text{Red}} \rightarrow \mathbb{R}$ (for Lagrangian mechanics) and $\Phi(q, p): T^*M_{\text{Blue}} \rightarrow \mathbb{R}$ (for Hamiltonian mechanics) represents Rayleigh’s dissipation function (describing internal frictional forces proportional to velocity).

So, let us try to formally match the Red and Blue vectorfields from Equations (1) with the general Lagrangian Equations (5) and the general Hamiltonian Equations (6):

$$\begin{aligned} \dot{L}_{x^a} + \Phi_{x^a} &= L_{x^a} + F_a \quad \Leftrightarrow \\ \text{Red: } \dot{R}^a &= cA_b^a B^b + c_b F_{cd}^{ab} B^c R^d + R^b \mathfrak{L}_R U_b^a + [R^a, B^a] + \delta R^a \text{ (H-L)}, \end{aligned}$$

and

$$\begin{aligned} \dot{q}^a &= H_{p_a} - \Phi_{p_a}, \quad \dot{p}_a = F_a - H_{q^a} + \Phi_{q^a} \quad \Leftrightarrow \\ \text{Blue: } \dot{B}^a &= \kappa C_b^a R^b + \kappa_b G_{cd}^{ab} B^c R^d + B^b \mathfrak{L}_B W_b^a + [B^a, R^a] + \delta B^a \text{ (H-L)}. \end{aligned}$$

By comparing the general forced-and-dissipative mechanics with our Red and Blue vectorfields, we make the following two observations. Firstly, we can see that there are no any covectors of external forces F_a in the Red and Blue (pure velocity) vectorfields, so we can reduce our matches to:

$$\begin{aligned} \dot{L}_{x^a} + \Phi_{x^a} &= L_{x^a} \quad \Leftrightarrow \\ \text{Red: } \dot{R}^a &= cA_b^a B^b + c_b F_{cd}^{ab} B^c R^d + [R^a, B^a] + R^b \mathfrak{L}_R U_b^a + \delta R^a \text{ (H-L)}, \end{aligned}$$

and

$$\begin{aligned} \dot{q}^a &= H_{p_a} - \Phi_{p_a}, \quad \dot{p}_a = -H_{q^a} + \Phi_{q^a} \quad \Leftrightarrow \\ \text{Blue: } \dot{B}^a &= \kappa C_b^a R^b + \kappa_b G_{cd}^{ab} B^c R^d + [B^a, R^a] + B^b \mathfrak{L}_B W_b^a + \delta B^a \text{ (H-L)}. \end{aligned}$$

Secondly, since the Red and Blue vectorfields are generalized from classical Lanchester equations (which are the 1st-order ODEs), there are no any covectors of inertial (internal) forces either. In other words, the Red and Blue vectorfields physically correspond to dynamics of highly viscous/dissipative fluids, in which inertial forces can be neglected, so we can make the second reduction as:

$$\begin{aligned} \Phi_{x^a} &= L_{x^a} \quad \Leftrightarrow \\ \text{Red: } \dot{R}^a &= cA_b^a B^b + c_b F_{cd}^{ab} B^c R^d + [R^a, B^a] + R^b \mathfrak{L}_R U_b^a + \delta R^a \text{ (H-L)}, \end{aligned}$$

and

$$\begin{aligned} \dot{q}^a &= H_{p_a} - \Phi_{p_a} \quad \Leftrightarrow \\ \text{Blue: } \dot{B}^a &= \kappa C_b^a R^b + \kappa_b G_{cd}^{ab} B^c R^d + [B^a, R^a] + B^b \mathfrak{L}_B W_b^a + \delta B^a \text{ (H-L)}. \end{aligned}$$

Therefore, since our Red and Blue vectorfields are pure velocity-vectorfields without internal or external force co-vectorfields, both Lagrangian and Hamiltonian equations are reduced to the 1st-order systems of ODEs: in Lagrangian formulation the 2nd-order (inertial force) term vanishes, and in Hamiltonian formulation the whole force equation vanishes (momenta still exist but their time derivatives vanish).

3. Computational Wargame: “Entropy Battle”

The computational wargame called the *Entropy Battle* (see **Figure 1**) is currently being implemented in C# language (on .Net 4.7), using Irrlicht 3D graphics engine and Bullet 3D physics engine, and implementing the metaphysics of wargaming outlined in the next subsection. The *core* version of the wargame simulates the aircraft battle scenario from [4] and [1] with 30 aircraft on each

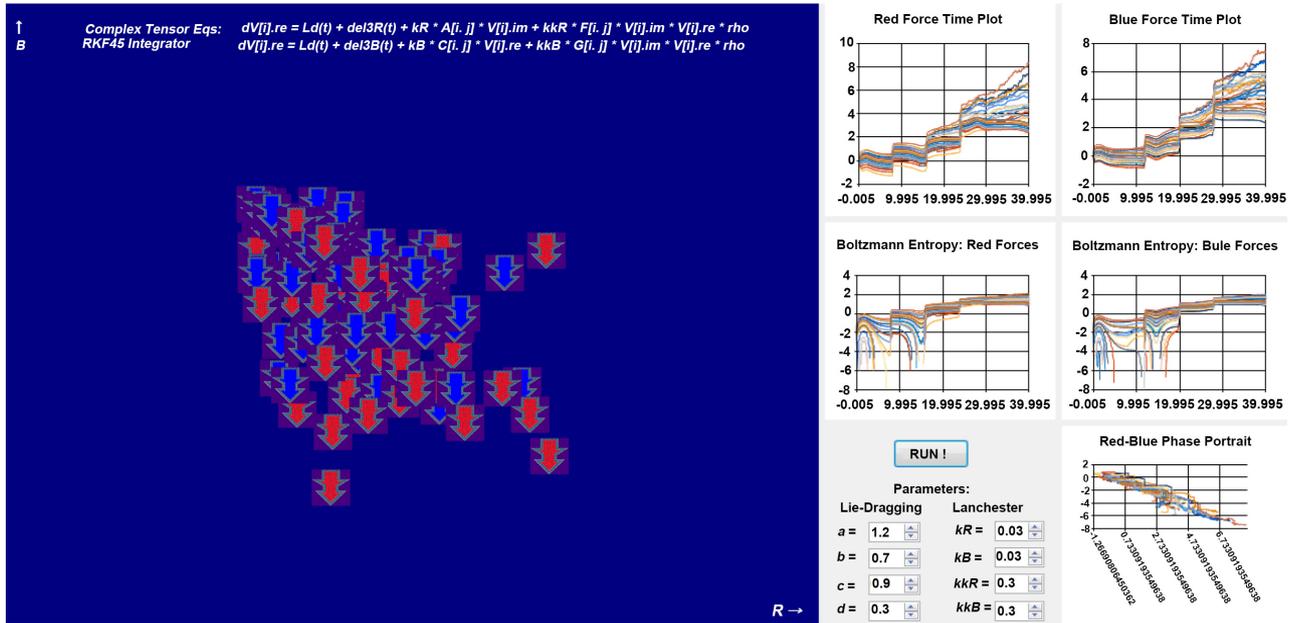


Figure 1. A prototype of the computational TCW-wargame called the *Entropy Battle*. The wargame is currently under development in the Joint and Operations Analysis Division, Defence Science & Technology Group, Australia, using the C# language, with Irrlicht 3D graphics engine and Bullet 3D physics engine. It implements a simplified version of the battle-vectorfield $\partial_t V^a(z^a, t)$, moving predominantly in the horizontal complex plane, solved by the complex-valued RKF45 (Cash-Karp) integrator and implementing the general metaphysics of wargaming.

side: Red aircraft starts in the bipartite formation and Blue aircraft starts in the tripartite formation. The battle is formally defined as a simplified version of the battle-vectorfield $\partial_t V^a(z^a, t)$, moving/flying predominantly in the horizontal complex plane \mathbb{C} . It is numerically solved in adaptive time steps using the complex-valued RKF45 (Cash-Karp) integrator, which is fast, accurate and almost symplectic.

The *extended* version of the Entropy Battle wargame has two levels:

Top level is the core aircraft battle, and

Bottom level has two scenarios (both formally defined by a simplified version of the battle-vectorfield $\partial_t V^a(z^a, t)$ moving in the complex plane):

- *Land battle* between Red and Blue land vehicles, and
- *Sea battle* between Red and Blue boats.

In both cases, the Entropy Battle wargame follows the general metaphysics of wargaming outlined as follows.

Metaphysics of Wargaming: Warfare Entropy and “Combat Signatures” in the Battlespace

- The stage for combat dynamics is the Red-Blue battlespace, which can be modeled by a dynamical concept of the phase-space. From a bird-view (or, from God’s Eye), the phase-space reduces to its 2D order-parameter subspace, the Red-Blue phase-plane, which is usually used in simulations.
- The concept of the phase-space (in our case spanned by the Red and Blue

forces) comes from Hamiltonian mechanics (when W.R. Hamilton formally unified Lagrangian mechanics and optics). It is also used in statistical mechanics. Besides, the 2D phase plane was the main analytical tool of H. Poincaré in his qualitative analysis of differential equations, from which both topology and chaos theory emerged. Finally, L. Boltzmann defined the entropy by coarse-graining the phase space. Every kind of entropy (including Boltzmann, Gibbs, Shannon, Kolmogorov-Sinai, Rényi, Bekenstein-Hawking, Kosko fuzzy, entanglement, topological, partition-function based, path-integral based, etc.) is essentially a logarithm of some more fundamental underlying (probabilistic, phase space or topological) measure, therefore it is itself an additive measure, which in our combat case gives:

$$\text{Total combat entropy} = \text{Red-entropy} + \text{Blue-entropy}.$$

- The cornerstone of Hamiltonian and statistical mechanics (as well as ergodic dynamics) is the key concept related to the Warfare Entropy and “Combat Signatures” in the Battlespace. It is the famous Liouville’s theorem: The flow of a conservative Hamiltonian vectorfield preserves the phase-space Volume (technically, Hamiltonian flow is a symplectomorphism: the Lie derivative of the volume form: $dRed \wedge dBlue$ along the [Red, Blue] vectorfield vanishes). This volume preservation necessarily implies various shape distortions (called “combat signatures”) and therefore uncertainty!
- Liouville’s theorem-based interpretation of the Warfare Entropy and “Combat Signatures”: If dynamics in the Red-Blue phase-plane stretches in the Red direction, it necessarily shrinks in the Blue direction, and vice versa. The stretching and shrinking distortions of the Combat Area cause rapid entropy growth and combat signatures in the 2D phase-plane. More generally, in higher Red-Blue phase-space dimensions, Liouville’s theorem causes Hamiltonian chaos, because there are so many possible ways for stretching and shrinking, each one reflected by entropy growth. There is no chaos in the 2D phase plane (theorem), but the entropy growth is still observable, since, e.g., the Kolmogorov-Sinai entropy is a sum of all Lyapunov exponents (both positive-chaotic and negative-nonchaotic).
- We can assume that, at least in a short time interval, the Red-Blue combat dynamics in the battlespace is conservative (no energy sources or sinks). Therefore, for a short time period, all combat dynamics can be derived from the so-called battle Hamiltonian (total combat energy function in an isolated region of battlespace)—at a certain entropy level. In the next short time period, we again have the conservative combat dynamics—at a higher entropy level, etc.
- Generalization/relaxation of Liouville’s theorem: the so-called Hamilton-Langevin framework has been proposed in [3], to include: delta-strikes, dissipation and random external forces. If the magnitude of these non-

conservative influences is not overwhelming, the entropic stretching-and-shrinking effect of Liouville's theorem is still visible.

- While slow changes of the battlefield are governed by Liouville's theorem, fast changes are governed by Onsager²—Prigogine's³ entropic, irreversible, non-equilibrium thermodynamics with the *arrow-of-time*.⁴ Sudden entropy growths in open combat Red-Blue systems reflect sudden energy dissipations due to impulsive Red-Blue crashes.
- In summary, general combat dynamics and wargaming necessarily includes both the *reversible* Hamiltonian-type dynamics (governed by Liouville's theorem) and *irreversible* Prigogine's non-equilibrium thermodynamics of open systems (exhibiting rapid entropy growth).

4. Conclusion

We have presented the Kähler dynamics approach to battlefields. It is the complex-dynamics synthesis of the combat dynamics series called the *tensor-centric warfare*, which includes tensor generalization of classical Lanchester-type combat equations, entropic Lie-dragging for modeling warfare uncertainty and symmetry, various (both deterministic and random) delta-strikes and missiles, and deep-learning at the battlefield. This synthesis is performed in the form of the complex battle-vectorfield, defined using the global framework of Kähler battle-manifolds. The proposed Red-Blue combat dynamics model is defined on the Kähler battle-manifold by a unique battle-vectorfield which is underpinned by the geometric Kähler-Ricci flow. This complex synthesis gives a new insight into the “physics of warfare” in terms of “hidden” Lagrangian and Hamiltonian structures of the battlefields. It also provides a convenient and efficient computational framework for entropic wargaming, in which the *Entropy Battle* is currently under development.

Acknowledgements

The authors are grateful to Dr. Tim McKay and Dr. Brandon Pincombe, Joint and Operations Analysis Division, Defence Science & Technology Group, Australia—for their constructive comments which have improved the quality of this paper.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

²Lars Onsager, 1968 Nobel Prize in Chemistry.

³Ilya Prigogine, 1977 Nobel Prize in Chemistry.

⁴When a glass or an egg is dropped on the floor, the arrow-of-time is obvious—once a glass or an egg is broken, there is no way to collect it back to its original state. However, all equations of classical and quantum physics are time-reversible, so they cannot model this simple crash-dissipation event. The only adequate physical/chemical principle at work is Onsager-Prigogine's Extended Second Law of Thermodynamics, reflecting the sudden entropy growth (burst) due to a dissipative crash.

References

- [1] Ivancevic, V., Pourbeik, P. and Reid, D. (2018) Tensor-Centric Warfare I: Tensor Lanchester Equations. *Intelligent Control and Automation*, **9**, 11-29. <https://doi.org/10.4236/ica.2018.92002>
- [2] Ivancevic, V., Reid, D. and Pourbeik, P. (2018) Tensor-Centric Warfare II: Entropic Uncertainty Modeling, *Intelligent Control and Automation*, **9**, 30-51.
- [3] Ivancevic, V., Pourbeik, P. and Reid, D. (2018) Tensor-Centric Warfare III: Combat Dynamics with Delta-Strikes. *Intelligent Control and Automation*, **9**, 107-122.
- [4] McLemore, C., Gaver, D. and Jacobs, P. (2016) Model for Geographically Distributed Combat Interactions of Swarming Naval and Air Forces. *Naval Research Logistics*, **63**, 562-576. <https://doi.org/10.1002/nav.21720>
- [5] Ivancevic, V. (2017) Kähler Dynamics for the Universal Multi-Robot Fleet. *Journal of Geometry and Symmetry in Physics*, **45**, 67-85. <https://doi.org/10.7546/jgsp-45-2017-67-85>
- [6] Ivancevic, V., Reid, D. and Scholz, J. (2014) Action-Amplitude Approach to Controlled Entropic Self-Organization. *Entropy*, **16**, 2699-2712. <https://doi.org/10.3390/e16052699>
- [7] Ivancevic, V. and Reid, D. (2015) Complexity and Control: Towards a Rigorous Behavioral Theory of Complex Dynamical Systems. World Scientific, Singapore.
- [8] Ivancevic, V., Reid, D. and Pilling, M. (2017) Mathematics of Autonomy: Mathematical Methods for Cyber-Physical-Cognitive Systems. World Scientific, New Jersey/Singapore. <https://doi.org/10.1142/10716>
- [9] Griffiths, P. and Harris, J. (1978) Principles of Algebraic Geometry, Pure and Applied Mathematics. Wiley-Interscience, New York.
- [10] Nakahara, M. (2003) Geometry, Topology and Physics. 2nd Edition, IOP, Bristol and Philadelphia.
- [11] Donaldson, S.K. (1999) Symmetric Spaces, Kähler Geometry and Hamiltonian Dynamics. *American Mathematical Society Translations: Series 2*, **196**, 13-33. <https://doi.org/10.1090/trans2/196/02>
- [12] Song, J. and Weinkove, B. (2012) Lecture Notes on the Kähler-Ricci Flow. arXiv:1212.3653, 94.
- [13] Yau, S.T. (2006) Structure of Three-Manifolds—Poincaré and Geometrization Conjectures. Morningside Center of Mathematics, Beijing.
- [14] Cao, H.-D. (2012) The Kähler-Ricci Flow on Fano Manifolds. arXiv:1212.6227.
- [15] He, W. (2015) On the Space of Kähler Potentials. *Communications on Pure and Applied Mathematics*, **68**, 332-343. <https://doi.org/10.1090/trans2/196/02>
- [16] Calabi, E. and Vesentini, E. (1960) On Compact, Locally Symmetric Kähler Manifolds. *Annals of Mathematics*, **71**, 472-507. <https://doi.org/10.2307/1969939>
- [17] Yau, S.T. (1978) On the Ricci Curvature of a Compact Kähler Manifold and the Complex Monge—Ampère Equation. *Communications on Pure and Applied Mathematics*, **31**, 339-441. <https://doi.org/10.1002/cpa.3160310304>
- [18] Ivancevic, V. and Ivancevic, T. (2011) Ricci Flow and Nonlinear Reaction-Diffusion Systems in Biology, Chemistry, and Physics. *Nonlinear Dynamics*, **65**, 35-54. <https://doi.org/10.1007/s11071-010-9872-6>
- [19] Hamilton, R.S. (1982) Three-Manifolds with Positive Ricci Curvature. *Journal of Differential Geometry*, **17**, 255-306. <https://doi.org/10.4310/jdg/1214436922>

- [20] Hamilton, R.S. (1986) Four-Manifolds with Positive Curvature Operator. *Journal of Differential Geometry*, **24**, 153-179. <https://doi.org/10.4310/jdg/1214440433>
- [21] Hamilton, R.S. (1988) The Ricci Flow on Surfaces. *Contemporary Mathematics*, **71**, 237-261,
- [22] Tian, G. and Zhang, Z. (2013) Regularity of Kähler-Ricci Flows on Fano Manifolds. arXiv:1310.5897 [math.DG].
- [23] Song, J., Phong, D.H. and Sturm, J. (2012) Complex Monge-Ampere Equations. *Surveys in Differential Geometry*, **17**, 327-411.

1. Appendix: Kähler Manifolds and Kähler-Ricci Flow

In this section, we give a brief review of Kähler manifolds (the main reference is [9]) and the Kähler-Ricci flow on them, which constitutes the geometric framework for the complex Red-Blue battle-vectorfield.

1.1. Geometry and Dynamics of Kähler Manifolds

Let $\mathcal{K} = \mathcal{K}^n$ be a compact (i.e., closed and bounded) complex n -manifold⁵ of complex dimension n (see [9] [10]). To be able to write various tensors on the manifold \mathcal{K} , we chose a local point $p \in \mathcal{K}$ with the neighborhood chart U that includes: 1) the holomorphic coordinates and their complex-conjugates: $\{z^i = x^i + iy^i, \bar{z}^i = \bar{x}^i + i\bar{y}^i\} \in U_p \subset \mathcal{K}$, 2) the natural basis of vectorfields in the tangent space $T_p\mathcal{K}$ at p : $\{\partial_i, \partial_{\bar{i}}\} \in T_p\mathcal{K}$ (using $\partial_i \equiv \partial/\partial z^i$), and 3) the dual basis of co-vectorfields (i.e., holomorphic 1-forms)⁶ in the cotangent space $T_p^*\mathcal{K}$ at p : $\{dz^i, d\bar{z}^i\} \in T_p^*\mathcal{K}$.

To make the complex manifold $\mathcal{K} = (\mathcal{K}, g) = (\mathcal{K}, \omega)$ into a Kähler n -manifold, we need to specify on it a Kähler metric g and its associated Kähler form ω , as follows (compare with [11] [12] [13] [14]). Consider a Hermitian metric⁷ g defined at each point $\{z^i, \bar{z}^i\} \in \mathcal{K}$ by a smooth positive-definite (1,1)-tensor field $g_{i\bar{j}} = g_{\bar{j}i}(z^i, \bar{z}^j)$, for $i, j = 1, \dots, n$, as:⁸

⁵Recall that a complex-valued function $f: \mathbb{C}^n \rightarrow \mathbb{C}$ is called holomorphic if-and-only-if (iff) $f = f_1 + if_2$ satisfies the Cauchy-Riemann relations for each holomorphic coordinate, $z^j = x^j + iy^j$:

$$\partial_{x^j} f_1 = \partial_{y^j} f_2, \quad \partial_{x^j} f_2 = -\partial_{y^j} f_1, \quad (\partial_{x^j} \equiv \partial/\partial x^j).$$

A complex manifold \mathcal{K} (like \mathbb{C}^n and Riemann surfaces) is a manifold with an atlas consisting of charts $t_{ij}: U_i \cap U_j \rightarrow \mathbb{C}^n$, such that the transition functions $t_{ij}(z)$ are holomorphic and satisfy the cocycle condition: $t_{ik}(z) = t_{ij}(z)t_{jk}(z)$ on triple overlaps $U_i \cap U_j \cap U_k$. An almost complex structure J is defined on a complex manifold \mathcal{K} as:

$$J\partial_{z^i} = i\partial_{z^i}, \quad J\partial_{\bar{z}^i} = -i\partial_{\bar{z}^i}, \quad J^2 = -1.$$

⁶The basis 1-forms (co-vectorfields) $\{dz^i, d\bar{z}^i\}$ are holomorphic de Rham differentials of the holomorphic coordinates:

$$dz^i = dx^i + idy^i \quad \text{and} \quad d\bar{z}^i = dx^i - idy^i.$$

They induce the holomorphic velocities:

$$\dot{z}^i = \dot{x}^i + i\dot{y}^i \quad \text{and} \quad \dot{\bar{z}}^i = \dot{x}^i - i\dot{y}^i.$$

⁷A Hermitian manifold (\mathcal{K}, g) is a complex manifold \mathcal{K} with a Hermitian metric tensor $g_{i\bar{j}} = g_{\bar{j}i}(z^i, \bar{z}^j)$, such that $(g^{\bar{i}j})$ is a positive-definite Hermitian matrix function, $(g^{\bar{i}j}) = (g^{ij})^*$. A Hermitian manifold (\mathcal{K}, g) becomes a Kähler manifold iff the almost complex structure J on it satisfies the condition: $\nabla_k J = 0$, where ∇_k is the Levi-Civita connection on (\mathcal{K}, g) , defined later in this section.

⁸The metric g given by (7) has the following dynamical interpretation: it defines the complex kinetic-energy Lagrangian $L(\dot{z}, \dot{\bar{z}}): \mathcal{K} \rightarrow \mathbb{C}$, in holomorphic coordinates $\{z^i, \bar{z}^i\} \in \mathcal{K}$ given as:

$$L(\dot{z}, \dot{\bar{z}}) = \frac{1}{2} g_{i\bar{j}} \dot{z}^i \otimes \dot{\bar{z}}^j,$$

from which the conservative Lagrangian equations read in the contravariant form:

$$\ddot{z}^i(t) + \Gamma_{jk}^i \dot{z}^j(t) \dot{\bar{z}}^k(t) = 0,$$

and in the standard covariant form (with indices denoting partial derivatives):

$$\dot{L}_{z^i} - L_{z^i} = 0.$$

$$g = g_{\bar{j}i} dz^i \otimes d\bar{z}^j > 0, \tag{7}$$

such that $(g_{\bar{j}i})$ is a positive-definite Hermitian matrix. Its inverse $g^{\bar{j}i}$ is given by the matrix $(g^{\bar{j}i}) = (g_{\bar{j}i})^{-1}$. Associated to the Hermitian metric g , there is a real positive-definite exterior (1,1)-form $\omega = \omega_{\bar{j}i}(z^i, \bar{z}^j)$ on \mathcal{K} , defined by:⁹

$$\omega = ig_{\bar{j}i} dz^i \wedge d\bar{z}^j > 0. \tag{8}$$

If the form ω is closed, $d\omega = 0$, then g is called the *Kähler metric* and ω is called the *Kähler form*. The fundamental closure condition: $d\omega = 0$ is called the *Kähler condition*, the global condition for any Kähler manifold \mathcal{K} , which is locally in $(z^i, \bar{z}^j) \in U \subset \mathcal{K}$ equivalent to the following metric symmetries:¹⁰

$$\partial_j g_{i\bar{k}} = \partial_i g_{j\bar{k}} \quad \text{and} \quad \partial_{\bar{j}} g_{k\bar{i}} = \partial_{\bar{i}} g_{k\bar{j}}, \tag{9}$$

(independent of the choice of local holomorphic coordinates $(z^i, \bar{z}^j) \in U \subset \mathcal{K}$). In (9), $\partial_j \equiv \partial$ and $\partial_{\bar{j}} \equiv \bar{\partial}$ are *Dolbeault's differential operators*, which are the additive components of the exterior derivative (de Rham differential) d on \mathcal{K} : $d = \partial + \bar{\partial}$.¹¹ In that case, as shown by E. Kähler

⁹From the Kähler form (8) with \bar{z}^i as canonical momenta, the *Hamiltonian formalism* can be derived from the complex kinetic-energy Hamiltonian $H(z, \bar{z}): \mathcal{K} \rightarrow \mathbb{C}$ in holomorphic coordinates $\{z^i, \bar{z}^i\} \in \mathcal{K}$ given as:

$$H(z, \bar{z}) = \frac{1}{2} g_{\bar{j}i} dz^i \wedge d\bar{z}^j,$$

from which the conservative Hamiltonian equations read (with indices denoting partial derivatives):

$$\dot{z}^i = H_{z^i}, \quad \dot{\bar{z}}^i = -H_{\bar{z}^i}.$$

¹⁰Since $d\omega = 0$, we have:

$$\begin{aligned} 0 &= (\partial + \bar{\partial}) ig_{\bar{j}i} dz^i \wedge d\bar{z}^j = i\partial_k g_{\bar{j}i} dz^k \wedge dz^i \wedge d\bar{z}^j + i\partial_{\bar{k}} g_{\bar{j}i} d\bar{z}^k \wedge dz^i \wedge d\bar{z}^j \\ &= \frac{1}{2} i(\partial_k g_{\bar{j}i} - \partial_i g_{\bar{k}j}) dz^k \wedge dz^i \wedge d\bar{z}^j + \frac{1}{2} i(\partial_{\bar{k}} g_{\bar{j}i} - \partial_{\bar{j}} g_{\bar{k}i}) d\bar{z}^k \wedge dz^i \wedge d\bar{z}^j, \end{aligned}$$

which implies:

$$\partial_j g_{i\bar{k}} = \partial_i g_{j\bar{k}} \quad \text{and} \quad \partial_{\bar{j}} g_{k\bar{i}} = \partial_{\bar{i}} g_{k\bar{j}}.$$

¹¹Recall that the *exterior derivative*, or de Rham differential, $d: \Omega^p(\mathcal{K}) \rightarrow \Omega^{p+1}(\mathcal{K})$ is the map on the space $\Omega^p(\mathcal{K})$ of exterior p -forms on the complex manifold \mathcal{K} , which generalizes standard gradient, curl and divergence operators from vector calculus. Being the additive components of the exterior derivative: $d = \partial + \bar{\partial}$, so that $df = \partial f + \bar{\partial} f$ for any smooth real function f on \mathcal{K} , Dolbeault's differential operators $(\partial, \bar{\partial})$ are the maps on the space $\Omega^{p,q}(\mathcal{K})$ of exterior forms on \mathcal{K} , such that $\partial: \Omega^{p,q}(\mathcal{K}) \rightarrow \Omega^{p+1,q}(\mathcal{K})$ and $\bar{\partial}: \Omega^{p,q}(\mathcal{K}) \rightarrow \Omega^{p,q+1}(\mathcal{K})$. They play the major role in the famous “ $\partial\bar{\partial}$ -lemma” and the associated Dolbeault cohomology (from the Hodge theory). Any function $f: \mathcal{K} \rightarrow \mathbb{C}$ is holomorphic iff $\bar{\partial} f = 0$ (the Cauchy-Riemann equations). In a local z^k -coordinate chart $U \subset \mathcal{K}$, for any holomorphic function $f \in U$, $(\partial, \bar{\partial})$ -operators are given by:

$$\begin{aligned} \partial f &= (\partial_{x^k} f - i\partial_{y^k} f) dz^k, & \bar{\partial} f &= (\partial_{x^k} f - i\partial_{y^k} f) d\bar{z}^k, \\ \text{such that: } \partial^2 &= 0, & \bar{\partial}^2 &= 0, & \partial\bar{\partial} + \bar{\partial}\partial &= 0. \end{aligned}$$

himself in 1933, the metric tensor $g_{j\bar{k}}$ can be written in terms of a real-valued smooth function $\varphi : \mathcal{K} \rightarrow \mathbb{R}$, called the *Kähler potential* (see below), as:¹²

$$g_{j\bar{k}} = \frac{\partial^2 \varphi}{\partial z^j \partial \bar{z}^k} \equiv \partial_j \bar{\partial}_k \varphi \equiv \partial \bar{\partial} \varphi \Rightarrow \omega_{j\bar{k}} = i \frac{\partial^2 \varphi}{\partial z^j \partial \bar{z}^k} \equiv i \partial_j \bar{\partial}_k \varphi \equiv i \partial \bar{\partial} \varphi.$$

We remark that the two differential expressions with the Kähler potential φ ,¹³ $g = \partial \bar{\partial} \varphi$ and $\omega = i \partial \bar{\partial} \varphi$, both governed by the $\partial \bar{\partial}$ -lemma (see below) constitute the *core* of the Kähler geometry, so that any other geometro-dynamical structure on (\mathcal{K}, g) , including the Kähler-Ricci flow and the Monge-Ampère equation, is derivable from them.

Holomorphic vectorfields and co-vectorfields (1-forms) are defined on (\mathcal{K}, g) by their appropriate holomorphic coordinate transformations (or, diffeomorphisms) in $T_p \mathcal{K}$ and $T_p^* \mathcal{K}$, respectively. Let $v = v^i \partial_i$ and $u = u^{\bar{i}} \bar{\partial}_{\bar{i}}$ be $T^{1,0}$ and $T^{0,1}$ vectorfields in $T_p \mathcal{K}$, such that $\partial_{\bar{j}} v^j = \partial_j u^{\bar{i}} = 0$, and let $\alpha = \alpha_i dz^i$ and $\beta = \beta_{\bar{i}} d\bar{z}^{\bar{i}}$ be (1,0) and (0,1) co-vectorfields in $T_p^* \mathcal{K}$, such that $\alpha_i d\bar{z}^{\bar{i}} = \beta_{\bar{i}} dz^i = 0$. If $\{\tilde{z}^i\} = \{\tilde{z}^1, \dots, \tilde{z}^n\}$ is another holomorphic coordinate system on \mathcal{K} , then on the overlap $\{z^i\} \cap \{\tilde{z}^i\} \in \mathcal{K}$ the following diffeomorphisms hold:

$$v^j = v^i \frac{\partial \tilde{z}^j}{\partial z^i}, \quad u^{\bar{j}} = u^{\bar{i}} \frac{\partial \tilde{z}^{\bar{j}}}{\partial \bar{z}^{\bar{i}}},$$

$$\tilde{\alpha}_j = \alpha_i \frac{\partial z^i}{\partial \tilde{z}^j}, \quad \tilde{\beta}_{\bar{j}} = \beta_{\bar{i}} \frac{\partial \bar{z}^{\bar{i}}}{\partial \tilde{z}^{\bar{j}}}.$$

The Kähler metric g induces the Levi-Civita connection on (\mathcal{K}, g) , given by the *Christoffel symbols* Γ^i_{jk} on (\mathcal{K}, g) , defined simply by:¹⁴

$$\Gamma^i_{jk} = g^{\bar{m}i} \partial_j g_{k\bar{m}}. \tag{10}$$

Γ^i_{jk} are not the components of a tensor; however, if g_{ij} and \hat{g}_{ij} are two Kähler metrics with Christoffel's symbols Γ^i_{jk} and $\hat{\Gamma}^i_{jk}$ then the difference $\Gamma^i_{jk} - \hat{\Gamma}^i_{jk}$ is a tensor. From the Kähler condition (9) it follows that Γ^i_{jk} are symmetric in the lower indices: $\Gamma^i_{jk} = \Gamma^i_{kj}$.

Using Christoffel's symbols Γ^i_{jk} , we can define the pair of *covariant derivatives* $(\nabla_k, \nabla_{\bar{k}})$ on (\mathcal{K}, g) , which act on smooth functions f on \mathcal{K} as: $\nabla_i f = \partial_i f, \nabla_{\bar{i}} f = \partial_{\bar{i}} f$. On the vectorfields (v, u) on $T\mathcal{K}$ and co-vectorfields

¹²Potential definition of the Kähler metric, $g_{j\bar{i}} = \partial \bar{\partial} \varphi$, implies that the Ricci curvature $\text{Ric}(g) = R_{j\bar{i}}$ (see below) is given by:

$$R_{j\bar{i}} = - \frac{\partial^2 \varphi}{\partial z^j \partial \bar{z}^{\bar{i}}} \log [\det(g_{j\bar{i}})] \equiv - \partial_j \bar{\partial}_{\bar{i}} \log [\det(g_{j\bar{i}})] \equiv - \partial \bar{\partial} \log [\det(g)],$$

and consequently, the solutions of the Einstein equation: $\text{Ric} = \lambda g$ (in components, $R_{j\bar{i}} = \lambda g_{j\bar{i}}$, for some real constant λ) can be defined by solving the scalar equation:

$$\det(\partial \bar{\partial} \varphi) = e^{-2\varphi}.$$

¹³For a real function $\varphi : \mathcal{K} \rightarrow \mathbb{R}$ the real (1,1)-form $i \partial \bar{\partial} \varphi \equiv i \partial_j \bar{\partial}_{\bar{k}} \varphi$ is the complex Hessian of φ .

¹⁴The Christoffel symbols Γ^i_{jk} on a Kähler manifold \mathcal{K} do not have their conjugate part. Because of this unique feature of Kähler manifolds, the Riemann and Ricci curvature tensors (defined below) are much simpler on \mathcal{K} than in standard Riemannian geometry.

(α, β) on $T^*\mathcal{K}$, the covariant derivatives $(\nabla_k, \nabla_{\bar{k}})$ act in the following way:¹⁵

$$\begin{aligned} \nabla_k v^i &= \partial_k v^i + \Gamma_{jk}^i v^j, & \nabla_{\bar{k}} v^i &= \partial_{\bar{k}} v^i, \\ \nabla_k u^{\bar{i}} &= \partial_k u^{\bar{i}}, & \nabla_{\bar{k}} u^{\bar{i}} &= \partial_{\bar{k}} u^{\bar{i}} + \overline{\Gamma_{jk}^i} u^{\bar{j}}, \\ \nabla_k \alpha_i &= \partial_k \alpha_i - \Gamma_{ik}^j \alpha_j, & \nabla_{\bar{k}} \alpha_i &= \partial_{\bar{k}} \alpha_i, \\ \nabla_k \beta_{\bar{i}} &= \partial_k \beta_{\bar{i}}, & \nabla_{\bar{k}} \beta_{\bar{i}} &= \partial_{\bar{k}} \beta_{\bar{i}} - \overline{\Gamma_{ik}^j} \beta_{\bar{j}}. \end{aligned}$$

In general, the Christoffel symbols Γ_{jk}^i are chosen so that both covariant derivatives of the metric tensor vanish: $\nabla_k g_{\bar{j}\bar{i}} = \nabla_{\bar{k}} g_{ij} = 0$. Similarly, a Hermitian manifold (\mathcal{K}, g) is a Kähler manifold iff the almost complex structure J satisfies: $\nabla_k J = \nabla_{\bar{k}} J = 0$.

The Laplacian (or, rather Laplace-Beltrami) operator Δ is defined in local coordinates $(z^i, \bar{z}^j) \in \mathcal{K}$ as:

$$\Delta \equiv \det(g_{\bar{j}\bar{i}})^{\frac{1}{2}} \partial_i \left(\det(g_{\bar{j}\bar{i}})^{\frac{1}{2}} g^{\bar{j}\bar{i}} \partial_{\bar{j}} \right).$$

Δ -action on smooth functions $f \in \mathcal{K}$ is given by:

$$\Delta f = g^{\bar{j}\bar{i}} \partial_i \partial_{\bar{j}} f = \text{Tr}(i\partial\bar{\partial}f),$$

where $\text{Tr}(\cdot) = \text{Tr}_\omega(\cdot)$ is the *trace* operator (i.e., contraction with $g^{\bar{j}\bar{i}}$).¹⁶ More generally, Δ -action on an arbitrary tensor T is defined in normal coordinates¹⁷ for g on (\mathcal{K}, g) as:

$$\Delta T = \frac{1}{2} (\nabla_k \nabla_{\bar{k}} + \nabla_{\bar{k}} \nabla_k) T.$$

A Kähler metric g defines a corresponding Riemannian metric g_R on (\mathcal{K}, g) , defined via its real and imaginary parts, as follows. In local coordinates

¹⁵We can extend the covariant derivatives $(\nabla_k, \nabla_{\bar{k}}) \in (\mathcal{K}, g)$ to act naturally on any type of tensor field on (\mathcal{K}, g) ; e.g., in case of covariant 2-tensors $S_{\bar{j}\bar{i}}$ and S_{ij} , we have:

$$\begin{aligned} \nabla_k S_{\bar{j}\bar{i}} &= \partial_k S_{\bar{j}\bar{i}} - \Gamma_{ik}^p S_{\bar{j}\bar{p}}, & \nabla_{\bar{k}} S_{\bar{j}\bar{i}} &= \partial_{\bar{k}} S_{\bar{j}\bar{i}} - \Gamma_{\bar{j}\bar{k}}^{\bar{p}} S_{\bar{p}\bar{i}}, \\ \nabla_k S_{ij} &= \partial_k S_{ij} - \Gamma_{ik}^p S_{pj} - \Gamma_{jk}^p S_{ip}, & \nabla_{\bar{k}} S_{ij} &= \partial_{\bar{k}} S_{ij}. \end{aligned}$$

Similarly, if T is a mixed 3-tensor with components $T_k^{\bar{j}\bar{i}}$ then we define its two covariant derivatives $\nabla_m T_k^{\bar{j}\bar{i}}$ and $\nabla_{\bar{m}} T_k^{\bar{j}\bar{i}}$, as:

$$\begin{aligned} \nabla_m T_k^{\bar{j}\bar{i}} &= \partial_m T_k^{\bar{j}\bar{i}} + \Gamma_{im}^l T_k^{\bar{j}\bar{l}} - \Gamma_{km}^l T_l^{\bar{j}\bar{i}}, \\ \nabla_{\bar{m}} T_k^{\bar{j}\bar{i}} &= \partial_{\bar{m}} T_k^{\bar{j}\bar{i}} + \overline{\Gamma_{im}^l} T_k^{\bar{j}\bar{l}}. \end{aligned}$$

¹⁶For example, the trace $\text{Tr}(\cdot)$ of a real $(1,1)$ -form $\alpha = i\alpha_{\bar{j}} dz^j \wedge d\bar{z}^{\bar{j}} \in \mathcal{K}$ is defined by:

$$\text{Tr}_\omega(\alpha) = g^{\bar{j}\bar{i}} \alpha_{\bar{j}} = n\omega^{n-1} \wedge \alpha / \omega^n.$$

Here, ω^n is the *volume form* on \mathcal{K} (see below), such that for any smooth function $f \in \mathcal{K}$, its L^p -norm (with respect to a Kähler metric g) is given by: $\left(\int_{\mathcal{K}} |f|^p \omega^n \right)^{1/p}$.

¹⁷A normal coordinate system for g centered at a point $p \in \mathcal{K}$ is a holomorphic coordinate system that satisfies: $g_{\bar{j}\bar{i}}(p) = \delta_{\bar{j}\bar{i}}$ and $\partial_k g_{\bar{j}\bar{i}}(p) = 0$, which implies that the Christoffel symbols Γ_{jk}^i of g vanish at p .

$\{z^i, \bar{z}^i\} \in \mathcal{K}$, we write $z^i = x^i + iy^i$, so that $\partial_{z^i} = \frac{1}{2}(\partial_{x^i} - i\partial_{y^i})$ and

$\partial_{\bar{z}^i} = \frac{1}{2}(\partial_{x^i} + i\partial_{y^i})$, which gives:

$$g_R(\partial_{x^i}, \partial_{x^j}) = g_R(\partial_{y^i}, \partial_{y^j}) = 2\text{Re}(g_{\bar{j}i}), \quad g_R(\partial_{x^i}, \partial_{y^j}) = 2\text{Im}(g_{\bar{j}i}).$$

The *Riemann curvature* tensor Rm of the Kähler metric $g \in (\mathcal{K}, g)$ is very simply defined in two forms, mixed and covariant, respectively:

$$R_{ik\bar{l}}^m = -\partial_{\bar{l}}\Gamma_{ik}^m \quad \text{and} \quad R_{\bar{j}k\bar{l}} = g_{m\bar{j}}R_{ik\bar{l}}^m.$$

Using (9) and (10), we have locally (in an open chart $\{z^i, \bar{z}^i\} \in U \subset \mathcal{K}$; see [12]):

$$R_{\bar{j}k\bar{l}} = -\partial_i\partial_{\bar{j}}g_{k\bar{l}} + g^{\bar{q}p}(\partial_i g_{k\bar{q}})(\partial_{\bar{j}}g_{p\bar{l}}). \tag{11}$$

The Riemann curvature tensor $R_{\bar{j}k\bar{l}}$ on (\mathcal{K}, g) has the following three symmetries:¹⁸

- 1) $\overline{R_{\bar{j}k\bar{l}}} = R_{\bar{j}\bar{l}k}$ (complex-conjugate);
- 2) $R_{\bar{j}k\bar{l}} - R_{\bar{k}j\bar{l}} - R_{\bar{l}i\bar{k}j}$ (I Bianchi identity); and
- 3) $\nabla_m R_{\bar{j}k\bar{l}} = \nabla_i R_{\bar{m}j\bar{k}l}$ (II Bianchi identity).

For any two nonzero vectorfields (v, u) on $T\mathcal{K}$, we say that (\mathcal{K}, g) has positive holomorphic *bisectional* curvature and positive holomorphic *sectional* curvature, respectively, if

$$R_{\bar{j}k\bar{l}}v^i\bar{v}^j u^k \bar{u}^l > 0 \quad \text{and} \quad R_{\bar{i}j\bar{k}l}v^i\bar{v}^j v^k \bar{v}^l > 0.$$

The trace of the Riemann curvature tensor $R_{\bar{i}j\bar{k}l}$ is the *Ricci curvature* tensor Rc , defined as:

$$R_{\bar{j}i} = g^{\bar{k}k} R_{\bar{j}k\bar{l}} = g^{\bar{k}k} R_{k\bar{l}i\bar{j}} = R_{k\bar{l}i\bar{j}}^k,$$

which locally (in an open chart $\{z^i, \bar{z}^i\} \in U \subset \mathcal{K}$; see [12] [14]) reads:

$$R_{\bar{j}i} = -\partial_i\partial_{\bar{j}} \log[\det(g_{\bar{j}i})] = -\partial\bar{\partial} \log[\det(g)].$$

Similarly, the trace of the Ricci curvature is the *scalar curvature*: $R = g^{\bar{j}i} R_{\bar{j}i}$.

A Kähler metric g defines a pointwise norm $|\cdot|_g$ on any tensor field on (\mathcal{K}, g) ; e.g., the squared norm of functions f on \mathcal{K} reads: $|\nabla f|^2 = g^{\bar{j}i}\partial_i f \partial_{\bar{j}} \bar{f}$, and similarly for the vectorfields $(v, u) \in T\mathcal{K}$ and co-vectorfields $(\alpha, \beta) \in T^*\mathcal{K}$ we have:¹⁹

¹⁸While the symmetries 1) and 2) follow immediately from (9) and (11), to show 3) we need to compute at a point $p \in (\mathcal{K}, g)$ in normal coordinates for g .

$$\nabla_m R_{\bar{j}k\bar{l}} = -\partial_m\partial_{\bar{j}}g_{k\bar{l}} = -\partial_i\partial_m\partial_{\bar{j}}g_{k\bar{l}} = \nabla_i R_{\bar{m}j\bar{k}l}.$$

¹⁹The norm $|\cdot|_g$ extends to any tensor field on \mathcal{K} ; e.g., if T is a tensor with components $T_k^{\bar{j}}$, its squared norm is given by:

$$|T|_g^2 = g^{\bar{m}k} g_{\bar{j}i} g_{p\bar{q}} T_k^{\bar{j}} \bar{T}_m^{\bar{p}},$$

which in case of the Ricci tensor Rc and the Riemann tensor Rm become, respectively:

$$|Rc|_g^2 = g^{\bar{i}r} g^{\bar{j}q} R_{\bar{j}i} R_{\bar{q}r} \quad \text{and} \quad |Rm|_g^2 = g^{\bar{q}i} g^{\bar{j}r} g^{\bar{k}s} g^{\bar{t}p} R_{\bar{j}k\bar{l}} R_{\bar{p}q\bar{r}t}.$$

$$|v|_g^2 = g_{\bar{j}i} v^i \bar{v}^{\bar{j}}, \quad |u|_g^2 = g_{\bar{j}i} u^i \bar{u}^{\bar{j}},$$

$$|\alpha|_g^2 = g^{\bar{j}i} \alpha_i \bar{\alpha}_{\bar{j}}, \quad |\beta|_g^2 = g^{\bar{j}i} \beta_{\bar{j}} \bar{\beta}_{\bar{i}}.$$

Associated to the Ricci curvature tensor Rc is the *Ricci form*, $Ric(g) \equiv Ric(\omega)$, a real closed (1,1)-form on \mathcal{K} , similar to the Kähler form ω , given by:

$$Ric(\omega) = iR_{\bar{j}i}(g) dz^i \wedge dz^{\bar{j}} = -i\partial\bar{\partial} \log[\det(g)], \tag{12}$$

which implies that $Ric(\omega)$ is closed: $dRic(\omega) = 0$.

The Riemann curvature tensor $R_{i\bar{j}k\bar{l}}$ arises when commuting covariant derivatives $(\nabla_k, \nabla_{\bar{l}}) \in (\mathcal{K}, g)$. Using the standard commutator definition: $[\nabla_k, \nabla_{\bar{l}}] = \nabla_k \nabla_{\bar{l}} - \nabla_{\bar{l}} \nabla_k$, we have the following commutation formulae for the vectorfields (v, u) on $T\mathcal{K}$ and co-vectorfields (α, β) on $T^*\mathcal{K}$:

$$[\nabla_k, \nabla_{\bar{l}}]v^m = R_{i\bar{k}l}^m v^i, \quad [\nabla_k, \nabla_{\bar{l}}]u^{\bar{m}} = -R_{\bar{j}kl}^{\bar{m}} u^{\bar{j}},$$

$$[\nabla_k, \nabla_{\bar{l}}]\alpha_i = -R_{i\bar{k}l}^m \alpha_m, \quad [\nabla_k, \nabla_{\bar{l}}]\beta_{\bar{j}} = R_{\bar{j}kl}^{\bar{m}} \beta_{\bar{m}},$$

which can naturally be extended to tensors of any type on \mathcal{K} . Also, when acting on any tensor, the covariant derivatives commute as: $[\nabla_i, \nabla_{\bar{j}}] = [\nabla_{\bar{i}}, \nabla_j] = 0$.

Now we come to the essential notion of cohomology of a Kähler manifold (\mathcal{K}, ω) , which is defined using the formalism of $(\partial, \bar{\partial})$ -operators. Recall that *de Rham's cohomology* group $H_d^2(\mathcal{K}, \mathbb{R})$, based on the exterior derivative $d = \partial + \bar{\partial}$,²⁰ considers a symplectic 2-form $\alpha \in \mathcal{K}$ which is globally *closed*: $d\alpha = 0$ and locally *exact*: $\alpha = d\eta$, for some canonical 1-form η (*Poincaré lemma*). Then the group $H_d^2(\mathcal{K}, \mathbb{R})$ is defined as the quotient space:²¹

$$H_d^2(\mathcal{K}, \mathbb{R}) = \frac{\{d\text{-closed real 2-forms}\}}{\{d\text{-exact real 2-forms}\}}.$$

Similarly, a (1,1)-form $\alpha \in (\mathcal{K}, \omega)$ is called $\bar{\partial}$ -closed if $\bar{\partial}\alpha = 0$ and $\bar{\partial}$ -exact if $\alpha = \bar{\partial}\eta$ for some (0,1)-form η .²² Therefore, a complexification of de Rham's group $H_d^2(\mathcal{K}, \mathbb{R})$ gives the *Dolbeault cohomology group* $H_{\bar{\partial}}^{1,1}(\mathcal{K}, \mathbb{R})$, defined as the quotient space:²³

²⁰The closure relation between these three derivative operators reads:

$$d(i\partial\bar{\partial}) = i(\partial + \bar{\partial})\partial\bar{\partial} = i(\partial^2\bar{\partial} - \partial\bar{\partial}^2) = 0.$$

²¹For general p -forms, the de Rham cohomology group $H_d^p(\mathcal{K}, \mathbb{R})$ is defined as the kernel over the image of the exterior derivative d

$$H_d^p(\mathcal{K}, \mathbb{R}) = \frac{\text{Ker}[d : \Omega^p(\mathcal{K}) \rightarrow \Omega^{p+1}(\mathcal{K})]}{\text{Im}[d : \Omega^p(\mathcal{K}) \rightarrow \Omega^{p+1}(\mathcal{K})]}.$$

²²This is the $\bar{\partial}$ -Poincaré lemma for complex manifolds, which says that a $\bar{\partial}$ -closed (0,1)-form is locally $\bar{\partial}$ -exact.

²³For general (p, q) -forms, the Dolbeault cohomology group $H_{\bar{\partial}}^{p,q}(\mathcal{K}, \mathbb{R})$ is defined as the kernel over the image of the $\bar{\partial}$ -operator:

$$H_{\bar{\partial}}^{p,q}(\mathcal{K}, \mathbb{R}) = \frac{\text{Ker}[\bar{\partial} : \Omega^{p,q}(\mathcal{K}) \rightarrow \Omega^{p,q+1}(\mathcal{K})]}{\text{Im}[\bar{\partial} : \Omega^{p,q}(\mathcal{K}) \rightarrow \Omega^{p,q+1}(\mathcal{K})]}.$$

$$H_{\bar{\partial}}^{1,1}(\mathcal{K}, \mathbb{R}) = \frac{\{\bar{\partial}\text{-closed real (1,1)-forms}\}}{\{\bar{\partial}\text{-exact real (1,1)-forms}\}}.$$

A Kähler form ω on (\mathcal{K}, ω) defines a nonzero element $[\omega]$ of $H_{\bar{\partial}}^{1,1}(\mathcal{K}, \mathbb{R})$. If a cohomology class $\alpha \in H_{\bar{\partial}}^{1,1}(\mathcal{K}, \mathbb{R})$ can be written as: $\alpha = [\omega]$, for some Kähler form ω , then we say that α is a *Kähler class* and write $\alpha > 0$.²⁴

As already mentioned, the famous $\partial\bar{\partial}$ -lemma (which is the holomorphic version of the classic *Poincaré lemma* that follows from Hodge theory) is the fundamental result of Kähler geometry. Let (\mathcal{K}, ω) be a compact Kähler manifold and suppose that $0 = [\alpha] \in H_{\bar{\partial}}^{1,1}(\mathcal{K}, \mathbb{R})$ for a real smooth $\bar{\partial}$ -closed (1,1)-form α on (\mathcal{K}, ω) . Then there exists a real-valued smooth function $\varphi \in (\mathcal{K}, \omega)$, called the *Kähler potential*, such that the form α is uniquely determined (up to the addition of a constant) as:²⁵

$$\alpha = i\partial_i\bar{\partial}_{\bar{j}}\varphi = i\partial\bar{\partial}\varphi > 0.$$

In other words, a real (1,1)-form α is $\bar{\partial}$ -exact iff it is $\partial\bar{\partial}$ -exact.²⁶ An immediate consequence is that if ω and ω_φ are two Kähler forms in the same Kähler class $[\omega] \in H_{\bar{\partial}}^{1,1}(\mathcal{K}, \mathbb{R})$, then we have:

$$\omega_\varphi = \omega + i\partial\bar{\partial}\varphi > 0,$$

for some smooth Kähler potential φ (which is uniquely determined up to a constant). In other words, two Kähler metrics $g_{\bar{j}}^i$ and $\tilde{g}_{\bar{j}}^i$ on (\mathcal{K}, g) belong to the same Kähler class iff

²⁴In other words, a Kähler form ω on \mathcal{K} defines a nonzero element $[\omega] \in H_{\bar{\partial}}^{1,1}(\mathcal{K}, \mathbb{R})$. If a cohomology class $\alpha \in H_{\bar{\partial}}^{1,1}(\mathcal{K}, \mathbb{R})$ can be written as $\alpha = [\omega]$ for some Kähler form ω on \mathcal{K} then we say that α is a *Kähler class* (and write $\alpha > 0$). Therefore, the Kähler class of ω is its cohomology class $[\omega] \in H_{\bar{\partial}}^{1,1}(\mathcal{K}, \mathbb{R})$. Alternatively, in terms of $H_d^2(\mathcal{K}, \mathbb{R})$, the Kähler class of ω is its cohomology class $[\omega] \in H_d^2(\mathcal{K}, \mathbb{R})$. Usually, all this is simply written: the Kähler class of ω is the cohomology class $[\omega] \in H^2(\mathcal{K}, \mathbb{R})$ represented by ω .

²⁵The simplest example of the Kähler potential is the case of the simplest Kähler n -manifold \mathbb{C}^n -the complex Euclid n -space, in which $\varphi \in \mathbb{C}^n$ is given by:

$$\varphi(z^i, \bar{z}^i) = \frac{1}{2} \sum_i^n z^i \bar{z}^i.$$

Another standard example is the following *entropic Kähler potential* (see [16]) with its Taylor expansion:

$$\begin{aligned} \varphi(z^i, \bar{z}^i) &= -\log\left(1 - \sum_i^n z^i \bar{z}^i + \frac{1}{4} \left|\sum_i^n (z^i)^2\right|^2\right) \\ &\approx \sum_i^n z^i \bar{z}^i + \frac{1}{4} \sum_i^n (z^i)^2 (\bar{z}^i)^2 + \sum_{i \neq j} \left[z^i \bar{z}^i z^j \bar{z}^j - \frac{1}{4} (z^i)^2 (\bar{z}^j)^2 - \frac{1}{4} (z^j)^2 (\bar{z}^i)^2 \right]. \end{aligned}$$

²⁶The $\partial\bar{\partial}$ -lemma is an extension of the two Poincaré lemmas (for d and $\bar{\partial}$). Basically, the $\partial\bar{\partial}$ -lemma says that a closed (1,1)-form α locally arises as $i\partial\bar{\partial}\varphi$ for some smooth function φ , called the Kähler potential. This is similar to the standard Poincaré lemma, which says that closed exterior forms, $d\alpha = 0$, are locally exact, $\alpha = d\varphi$, where “exactness” means d (something), which is true on all smooth manifolds. The $\bar{\partial}$ -Poincaré lemma, in which “exactness” means $\bar{\partial}$ (something), is true on all complex manifolds. The $\partial\bar{\partial}$ -lemma, in which “exactness” means $\partial\bar{\partial}$ (something), is true only on Kähler manifolds. Therefore, Kähler manifolds are the most important of all: $\partial\bar{\partial}$ -lemma \Rightarrow $\bar{\partial}$ -Poincaré lemma \Rightarrow Poincaré lemma.

$$g_{\bar{j}} = \tilde{g}_{\bar{j}} + \partial_i \partial_{\bar{j}} \varphi.$$

The volume form: $\omega^n (= n! \omega^{[n]})$ and the standard volume Vol_ω on (\mathcal{K}, ω) are given, respectively, by:

$$\omega^{[n]} = i^n \det(g) dz^1 \wedge dz^{\bar{1}} \wedge \dots \wedge dz^n \wedge dz^{\bar{n}}, \quad \text{Vol}_\omega = \int_{\mathcal{K}} \omega^{[n]},$$

so that: $\partial \bar{\partial} \log(\omega^n) = \partial \bar{\partial} \log[\det(g)]$. By the universal *Stokes theorem*, if ω and $\tilde{\omega}$ are two Kähler forms in the same Kähler class $[\omega] \in H_{\bar{\partial}}^{1,1}(\mathcal{K}, \mathbb{R})$ then: $\text{Vol}_\omega = \text{Vol}_{\tilde{\omega}}$. The total scalar curvature is determined by the Ricci form $\text{Ric}(\omega)$ as [14]:

$$\int_{\mathcal{K}} R \omega^{[n]} = \int_{\mathcal{K}} \text{Ric}(\omega) \wedge \omega^{[n-1]},$$

and it depends only on the Kähler class $[\omega] \in H_{\bar{\partial}}^{1,1}(\mathcal{K}, \mathbb{R})$ and the *first Chern class*, $c_1(\mathcal{K})$, defined as the cohomology class of the Ricci form: $[\text{Ric}(\omega)] \in H_{\bar{\partial}}^{1,1}(\mathcal{K}, \mathbb{R})$.

The space $\mathcal{K}_{[\omega]}$ of Kähler forms ω on (\mathcal{K}, ω) with the same Kähler class $[\omega] \in H_{\bar{\partial}}^{1,1}(\mathcal{K}, \mathbb{R})$ is given by:

$$\mathcal{K}_{[\omega]} = \{[\omega] \in H^2(\mathcal{K}, \mathbb{R}) \mid \omega + i \partial \bar{\partial} \varphi > 0\},$$

and the associated functional space \mathcal{H} of Kähler potentials $\varphi \in (\mathcal{K}, \omega)$ in the class $[\omega]$ is given by [15]:

$$\mathcal{H} = \{\varphi \in C^\infty(\mathcal{K}, \mathbb{R}) \mid \omega_\varphi = \omega + i \partial \bar{\partial} \varphi > 0\},$$

for which the geodesic equation (w.r.t. ω_φ) reads:²⁷

$$\ddot{\varphi} - |\nabla \dot{\varphi}|^2 = 0, \quad \varphi(0) = \varphi_0.$$

Based on the sign of their first Chern class $c_1(\mathcal{K}) = [\text{Ric}(\omega)] \in H_{\bar{\partial}}^{1,1}(\mathcal{K}, \mathbb{R})$, all compact Kähler manifolds (\mathcal{K}, ω) can be classified into the following three categories:

- (\mathcal{K}, ω) with positive first Chern class, $c_1(\mathcal{K}) > 0$, is called the *Fano manifold* in which $[\text{Ric}(\omega)] = \pi c_1(\mathcal{K})$. It admits *Kähler-Ricci solitons*, metrics for which:

$$\text{Ric}(\omega) - \omega = \mathcal{L}_v \omega,$$

where \mathcal{L}_v is the Lie derivative along a holomorphic vector field $v = v^a \in \mathcal{K}$.

- (\mathcal{K}, ω) with vanishing first Chern class, $c_1(\mathcal{K}) = 0$, is called the *Calabi-Yau manifold*, the basic geometric object in string theory.
- (\mathcal{K}, ω) with negative first Chern class, $c_1(\mathcal{K}) < 0$, is called the *Kähler-Einstein manifold*, which admits the Kähler-Einstein metric g defined by:

$$R_{\bar{j}} = \lambda g_{\bar{j}} \quad (\text{or } \text{Ric}(\omega) = \lambda \omega), \quad \text{with } \lambda = \frac{2\pi}{\text{Vol}_\omega} \int_{\mathcal{K}} c_1(\mathcal{K}) \wedge \omega^{n-1}. \quad (13)$$

In addition, if $\text{Ric}(g) = 0$ on (\mathcal{K}, g) then g is a Ricci-flat metric. In that

²⁷The geodesic equation is equivalent [11] to the homogeneous complex Monge-Ampère equation (see below):

$$\left(\ddot{\varphi} - |\nabla \dot{\varphi}|^2\right) \frac{\omega_\varphi^n}{\omega^n} = 0.$$

case, according to the *Calabi conjecture* (see [16]) proven by S.-T. Yau [17], the first Chern class must also vanish: $c_1(\mathcal{K}) = 0$.²⁸

1.2. Kähler-Ricci Flow

Now we are ready to introduce our main actor, the *Kähler-Ricci flow* (see [6] [18] and the references therein) on a Kähler manifold (\mathcal{K}, g) . For this, we firstly recall that the real-valued *Ricci flow* on a Riemannian manifold M (introduced by R. Hamilton [19] [20] [21] and subsequently used by G. Perelman to prove the 100-year old *Poincaré Conjecture*, the only solved Clay Research's Millennium Prize problem), is governed by the nonlinear evolution equation of the Riemannian metric (in real components $i, j = 1, \dots, n$):

$$\partial_t g_{ij}(t) = -2R_{ij}(t), \quad g_{ij}(0) = g_0, \quad (14)$$

which in local harmonic coordinates on M can be rewritten in terms of the Laplace-Beltrami operator Δ as:

$$\partial_t g_{ij}(t) = \Delta g_{ij} + Q_{ij}(g_{ij}, \partial g_{ij}), \quad g_{ij}(0) = g_0, \quad (15)$$

where the tensor function $Q_{ij}(g_{ij}, \partial g_{ij})$ is quadratic in g_{ij} and its first order partial derivatives, ∂g_{ij} . Later, in [18], Equations (14)-(15) were proposed as a general model for a wide range of (real) nonlinear reaction-diffusion systems.

The Ricci flow (14) has a unique solution, called a *gradient Ricci soliton*, only in case of Einstein manifolds, such that $R_{ab} = \lambda g_{ab}$, which can be *shrinking* if $\lambda > 0$, *steady* if $\lambda = 0$ and *expanding* if $\lambda < 0$.²⁹

The complexification of the real Ricci flow (14), from a Riemannian manifold (M, g_{ij}) of real dimension n to the Kähler manifold $(\mathcal{K}, g_{i\bar{j}})$ of complex dimension n , is called the *Kähler-Ricci flow* (KRF), given by (see, e.g. [14] and [12]):

$$\partial_t \omega = -\text{Ric}(\omega), \quad \omega(0) = \omega_0, \quad (16)$$

²⁸The Calabi-Yau theorem says that if (\mathcal{K}, ω) is a compact Kähler manifold and α is a real (1,1)-form representing $c_1(\mathcal{K})$ then there exists a unique Kähler metric g on \mathcal{K} with $[g] = [\omega]$, such that $\text{Ric}(g) = 2\pi\alpha$.

²⁹A Riemannian metric g is called *Einstein metric* if the Ricci curvature of the metric is some multiple of the metric, that is, the Ricci tensor $R_{ab}(g)$ is defined as $R_{ab} = \lambda g_{ab}$ for some constant λ . A smooth manifold (M, g) with an Einstein metric g is called an *Einstein manifold*.

If the initial metric $g(0)$ is *Ricci flat* ($R_{ab} = 0$), then $\partial_t g_{ab} = 0$, so $g_{ab} = \text{const}$ under (12). Hence any Ricci flat metric is a stationary solution of the Ricci flow (12). If the initial metric is an Einstein metric of positive scalar curvature ($R > 0$), then the metric will *shrink* homothetically under the Ricci flow (12) by a time-dependent factor, and if the initial metric is an Einstein metric of negative scalar curvature ($R < 0$), the metric will *expand* homothetically for all times.

Therefore, the following simple and conformal solution of the Ricci flow (12) uniquely exists on Einstein n -manifolds (with $R_{ab} = \lambda g_{ab}$):

$$g_{ab}(x, t) = \rho^2(t) g_{ab}(x, 0), \quad \text{or simply: } g(t) = \rho^2(t) g(0),$$

where in case of the positive scalar curvature ($R > 0$), the function $\rho(t)$ satisfies the ODE:

$$\dot{\rho} = -\lambda/p, \quad \text{with the shrinking solution: } \rho^2(t) = 1 - 2\lambda t,$$

and in case of the negative scalar curvature ($R < 0$), the function $\rho(t)$ satisfies the ODE:

$$\dot{\rho} = \lambda/p, \quad \text{with the expanding solution: } \rho^2(t) = 1 + 2\lambda t.$$

with the extended form:³⁰

$$\partial_t \omega = -\text{Ric}(\omega) - \lambda \omega, \quad \omega(0) = \omega_0, \quad (17)$$

where the real constant λ is either 0 or 1. The case $\lambda = 1$ gives a rescaling of (16) called the *normalized KRF*.

In particular, a Fano n -manifold (\mathcal{K}, g) with positive first Chern class, $c_1(\mathcal{K}) > 0$, in which $[\text{Ric}(\omega)] = \pi c_1(\mathcal{K})$, admits the normalized KRF (see [14] [22]) with the time-dependent Ricci form (12) $\text{Ric}(\omega) = \text{Ric}[\omega(t)]$, given by:³¹

$$\partial_t g_{ij}(t) = g_{ij}(t) - \text{Ric}[\omega(t)], \quad g_{\bar{j}}(0) = g_0, \quad (18)$$

which is, starting from some smooth initial Kähler metric tensor g_0 given locally (in an open chart $U \subset \mathcal{K}$) by:

$$\partial_t g_{\bar{j}}(t) = g_{\bar{j}}(t) - R_{\bar{j}}(t), \quad g_{\bar{j}}(0) = g_0.$$

The normalized KRF (18) preserves the Kähler class $[\omega]$. It has a global solution $g(t) \equiv \omega(t)$ when $g_0 = g_{\bar{j}}(0)$ has $[\omega] = 2\pi c_1(\mathcal{K})$ as its Kähler class [which is written as $g_0 \in 2\pi c_1(\mathcal{K})$].

In terms of time-dependent Kähler potentials $\varphi = \varphi(t)$, the KRF (18) can be expressed as:

$$\partial_t \varphi(t) = \varphi(t) + \log(\omega_\varphi^n / \omega^n) - g(t), \quad \varphi(0) = \varphi_0,$$

where the time-dependent Kähler metric $g = g(t)$ is defined by:

$$i\partial\bar{\partial}g(t) = \text{Ric}[\omega(t)] - \omega(t) \quad \text{and} \quad \int_{\mathcal{K}} (e^{g(t)} - 1) \omega^n = 0.$$

The corresponding evolutions of the Ricci curvature $R_{\bar{j}} = R_{\bar{j}}(t)$ and the scalar curvature $R = R(t)$ on (\mathcal{K}, g) are governed, respectively by:

$$\partial_t R_{\bar{j}} = \Delta R_{\bar{j}} + R_{\bar{j}p\bar{q}} R_{q\bar{p}} - R_{\bar{p}} R_{\bar{j}\bar{p}}, \quad \partial_t R = \Delta R + R_{\bar{j}} R_{\bar{j}} - R,$$

starting from some smooth initial Ricci and scalar curvatures, $R_{\bar{j}}(0)$ and $R(0)$.

The evolution of the scalar curvature R can be also expressed in terms of the Ricci form as:

$$\partial_t R = \Delta R + |\text{Ric}(\omega)|^2 + \lambda R, \quad R(0) = R_0,$$

and it has a lower bound (determined by the real constant: $C = -\inf_{\mathcal{K}} R(0) - \lambda n$; see [12]):

$$R(t) \geq -\lambda n - Ce^{-\lambda t}.$$

The corresponding time evolution of the *trace* of the metric, $\text{Tr}(\omega) = \text{Tr}_\omega(g)$,

³⁰Similar to the KRF is the *Calabi flow* determined by the scalar curvature: $\partial_t \omega = i\partial\bar{\partial}R$.

³¹The KRF (18) preserves the Kähler class $[\omega]$. It has a global solution $g(t) \equiv \omega(t)$ when $g_0 = g_{\bar{j}}(0)$ has $[\omega] = 2\pi c_1(\mathcal{K})$ as its Kähler class, which is written as $g_0 \in 2\pi c_1(\mathcal{K})$. In particular by the $\partial\bar{\partial}$ -Lemma, there exists a family of real-valued functions $u(t)$, called Ricci potentials of the metric $g(t)$, which are special Kähler potentials. They are determined by the volume V_R of the KRF (18):

$$g_{\bar{j}} - R_{\bar{j}} = \partial_i \partial_{\bar{j}} u, \quad \frac{1}{V_R} \int_{\mathcal{K}} e^{-u(t)} dv_g = 1, \quad V_R = \int_{\mathcal{K}} dv_g.$$

computed in normal coordinates for the metric g (see [12]), and its lower bound, are given respectively by:

$$\begin{aligned} \partial_t \text{Tr}(\omega) &= -g^{\bar{j}i} R_{\bar{j}i} - \lambda \text{Tr}(\omega), \quad \text{Tr}(\omega)|_{\omega_0} = \text{Tr}(\omega_0), \\ (\partial_t - \Delta) \log[\text{Tr}(\omega)] &\leq C \text{Tr}(\omega) - \lambda. \end{aligned}$$

In general, the existence of the KRF in a time interval $t \in [0, t_1)$ can be established as follows: if $\omega(t)$ is a solution of the KRF:

$$\partial_t \omega(t) = -\text{Ric}[\omega(t)], \quad \omega(0) = \omega_0, \tag{19}$$

then the corresponding cohomology class $[\omega(t)]$ evolves on (\mathcal{K}, g) according to the following ODE:

$$\begin{aligned} \partial_t [\omega(t)] &= -c_1(\mathcal{K}), \quad [\omega(0)] = [\omega_0], \quad \text{with the solution:} \\ [\omega(t)] &= [\omega_0] - tc_1(\mathcal{K}) = [i\partial\bar{\partial}\varphi(0)] - tc_1(\mathcal{K}), \quad \text{for } [t \in [0, t_1)]. \end{aligned} \tag{20}$$

So, the KRF (19) exists for $t \in [0, t_1)$ iff $[\omega_0] - tc_1(\mathcal{K}) > 0$ (see [12] [14]).

In particular, the extensions of the Kähler-Einstein (KE) metric:

$$R_{\bar{j}i} = \lambda g_{\bar{j}i} \Leftrightarrow \text{Ric}(\omega) = \lambda \omega,$$

are the Kähler-Ricci (KR) solitons: a time-dependent Kähler metric $g(t) = g_{\bar{j}i}(t)$ is called a *gradient KR soliton* if there exists a real smooth Kähler potential φ on (\mathcal{K}, g) such that:

$$R_{\bar{j}i} = \lambda g_{\bar{j}i} - \partial_i \partial_{\bar{j}} \varphi \quad \text{and} \quad \nabla_i \nabla_{\bar{j}} \varphi = 0 \Leftrightarrow \nabla \varphi = (g^{\bar{j}i} \partial_{\bar{j}} \varphi) \partial_i.$$

Similar to the real Ricci flow case, this soliton is called *shrinking* if $\lambda > 0$, *steady* if $\lambda = 0$, and *expanding* if $\lambda < 0$, and the gradient vectorfield $\nabla \varphi$ is holomorphic. If the Kähler manifold (\mathcal{K}, g) admits a KE metric (or, a KR soliton) g then the first Chern class $c_1(\mathcal{K})$ is necessarily definite: $\pi c_1(\mathcal{K}) = \lambda [\omega_g]$.³²

At the end of this section, we remark that it was shown by [17] [23] that the KRF (19) can be rewritten as the (parabolic, complex) *Monge-Ampère equation*.³³

³²For any Kähler metric $g \in 2\pi c_1(\mathcal{K})$ with scalar curvature R and for any smooth function u on \mathcal{K} , the *Kähler-Perelman (KP) entropy* $\mathcal{E} \equiv \mathcal{E}(g)$ is defined by:

³³On \mathbb{C}^n the complex Monge-Ampère operator for the smooth potential φ is defined as the determinant of the complex Hessian:

$$\det(\partial, \partial_{\bar{j}} \varphi) \equiv \det \left(\frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} \right), \quad \text{with } (\partial, \partial_{\bar{j}} \varphi) \geq 0.$$

However, on a compact Kähler manifold (\mathcal{K}, g) , the condition $(\partial, \partial_{\bar{j}} \varphi) \geq 0$ would imply that φ is constant, so the Hessian $(\partial, \partial_{\bar{j}} \varphi)$ is naturally replaced by $(g_{\bar{j}i} + \partial, \partial_{\bar{j}} \varphi)$, to define the complex Monge-Ampère operator on (\mathcal{K}, g) as:

$$\det(g_{\bar{j}i} + \partial, \partial_{\bar{j}} \varphi) \equiv \det \left(g_{\bar{j}i} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} \right), \quad \text{with } (\partial, \partial_{\bar{j}} \varphi) \geq 0.$$

Using this operator, the complex Monge-Ampère equation is defined as:

$$\partial, \varphi = \log \frac{\det(g_{\bar{j}i} + \partial, \partial_{\bar{j}} \varphi)}{\det(g)} = \log \left[\frac{(\omega_g + i\partial\bar{\partial}\varphi)^n}{\omega^n} \right],$$

since: $\partial\bar{\partial} \log(\omega^n) = \partial\bar{\partial} \log[\det(g)]$.

$$\partial_t \varphi = \log \left[\left(\omega_\varphi + i\partial\bar{\partial}\varphi \right)^n / \omega^n \right], \quad \omega_0 + i\partial\bar{\partial}\varphi > 0, \quad \varphi(0) = \varphi_0,$$

since we have (see [12]):

$$\partial_t \omega(t) = \partial_t \omega_\varphi + i\partial\bar{\partial}(\partial_t \varphi) = \partial_t \left(\omega_\varphi + i\partial\bar{\partial}\varphi \right) = -\text{Ric}[\omega(t)].$$

Similarly, the normalized KRF (18) with $\lambda = 1$, that is:

$$\partial_t \omega(t) = -\text{Ric}[\omega(t)] - \omega(t), \quad \omega(0) = \omega_0,$$

can be rewritten as a *normalized* (parabolic, complex) Monge-Ampère equation:

$$\partial_t \varphi = \log \left[\left(\omega_0 + i\partial\bar{\partial}\varphi \right)^n / \omega^n \right] - \varphi, \quad \omega_0 + i\partial\bar{\partial}\varphi > 0, \quad \varphi(0) = \varphi_0.$$

For more technical details on the Kähler-Ricci flow, see e.g., [12] [14] and the references therein.