Advances in Pure Mathematics, 2011, 1, 367-377
doi:10.4236/apm.2011.16066 Published Online November 2011 (http://www.SciRP.org/journal/apm)

The Localization of Commutative Bounded BCK-Algebras

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Received August 11, 2011; *revised October* 20, 2011; *accepted October* 30, 2011

Abstract

In this paper we develop a theory of localization for bounded commutative BCK-algebras. We try to extend some results from the case of commutative Hilbert algebras (see [1]) to the case of commutative BCK-algebras.

Keywords: BCK-Algebra, Commutative BCK-Algebra, Algebra of Fractions, Maximal Algebra of Quotients, $\vee$-Closed System, Topology, Localization Algebra

1. Introduction

In 1966, Y. Imai and K. Iséki introduced a new notion called a *BCK-algebra* (see [2]). This notion is originated from two different ways. One of the motivations is based on the set theory (where the set difference operation play the main role) and another motivation is from classical and non-classical propositional calculi (see [2]). There are some systems which contain the only implication functor among the logical functors. These examples are the systems of positive implicational calculus, weak positive implicational calculus by A. Church, and BCI, BCK-systems by C. A. Meredith (see [3]).

In this paper we develop a theory of localization for commutative (bounded) BCK-algebras, and then we deal with generalizations of results which are obtained in the paper [1] for case of Hilbert algebras. For some informal explanations of the theory of localization for others categories of algebras see [4,5].

The paper is organized as follows: in Section 2 we recall the basic definitions and put in evidence many rules of calculus in (commutative) BCK-algebras which we need in the rest of paper. In Section 3 we introduce the *commutative BCK-algebra of fractions relative to a* $\vee$*-closed system*. In Section 4 we develop a theory for multipliers on a commutative (bounded) BCK-algebra. In Section 5 we define the notions of *BCK-algebras of fractions* and *maximal BCK-algebra of quotients* for a commutative (bounded) BCK-algebra. In the last part of this section is proved the existence of the maximal BCK-algebra of quotients (Theorem 29). In Section 6 we develop a theory of *localization for commutative* (*bounded*) *BCK-algebras*. So, for commutative (bounded) BCK-algebra A we define the notion of *localization BCK-algebra relative to a topology* F *on A*. In Section 7 we describe the localization BCK-algebra A_F in some special instances.

2. Preliminaries

In this paper the symbols $\Rightarrow$ and $\Leftrightarrow$ are used for logical implication, respectively logical equivalence.

Definition 1 ([6]) A *BCK-algebra* is an algebra $(A,\to,1)$ of type (2,0) such that the following axioms are fulfilled for every $x, y, z \in A$:

(a_1) $x \to x = 1$;

(a_2) If $x \to y = y \to x = 1$, then $x = y$;

(**B**) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$;

(**C**) $x \to (y \to z) = y \to (x \to z)$;

(**K**) $x \to (y \to x) = 1$.

In [7] it is proved that the system of axioms {a_1, a_2, **B**, **C**, **K**} is equivalent with the system {a_2, a_3, a_4, **B**}, where:

(a_3) $x \to 1 = 1$;

(a_4) $1 \to x = x$.

For examples of BCK-algebras see [6-8]. If A is a BCK-algebra, then the relation $\le$ defined by $x \le y$ iff $x \to y = 1$ is a partial order on A (which will be called the *natural order* on A; with respect to this order 1 is the largest element of A. A will be called *bounded* if A has a smallest element 0; in this case for $x \in A$ we denote $x^* = x \to 0$. If $(x \to y) \to y = (y \to x) \to x$ for every $x, y \in A$, then A is called *commutative* (see [5,9,10]).

We have the following rules of calculus in a BCK-algebra A (see [6,7]):

(c_1) $x \le y \to x$;

(c_2) $x \le (x \to y) \to y$;
(c_3) $((x \to y) \to y) \to y = x \to y$;
(c_4) $x \to y \le (z \to x) \to (z \to y) \le z \to (x \to y)$;
(c_5) If $x \le y$, then for every $z \in A$, $z \to x \le z \to y$ and $y \to z \le x \to z$.

Proposition 1 ([9], p. 5) If A is a commutative BCK-algebra, then relative to the natural ordering, A is a join-semilattice, where for $x, y \in A$:

$$x \vee y = (x \to y) \to y = (y \to x) \to x.$$

Lemma 2 Let A be a commutative BCK-algebra. For every $x, y, z \in A$ there exists $(x \to z) \wedge (y \to z)$ and
(c_6) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.

Proof. Since $x, y \le x \vee y$ by (c_5) we deduce that $(x \vee y) \to z \le x \to z, y \to z$. Let now $t \in A$ such that $t \le x \to z, y \to z$. Then $x, y \le t \to z \Rightarrow x \vee y \le t \to z$ $\Rightarrow t \le (x \vee y) \to z$, that is,
$(x \vee y) \to z = (x \to z) \wedge (y \to z)$. □

In [9] (Theorem 8) and [8] (Remark 2.1.32) it is proved the following result:

Theorem 3 If A is a BCK-algebra, then the following assertions are equivalent:

1) For every $x, y, z \in A$,

$$x \to (y \to z) = (x \to y) \to (x \to z);$$

2) For every $x, y \in A$, $x \to (x \to y) = x \to y$;

3) For every $x, y \in A$,

$$(x \to y) \to ((y \to x) \to x) = (y \to x) \to ((x \to y) \to y).$$

A BCK-algebra which verify one of the above equivalent conditions is called *Hilbert algebra* (or *positive implicative BCK-algebra*).

If A is a bounded BCK-algebra, we have the following rules of calculus in A (see [6]):
(c_7) If $x \le y$, then $y^* \le x^*$;
(c_8) $x^* = x^{***}, x \le x^{**}$;
(c_9) $x \to y^* = y \to x^*, (x \to y^{**})^{**} = x \to y^{**}$.

Remark 1 If A is a bounded commutative BCK-algebra, then for every $x \in A$,

$$(x \to 0) \to 0 = (0 \to x) \to x \Leftrightarrow x^{**} = x,$$

that is, A is an *involutive* BCK-algebra (see [6], p. 115 and [9], p. 10).

For $x_1, \cdots, x_n, x \in A$ ($n \ge 1$) we will define

$$(x_1, \cdots, x_n; x) = x_1 \to (x_2 \to \cdots (x_n \to x) \cdots).$$

For two elements $x, y \in A$ and a natural number $n \ge 1$ we denote $x \to_n y = (x, x, \cdots, x; y)$ where n indicates the number of occurrences of x. Clearly, if A is a Hilbert algebra, then $x \to_n y = x \to y$, for every $n \ge 1$.

Let A be a BCK-algebra. *A deductive system* (or *i-filter*) of A is a nonempty subset D of A such that $1 \in D$ and for every $x, y \in A$, if $x, x \to y \in D$, then $y \in D$. It is clear that if D is a deductive system, $x \le y$ and $x \in D$, then $y \in D$. We denote by $Ds(A)$ the set of all deductive systems of A. For a nonempty subset $X \subseteq A$, we denote by $\langle X \rangle = \cap \{D \in Ds(A) : X \subseteq D\}$ ($\langle X \rangle$ is called the *deductive system generated by* X). It is known that
$\langle X \rangle = \{x \in A : (x_1, \cdots, x_n; x) = 1, \text{for some } x_1, \cdots, x_n \in X\}$.
In particular for $a \in A$, we denote by $\langle a \rangle$ the deductive system generated by $\{a\}$ ($\langle a \rangle$ is called *principal* and $\langle a \rangle = \{x \in A : a \to_n x = 1, \text{for some } n \ge 1\}$).

Lemma 4 Let A be a bounded BCK-algebra and $x, y \in A$ such that there exists $x \vee y$ in A. Then there exists $x^* \wedge y^*$ and $x^* \wedge y^* = (x \vee y)^*$.

Proof. Clearly, $(x \vee y)^* \le x^*, y^*$. Let $t \in A$ such that $t \le x^*, y^*$. Then
$x, y \le t^* \Rightarrow x \vee y \le t^* \Rightarrow t^{**} \le (x \vee y)^*$. From ($c_8$) we deduce that $t \le t^{**} \le (x \vee y)^* \Rightarrow t \le (x \vee y)^*$, that is,
$(x \vee y)^* = x^* \wedge y^*$. □

Definition 2 ([7], p. 944) Let A be a bounded BCK-algebra. An element $x \in A$ is called *boolean* if $< x > \cap < x^* > = \{1\}$ (clearly, $\langle x \rangle \cup \langle x^* \rangle = A$).

We denote by $B(A)$ the set of all boolean elements of A; clearly, $0, 1 \in B(A)$.

Lemma 5 ([7]) Let A be a BCK-algebra. Then for every $x, y \in A$, $x \vee y = 1 \Leftrightarrow < x > \cap < y > = \{1\}$.

Corollary 6 For a bounded BCK-algebra $x \in B(A)$ iff $x \vee x^* = 1$.

Remark 2 If $x \in B(A)$, that is, $x \vee x^* = 1$, then from Lemma 4 we deduce that
$x^* \wedge x^{**} = (x \vee x^*)^* = 1^* = 0$, hence
$x \wedge x^* \le x^{**} \wedge x^* = 0 \Rightarrow x \wedge x^* = 0$, that is, x^* is the complement of x in A.

Boolean elements also satisfy several interesting properties which can be proved using above corollary and some arithmetical calculus:

Proposition 7 ([7]) Let A be a bounded BCK-algebra. Then for every $a \in B(A)$ and $x, y \in A$ we have:
(c_{10}) $a^* \in B(A)$;
(c_{11}) $a \to (a \to x) = a \to x$;
(c_{12}) $a \to (x \to y) = (a \to x) \to (a \to y)$;
(c_{13}) $a \to a^* = a^*, a^* \to a = a$;
(c_{14}) $a^{**} = a$;
(c_{15}) $(a \to x) \to a = a$;
(c_{16}) $(a \to x) \to x \le (x \to a) \to a$;
(c_{17}) $((a \to x) \to a^*) \to a^* = a \to x^{**}$;
(c_{18}) If $b \in B(A)$, then $(a \to b) \to b = (b \to a) \to a$;
(c_{19}) $a^* \to x = a \vee x = (a \to x) \to x$;
(c_{20}) $(a \to x^*)^* = a \wedge x^{**}$.

Corollary 8 ([7]) Let A be a bounded BCK-algebra. Then

1) If $a \in B(A)$, then $\langle a \rangle = [a) = \{x \in A : a \le x\}$;
2) For $a, b \in B(A)$, $a \to b \in B(A)$;
3) $(B(A), \to, 0, 1)$ is a Boolean algebra (where for $a, b \in B(A)$, $a \vee b = a^* \to b$ and $a \wedge b = (a \to b^*)^*$.

Corollary 9 Let A be a commutative BCK-algebra. For every $a \in B(A)$ and $y, z \in A$ we have:
(c_{21}) $a \vee (y \to z) = (a \vee y) \to (a \vee z)$.

Proof. By (c_6) we have

$$(a \vee y) \to (a \vee z) = (a \to (a \vee z)) \wedge (y \to (a \vee z))$$
$$= 1 \wedge (y \to (a \vee z)) = y \to (a \vee z) = y \to ((a \to z) \to z)$$

so (c_{21}) is equivalent with (*) $a \vee (y \to z) = y \to (a \vee z)$. Clearly, $a \le a \vee z \le y \to (a \vee z)$ and from $z \le a \vee z \Rightarrow y \to z \le y \to (a \vee z)$. So to prove (*) let $t \in A$ such that $a \le t$ and $y \to z \le t$. We have the intention to prove that

$$y \to (a \vee z) \le t \Leftrightarrow y \to ((a \to z) \to z) \le t$$
$$\Leftrightarrow (**) (a \to z) \to (y \to z) \le t.$$

Indeed, from

$$y \to z \le t \Rightarrow (a \to z) \to (y \to z)$$
$$\le (a \to z) \to t \overset{(B)}{\le} (t \to a) \to [(a \to z) \to a] \quad \square$$
$$\overset{(c_{15})}{=} (t \to a) \to a = (a \to t) \to t = 1 \to t = t.$$

Proposition 10 Let A be a commutative BCK-algebra. Then for every $a, b \in B(A)$ and $x \in A$ we have:
(c_{22}) $(a \vee x) \to (b \vee x) = (a \to b) \vee x$.

Proof. By (c_6) we have

$$(a \vee x) \to (b \vee x)$$
$$= [a \to (b \vee x)] \wedge [x \to (b \vee x)]$$
$$= [a \to (b \vee x)] \wedge 1 = a \to ((x \to b) \to b).$$

Also

$$(a \to b) \vee x = (x \to (a \to b)) \to (a \to b)$$
$$\overset{(C)}{=} (a \to (x \to b)) \to (a \to b) \quad \square$$
$$\overset{(c_{12})}{=} a \to ((x \to b) \to b).$$

Definition 3 If A_1, A_2 are BCK-algebras, then $f : A_1 \to A_2$ is called *morphism of BCK-algebras* if $f(x \to y) = f(x) \to f(y)$, for every $x, y \in A_1$ (if A_1, A_2 are bounded BCK-algebras, then we add the condition $f(0) = 0$).

3. Commutative BCK-Algebra of Fractions Relative to a $\vee$-Closed System

In this section by A we denote a commutative bounded BCK-algebra.

Definition 4 A nonempty subset S of A will be called $\vee$-*closed system* of A if $0 \in S$ and $x \vee y \in S$ for every $x, y \in S$.

For a $\vee$-closed system $S \subseteq A$ we define the binary relation θ_S on A by $(x, y) \in \theta_S$ iff there is $s \in S \cap B(A)$ such that $s \vee x = s \vee y$.

Proposition 11 The relation θ_S is a congruence on A.

Proof. Clearly θ_S is an equivalence relation on A. To prove the compatibility of θ_S with the operation →, let $x, y, z \in A$ such that $(x, y) \in \theta_S$ (hence there is $s \in S \cap B(A)$ such that $s \vee x = s \vee y$). By (c_{21}) we deduce

$$s \vee (z \to x) = (s \vee z) \to (s \vee x)$$
$$= (s \vee z) \to (s \vee y) = s \vee (z \to y),$$

and similarly, $s \vee (x \to z) = s \vee (y \to z)$, that is, $(z \to x, z \to y) \in \theta_S$ and $(x \to z, y \to z) \in \theta_S$. $\square$

We denote $A[S] = A/\theta_S$; the commutative BCK- algebra $A[S]$ will be called *BCK-algebra of fractions of A relative to S*. For $x \in A$ we denote by $[x]_{\theta_S}$ the equivalence class of x relative to θ_S. Clearly, in $A[S]$, **1** = $[1]_{\theta_S} = \{x \in A : (x, 1) \in \theta_S\}$ = { $x \in A$: there is $s \in S \cap B(A)$ such that $s \vee x = 1$ },
0 = $[0]_{\theta_S} = \{x \in A : (x, 0) \in \theta_S\}$ = { $x \in A$: there is $s \in S \cap B(A)$ such that $s \vee x = s$ } = { $x \in A$: there is $s \in S \cap B(A)$ such that $x \le s$ } and for every $x, y \in A$, $[x]_{\theta_S} \to [y]_{\theta_S} = [x \to y]_{\theta_S}$.

Proposition 12 $A[S]$ is a bounded commutative BCK-algebra, when **0** = $[s]_{\theta_S}$ with $s \in S \cap B(A)$.

Proof. Clearly, if $s, t \in S \cap B(A)$, since $r = s \vee t \in S \cap B(A)$ and $r \vee s = r \vee t \Rightarrow [s]_{\theta_S} = [t]_{\theta_S}$. To prove that, for $s \in S \cap B(A)$, $[s]_{\theta_S}$ = **0**, let $x \in A$. We have

$$[s]_{\theta_S} \le [x]_{\theta_S} \Leftrightarrow [s]_{\theta_S} \vee [x]_{\theta_S} = [x]_{\theta_S} \Leftrightarrow [s \vee x]_{\theta_S} = [x]_{\theta_S}$$

which is true since $s \vee (s \vee x) = s \vee x$. $\square$

We denote by $p_S : A \to A[S]$ the canonical surjective morphism of BCK-algebras (defined by $p_S(x) = [x]_{\theta_S}$, for every $x \in A$).

Remark 3 Since for every $s \in S \cap B(A)$, } $s \vee s = s \vee 0$ we deduce that $p_S(S \cap B(A)) = \{0\}$.

Proposition 13 If $x \in A$, then $[x]_{\theta_S} \in B(A[S])$ iff there exists $s \in S \cap B(A)$ such that $x \vee x^* \vee s = 1$. So, if $x \in B(A)$, then $[x]_{\theta_S} \in B(A[S])$.

Proof. For $x \in A$, we have

$$[x]_{\theta_S} \in B(A[S]) \Leftrightarrow [x]_{\theta_S} \vee ([x]_{\theta_S})^*$$
$$= 1 \Leftrightarrow [x \vee x^*]_{\theta_S} = 1 \Leftrightarrow$$

there exists $s \in S \cap B(A)$ such that $x \vee x^* \vee s = 1 \vee s = 1$. If $x \in B(A)$, since

$x \vee x^* \vee 0 = 1$ and $0 \in S \cap B(A)$, we deduce that $[x]_{\theta_S} \in B(A[S])$. □

$A[S]$ verify the following property of universality:

Theorem 14 For every bounded commutative BCK-algebra B and every morphism of bounded BCK-algebras $f : A \to B$ such that $f(S \cap B(A)) = \{0\}$, there exists a unique morphism of bounded BCK-algebras $f' : A[S] \to B$ such that $f' \circ p_S = f$.

Proof. Let $x, y \in A$ such that $[x]_{\theta_S} = [y]_{\theta_S}$. Then there is $s \in S \cap B(A)$ such that

$$s \vee x = s \vee y \Rightarrow f(s \vee x) = f(s \vee y)$$
$$\Rightarrow f(s) \vee f(x) = f(s) \vee f(y)$$
$$\Rightarrow 0 \vee f(x) = 0 \vee f(y) \Rightarrow f(x) = f(y).$$

So, $f' : A[S] \to B$ defined for $x \in A$ by $f'([x]_{\theta_S}) = f(x)$ is correct defined. Clearly, f' is morphism of bounded BCK-algebras and $f' \circ p_S = f$. The unicity of f' follows from the fact that p_S is onto. □

Example 1 If *A* is a bounded commutative BCK-algebra and $S = \{0\}$ or S is such that $0 \in S$ and $S \cap (B(A) \setminus \{0\}) = \varnothing$, then for $x, y \in A$, $(x, y) \in \theta_S \Leftrightarrow x \vee 0 = y \vee 0 \Leftrightarrow x = y$, hence $A[S] = A$.

Example 2 If *A* is a bounded commutative BCK-algebra and S is an $\vee$-closed system system such that $1 \in S$ (for example $S = A$ or $S = B(A)$), then for every $x, y \in A$, $(x, y) \in \theta_S$ (since $x \vee 1 = y \vee 1$ and $1 \in S \cap B(A)$, hence in this case $A[S] = \{1\}$.

Definition 5 $A[S]$ is called the *BCK-algebra of fractions* of A relative to S.

4. Multipliers on a Commutative Bounded BCK-Algebra

The concept of *maximal lattice of quotients* for a distributive lattice was defined by J. Schmid in [11,12] (taking as a guide-line the construction of *complete ring of quotients* by partial morphisms introduced by G. Findlay and J. Lambek (see [13], p. 36). The central role in the construction of the maximal lattice of quotients for a distributive lattice due to J. Schmidt in [11] and [12] is played by the concept of *multiplier* for a distributive lattice defined by W. H. Cornish in [14].

In this section we develop a theory for multipliers on a commutative bounded BCK-algebra A.

Definition 6 A subset $T \subseteq A$ is called $\vee$-*subset* of *A* if for every $a \in A$ and $x \in T$ we have $a \vee x \in T$.

We denote by $T(A)$ the set of all $\vee$-subsets of *A*. Clearly $Ds(A) \subseteq T(A)$ (and more generally, if denote by $I(A)$ the set of all increasing subsets of *A*, then $I(A) \subseteq T(A)$).

Remark 4 Clearly, if $D_1, D_2 \in T(A)$, then $D_1 \cap D_2 \in T(A)$.

Lemma 15 If $D \in T(A)$, then

1) $1 \in D$;

2) If $x \le y$ and $x \in D$, then $y \in D$.

Proof. (i). If $x \in D$, since $1 \in A$, then $1 = 1 \vee x \in D$.

3) We have $y = y \vee x$. □

Definition 7 By *partial strong multiplier* on *A* we mean a map $f : D \to A$, where $D \in T(A)$, such that:

(sm_1) For every $x \in D$ and $e \in B(A)$,

$$f(e \vee x) = e \vee f(x);$$

(sm_2) For every $x \in D$, $x \le f(x)$;

(sm_3) If $e \in D \cap B(A)$, then $f(e) \in B(A)$;

(sm_4) For every $x \in D$ and $e \in D \cap B(A)$,

$$f(e) \vee x = e \vee f(x).$$

By $dom(f) \in T(A)$ we denote the domain of f; if $dom(f) = A$, we called f *total*.

To simplify the language, we will use *strong multiplier* instead *partial strong multiplier* using *total* to indicate that the domain of a certain multiplier is *A*.

Examples

1) The maps $\mathbf{0}, \mathbf{1} : A \to A$ defined by $\mathbf{0}(x) = x$ and respectively $\mathbf{1}(x) = 1$, for every $x \in A$ are total strong multipliers on A.

2) For $a \in B(A)$ and $D \in T(A)$, the map $f_a : D \to A$ defined by $f_a(x) = a \vee x$, for every $x \in D$ is a strong multiplier on *A* (called *principal*).

If $dom(f_a) = A$, we denote f_a by $\overline{f_a}$.

Remark 5 If $f : D \to A$ is a strong multiplier on *A* (with $D \in T(A)$), then $f(1) = 1$. Indeed, if in (sm_1) we put $e = 1$, we obtain that for every $x \in D$, $f(1 \vee x) = 1 \vee f(x) \Leftrightarrow f(1) = 1$.

For $D \in T(A)$, we denote

$$M(D, A) = \{ f : D \to A : f \textit{ is a strong multiplier on } A \}$$

and $M(A) = \bigcup_{D \in T(A)} M(D, A)$.

For $D_1, D_2 \in T(A)$ and $f_i \in M(D_i, A)$, $i = 1, 2$, we define $f_1 \to f_2 : D_1 \cap D_2 \to A$ by $(f_1 \to f_2)(x) = f_1(x) \to f_2(x)$, for every $x \in D_1 \cap D_2$.

Lemma 16 $f_1 \to f_2 \in M(D_1 \cap D_2, A)$.

Proof. If $x \in D_1 \cap D_2$ and $e \in B(A)$, then

$$(f_1 \to f_2)(e \vee x) = f_1(e \vee x) \to f_2(e \vee x)$$
$$= (e \vee f_1(x)) \to (e \vee f_2(x)) \overset{(c_{21})}{=} e \vee (f_1(x) \to f_2(x))$$
$$= e \vee (f_1 \to f_2)(x),$$

$$(f_1 \to f_2)(x) = f_1(x) \to f_2(x) \ge f_2(x) \overset{(sm_2)}{\ge} x,$$
$$(f_1 \to f_2)(e) = f_1(e) \to f_2(e) \in B(A)$$

by Corollary 8 (since $f_1(e), f_2(e) \in B(A)$ and if

$$e \in D_1 \cap D_2 \cap B(A),$$

$$e \vee (f_1 \to f_2)(x) = e \vee (f_1(x) \to f_2(x))$$

$$\overset{(c_{21})}{=} (e \vee f_1(x)) \to (e \vee f_2(x))$$

$$\overset{(sm_4)}{=} (x \vee f_1(e)) \to (x \vee f_2(e))$$

$$\overset{(c_{22})}{=} x \vee (f_1(e) \to f_2(e)) = x \vee (f_1 \to f_2)(e),$$

that is, $f_1 \to f_2 \in M(D_1 \cap D_2, A)$. □

Corollary 17 $(M(A), \to, 0, 1)$ is a bounded commutative BCK-algebra.

Proof. The fact that $M(A)$ is a commutative BCK-algebra follows from Lemma 16. If $D \in T(A)$, $f \in M(D, A)$ and $x \in D$, then $0(x) \le x \le f(x) \le 1 \le 1(x)$ and since the relation of order on $M(A)$ is given by $f_1 \le f_2$ iff $f_1(x) \le f_2(x)$ for every $x \in dom(f_1) \cap dom(f_2)$, we deduce that $0 \le f \le 1$, that is, $M(A)$ is bounded. □

Lemma 18 The map $v_A : B(A) \to M(A)$ defined by $v_A(a) = \overline{f_a}$ for every $a \in B(A)$ is a morphism of bounded BCK-algebras.

Proof. If $a, b \in B(A)$ and $x \in A$, then

$$\left(\overline{f_a} \to \overline{f_b}\right)(x) = \overline{f_a}(x) \to \overline{f_b}(x)$$

$$= (a \vee x) \to (b \vee x) \overset{(c_{22})}{=} (a \vee b) \to x = \overline{f_{a \to b}}(x),$$

so, $v_A(a) \to v_A(b) = v_A(a \to b)$

and $v_A(0) = \overline{f_0} = 0$. □

Definition 8 $D \subseteq A$ is called *regular* if for every $x, y \in A$ such that $e \vee x = e \vee y$ for every $e \in D \cap B(A)$, then $x = y$.

For example, a bounded BCK-algebra A is regular since if $x, y \in A$ such that $e \vee x = e \vee y$ for every $e \in A \cap B(A) = B(A)$, then in particular, for $e = 0$ we obtain $x \vee 0 = y \vee 0 \Rightarrow x = y$.

If A is bounded, $D \in T(A)$ and $0 \in D$, then D is regular. We denote

$$R(A) = \{D \subseteq A : D \textit{ is a regular subset of } A\}.$$

Lemma 19 If $D_1, D_2 \in T(A) \cap R(A)$, then $D_1 \cap D_2 \in T(A) \cap R(A)$.

Proof. By Remark 4, $D_1 \cap D_2 \in T(A)$. Let $x, y \in A$ such that $e \vee x = e \vee y$ for every $e \in D_1 \cap D_2 \cap B(A)$.

For every $e_i \in D_i \cap B(A)$, $i = 1, 2$, since $e_1 \vee e_2 \in D_1 \cap D_2 \cap B(A)$ we have

$$(e_1 \vee e_2) \vee x = (e_1 \vee e_2) \vee y \Rightarrow e_1 \vee (e_2 \vee x)$$
$$= e_1 \vee (e_2 \vee y) \Rightarrow e_2 \vee x = e_2 \vee y \Rightarrow x = y,$$

so $D_1 \cap D_2 \in R(A)$. □

We denote

$$M_r(A) = \{f \in M(A) : dom(f) \in T(A) \cap R(A)\}.$$

Corollary 20 $M_r(A)$ is a BCK-subalgebra of $M(A)$.

Proposition 21 $M_r(A)$ is a Boolean subalgebra of $M(A)$.

Proof. Let $f : D \to A$ be a strong multiplier on A with $D \in T(A) \cap R(A)$. Then $f^* : D \to A$, $f^*(x) = (f \to 0)(x) = f(x) \to x$, for $x \in D$.

We have

$$(f \vee f^*)(x) = f(x) \vee (f(x) \to x)$$
$$= [(f(x) \to x) \to f(x)] \to f(x)$$

Then for $e \in D \cap B(A)$ and $x \in D$ we have

$$e \vee [f \vee f^*](x) = e \vee [[(f(x) \to x) \to f(x)] \to f(x)]$$

$$\overset{(c_{21})}{=} [[((e \vee f(x)) \to (e \vee x)) \to (e \vee f(x))] \to (e \vee f(x))]$$

$$\overset{(sm_4)}{=} [[((x \vee f(e)) \to (e \vee x)) \to (x \vee f(e))] \to (x \vee f(e))]$$

$$\overset{(c_{22})}{=} x \vee [[(f(e) \to e) \to f(e)] \to f(e)] = x \vee [(f(e) \to e) \vee f(e)]$$

$$= x \vee [(f(e))^* \vee e \vee f(e)] = x \vee 1 = 1 = e \vee 1 = e \vee 1(x).$$

Since $D \in R(A)$ we deduce that $(f \vee f^*)(x) = 1(x)$, hence $f \vee f^* = 1$, that is, $M_r(A)$ is a Boolean algebra (by Corollary 6). □

Remark 6 The axioms sm_3, sm_4 were necessary in the proof of Proposition 21.

Definition 9 Given two strong multipliers f_1, f_2 on A, we say that f_1 *extends* f_2 if $dom(f_2) \subseteq dom(f_1)$ and $f_1(x) = f_2(x)$, for all $x \in dom(f_2)$; we write $f_2 \le f_1$ if f_1 extends f_2. A strong multiplier f is called *maximal* if f can not be extended to a strictly larger domain.

Lemma 22 1) If $f_1, f_2 \in M(A)$, $f \in M_r(A)$ and

$f \le f_1$, $f \le f_2$, then f_1 and f_2 coincide on the $dom(f_1) \cap dom(f_2)$;

2) Every strong multiplier $f \in M_r(A)$ can be extended to a maximal strong multiplier. More precisely, each principal strong multiplier f_a with $a \in B(A)$ and $dom(f_a) \in T(A) \cap R(A)$ can be uniquely extended to the total strong multiplier $\overline{f_a}$ and each non-principal strong multiplier can be extended to a maximal non-principal one.

Proof. 1) If by contrary, there exists $t \in dom(f_1) \cap dom(f_2)$ such that $f_1(t) \ne f_2(t)$, since $dom(f) \in R(A)$, then there exists $t' \in dom(f) \cap B(A)$ such that $t' \vee f_1(t) \ne t' \vee f_2(t) \Leftrightarrow f_1(t' \vee t) \ne f_2(t' \vee t)$ which is contradictory, since $t' \vee t \in dom(f)$.

2) We first prove that f_a with $a \in B(A)$ can not be extended to a non-principal strong multiplier. Let $D = dom(f_a) \in T(A) \cap R(A)$, $f_a : D \to A$ and suppose by contrary that there exists $D' \in T(A), D \subseteq D'$, (hence $D' \in T(A) \cap R(A)$) and a non-principal strong multiplier $f \in M(D', A)$ which extends f_a. Since f is non-principal, there exists $x_0 \in D', x_0 \notin D$ such that $f(x_0) \ne a \vee x_0$. Since $D \in R(A)$, then there exists $t \in D \cap B(A)$ such that

$$t \vee f(x_0) \ne t \vee (a \vee x_0) \Leftrightarrow f(t \vee x_0) \ne a \vee (t \vee x_0),$$

which is contradictory since $f_a \le f$. Hence f_a is uniquely extended by $\overline{f_a}$.

Now, let $f \in M_r(A)$ be non-principal and

$$M_f = \{(D, g) : D \in T(A), g \in M(D, A), dom(f) \subseteq D \text{ and } g_{|dom(f)} = f\}$$

(clearly, if $(D, g) \in M_f$, then $D \in T(A) \cap R(A)$).

The set M_f is ordered by $(D_1, g_1) \le (D_2, g_2)$ iff $D_1 \subseteq D_2$ and $g_{2|D_1} = g_1$. Let $\{(D_k, g_k) : k \in K\}$ be a chain in M_f. Then $D' = \bigcup_{k \in K} D_k \in T(A)$ and $dom(f) \subseteq D'$. So, $g' : D' \to A$ defined by $g'(x) = g_k(x)$ if $x \in D_k$ is correctly defined (since if $x \in D_k \cap D_t$ with $k, t \in K$, then by 1), $g_k(x) = g_t(x)$).

Clearly, $g' \in M(D', A)$ and $g'_{|dom(f)} = f$ (since if $x \in dom(f) \subseteq D'$, then $x \in D'$ and so there exists $k \in K$, such that $x \in D_k$, hence $g'(x) = g_k(x) = f(x)$).

So, (D', g') is an upper bound for the family $\{(D_k, g_k) : k \in K\}$, hence by Zorn's lemma, M_f contains at least one maximal strong multiplier h which extends f. Since f. is non-principal and h extends f, h is also non-principal. □

On the Boolean algebra $M_r(A)$ we consider the relation ρ_A defined by $(f_1, f_2) \in \rho_A$ iff f_1 and f_2 coincide on the intersection of their domains.

Lemma 23 ρ_A is a congruence on $M_r(A)$.

Proof. The reflexivity and the symmetry of ρ_A are immediately; to prove the transitivity of ρ_A let $(f_1, f_2), (f_2, f_3) \in \rho_A$. Therefore f_1, f_2, and respectively f_2, f_3 coincide on the intersection of their domains. If by contrary, there exists $x_0 \in dom(f_1) \cap dom(f_3)$ such that $f_1(x_0) \ne f_3(x_0)$, since $dom(f_2) \in R(A)$, there exists $t \in dom(f_2) \cap B(A)$ such that

$$t \vee f_1(x_0) \ne t \vee f_3(x_0) \Leftrightarrow f_1(t \vee x_0) \ne f_3(t \vee x_0)$$

which is contradictory, since $t \vee x_0 \in dom(f_1) \cap dom(f_2) \cap dom(f_3)$. The compatibility of ρ_A with $\to$ on $M_r(A)$ is immediately. □

For $f \in M_r(A)$ we denote by $[f]$ the congruence class of f modulo ρ_A and $A'' = M_r(A)/\rho_A$.

Remark 7 From Proposition 21 we deduce that A'' is a Boolean algebra.

Lemma 24 The map $\overline{v}_A : B(A) \to A''$ defined by $\overline{v}_A(a) = [\overline{f_a}]$ is an injective morphism of Boolean algebras and $\overline{v}_A(B(A)) \in R(A'')$.

Proof. The fact that $\overline{v}_A$ is a morphism of Boolean algebras follows from Lemma 18. To prove the injectivity of $\overline{v}_A$ let $a, b \in B(A)$ such that $\overline{v}_A(a) = \overline{v}_A(b)$. Then $[\overline{f_a}] = [\overline{f_b}] \Leftrightarrow (\overline{f_a}, \overline{f_b}) \in \rho_A \Leftrightarrow \overline{f_a}(x) = \overline{f_b}(x)$, or every $x \in A \Leftrightarrow x \vee a = x \vee b$, for every $x \in A$, hence for $x = 0$ we obtain that $0 \vee a = 0 \vee b \Rightarrow a = b$. To prove $\overline{v}_A(B(A)) \in R(A'')$, if by contrary there exist $f_1, f_2 \in M_r(A)$ such that $[f_1] \ne [f_2]$ (that is there exists $x_0 \in dom(f_1) \cap dom(f_2)$ such that $f_1(x_0) \ne f_2(x_0)$) and

$$[f_1] \vee [\overline{f_a}] = [f_2] \vee [\overline{f_a}] \Leftrightarrow [f_1 \vee \overline{f_a}] = [f_2 \vee \overline{f_a}]$$

for every $a \in B(A) \Leftrightarrow f_1(x) \vee a \vee x = f_2(x) \vee a \vee x$, for every $a \in B(A)$ and every $x \in dom(f_1) \cap dom(f_2)$. For $a = 0$ and $x = x_0$ we obtain that $f_1(x_0) \vee x_0 = f_2(x_0) \vee x_0 \Leftrightarrow f_1(x_0) = f_2(x_0)$ which is contradictory. □

Remark 8 Since for every $a \in B(A)$, $\overline{f_a}$ is the unique maximal strong multiplier on $[\overline{f_a}]$ (by Lemma 22) we can identify $[\overline{f_a}]$ with $\overline{f_a}$. So, since $\overline{v}_A$ is injective morphism of Boolean algebras, the elements of $B(A)$ can be identified with the elements of the set $\{\overline{f_a} : a \in B(A)\}$.

Lemma 25 In view of the identifications made above, if $[f] \in A''$ (with $f \in M_r(A)$ and $D = dom(f) \in T(A) \cap R(A)$), then

$$D \cap B(A) \subseteq \{a \in B(A) : \overline{f_a} \vee [f] \in B(A)\}.$$

Proof. Let $a \in D \cap B(A)$. If by contrary, $\overline{f_a} \vee [f] \notin B(A)$ then $\overline{f_a} \vee f$ is a non-principal strong multiplier. Then by Lemma 22, (2), $\overline{f_a} \vee f$ can be extended to a non-principal maximal strong multiplier

$\overline{f}:\overline{D}\to A$ with $\overline{D}\in T(A)$. Thus, $D\subseteq\overline{D}$ and for every $x\in D$,

$$\overline{f}(x)=\left(\overline{f_a}\vee f\right)(x)=a\vee x\vee f(x)=a\vee f(x).$$

Since $a\in D\cap B(A)$, then

$\overline{f}(x)=f(a\vee x)\overset{(sm_4)}{=}x\vee f(a)$, that is, $\overline{f}_{|D}$ is principal which is contradictory with the assumption that $\overline{f}$ is non-principal. □

5. Maximal Commutative BCK-Algebra of Quotients

The goal of this section is to define (taking as a guideline the case of distributive lattices) the notions of *BCK-algebra of fractions* and *maximal BCK-algebra of quotients* for a commutative bounded BCK-algebra. For some informal explanations of notions of fraction see [13] and [5].

Definition 10 A bounded commutative BCK-algebra A' is called *BCK-algebra of fractions of A* if:

(f_1) $B(A)$ is a BCK-subalgebra of A';

(f_2) For every $a',b',c'\in A', a'\neq b'$, there exists $a\in B(A)$ such that $a\vee a'\neq a\vee b'$ and $a\vee c'\in B(A)$.

As a notational convenience, we write $A\prec A'$ to indicate that A' is a BCK-algebra of fractions of A. So, $B(A)\prec B(A)$ (since for $a',b',c'\in B(A)$ with $a'\neq b'$, if consider $0\in B(A)$, then $a'=a'\vee 0\neq b'\vee 0=b'$ and $c'=c'\vee 0\in B(A)$).

Definition 11 $Q(A)$ is the *maximal* (*commutative*) *BCK-algebra of quotients of A* if $A\prec Q(A)$ and for every commutative and bounded BCK-algebra A' with $A\prec A'$, there exists a monomorphism of BCK-algebras $i:A'\to Q(A)$.

Proposition 26 Let *A* be a commutative and bounded BCK-algebra such that $A\prec A'$. Then A' is a Boolean algebra.

Proof. If by contrary, A' is not a Boolean algebra, then by Corollary 6, there exists $x\in A'$ such that $x\vee x^*\neq 1$. Since $A\prec A'$, then there exists $e\in B(A)$, such that $e\vee x\in B(A)$ and $e\vee\left(x\vee x^*\right)\neq e\vee 1=1$. Then, by Lemma 4,

$$(e\vee x)\vee(e\vee x)^*=1\Rightarrow(e\vee x)\vee\left(e^*\wedge x^*\right)=1$$
$$\Rightarrow 1\leq\left(e\vee x\vee e^*\right)\wedge\left(e\vee x\vee x^*\right)$$
$$\Rightarrow 1\leq 1\wedge\left(e\vee x\vee x^*\right)\Rightarrow e\vee x\vee x^*=1,$$

a contradiction! □

Remark 9 If *A* is a Boolean algebra, then $B(A)=A$. By Proposition 26, $Q(A)$ is a Boolean algebra and the axioms sm_1-sm_4 are equivalent with sm_1, hence $Q(A)$ is in this case just the classical Dedekind-MacNeille completion of *A* (see [12], p. 687). In contrast to the general situation, the Dedekind-MacNeille completion of a Boolean algebra is again distributive and, in fact, is a Boolean algebra (see [15], p. 239).

Lemma 27 Let $A\prec A'$; then for every $a',b'\in A'$, $a'\neq b'$, and any finite sequence $c_1',\cdots,c_n'\in A'$, there exists $a\in B(A)$ such that $a\vee a'\neq a\vee b'$ and $a\vee c_i'\in B(A)$ for $i=1,2,\cdots,n$ $(n\geq 2)$.

Proof. Assume lemma holds true for $n-1$. So we may find $b\in B(A)$ such that $b\vee a'\neq b\vee b'$ and $b\vee c_i'\in B(A)$ for $i=1,2,\cdots,n-1$. Since $A\prec A'$, we find $c\in B(A)$ such that $c\vee(b\vee a')\neq c\vee(b\vee b')$ and $c\vee c_n'\in B(A)$. The element $a=b\vee c\in B(A)$ has the required properties. □

Lemma 28 Let $A\prec A'$ and $a'\in A'$. Then

$$D_{a'}=\{a\in B(A):a\vee a'\in B(A)\}\in T(B(A))\cap R(A).$$

Proof. If $a\in B(A)$ and $x\in D_{a'}$, then $x\vee a'\in B(A)$ and since $(a\vee x)\vee a'=a\vee(x\vee a')\in B(A)$ it follows $a\vee x\in D_{a'}$, hence $D_{a'}\in T(B(A))$. To prove $D_{a'}\in R(A)$ consider $x,y\in A$ such that $e\vee x=e\vee y$, for every $e\in D_{a'}\cap B(A)$. If by contrary, $x\neq y$, since $A\prec A'$, there exists $a_0\in B(A)$ such that $a_0\vee a'\in B(A)$ (that is, $a_0\in D_{a'}$) and $a_0\vee x\neq a_0\vee y$, which is contradictory. □

Theorem 29 A'' (defined in Section 4) is the maximal (commutative) BCK-algebra of quotients $Q(A)$ of *A*.

Proof. The fact that $B(A)$ is a BCK-subalgebra (Boolean subalgebra) of $Q(A)$ follows from Lemma 24 and Remark 8. To prove $A\prec Q(A)$, let $[f],[g],[h]\in Q(A)$ with $f,g,h\in M_r(A)$ such that $[g]\neq[h]$ (that is, there exists $x_0\in dom(g)\cap dom(h)$ such that $g(x_0)\neq h(x_0)$).

Put $D=dom(f)\in T(A)\cap R(A)$ and $D_{[f]}=\left\{a\in B(A):\overline{f_a}\vee[f]\in B(A)\right\}$.

Then by Lemma 25, $D\cap B(A)\subseteq D_{[f]}$. If suppose that for every $a\in D\cap B(A)$, $\overline{f_a}\vee[g]=\overline{f_a}\vee[h]$, then $\left[\overline{f_a}\vee g\right]=\left[\overline{f_a}\vee h\right]$, hence for every $x\in dom(g)\cap dom(h)$ we have $\left(\overline{f_a}\vee g\right)(x)=\left(\overline{f_a}\vee h\right)(x)\Leftrightarrow$ (analogously than as in the proof of Lemma 24)

$\Leftrightarrow a\vee x\vee g(x)=a\vee x\vee h(x)\Leftrightarrow a\vee g(x)=a\vee h(x)$.

Since $D\in R(A)$ we deduce that $g(x)=h(x)$ for every $x\in dom(g)\cap dom(h)$ so $[g]=[h]$, which is contradictory. Hence, if $[g]\neq[h]$, then there exists $a\in D\cap B(A)$, such that $\overline{f_a}\vee[g]\neq\overline{f_a}\vee[h]$. But for this $a\in D\cap B(A)$ we have $\overline{f_a}\vee[f]\in B(A)$ (since $D\cap B(A)\subseteq D_{[f]}$) hence $A\prec Q(A)$.

To prove the maximality of $Q(A)$, let A' be a

bounded commutative BCK-algebra such that $A \prec A'$, thus $B(A) \subseteq B(A')$; Then A' is embedded in $Q(A)$ by $i: A' \to Q(A)$ defined by $i(a') = [f_{a'}]$, for every $a' \in A'$, where $dom(f_{a'}) \in D_{a'}$ (see Lemma 28).
Clearly, $f_{a'} \in M_r(A)$ (by Lemma 28) and i is a morphism of BCK-algebras (see Lemma 18). To prove the injectivity of i, let $a', b' \in A'$, such that
$[f_{a'}] = [f_{b'}] \Leftrightarrow f_{a'}(x) = f_{b'}(x)$ for every $x \in D_{a'} \cap D_{b'}$.
If $a' \neq b'$, by Lemma 27 (since $A \prec A'$), there exists $a \in B(A)$ such that $a \vee a', a \vee b' \in B(A)$ and $a \vee a' \neq a \vee b'$ which is contradictory (since $a \vee a', a \vee b' \in B(A)$ implies $a \in D_{a'} \cap D_{b'}$). □

Remark 10 1. If A is a BCK-algebra with $B(A) = \{0,1\} = L_2$ and $A \prec A'$ then $A' = \{0,1\}$, hence $Q(A) \approx L_2$. Indeed, if $a, b, c \in A'$, with $a \neq b$, then there exists $e \in B(A)$ such that $e \vee a \neq e \vee b$, (hence $e \neq 1$) and $e \vee c \in B(A)$. Clearly, $e = 0$, hence $c \in B(A)$, that is $A' = B(A)$. As examples of BCK-algebras with this property we have local BCK-algebras and BCK-chains.

2. More general, if A is a BCK-algebra such that $B(A)$ is finite, if $A \prec A'$ then $A' = B(A)$, hence $Q(A) = B(A)$

Indeed, $B(A) \subseteq A'$ and consider $a \in A'$. $B(A)$ being finite, there exists a smallest element $e_a \in B(A)$ such $e_a \vee a \in B(A)$. Suppose $e_a \vee a \neq a$, then there would exists $e \in B(A)$ such that $e \vee (e_a \vee a) \neq e \vee a$ and $e \vee a \in B(A)$. But $e \vee a \in B(A)$ implies $e_a \leq e$, and thus we obtain $e \vee (e_a \vee a) \neq e \vee a \Leftrightarrow e \vee a \neq e \vee a$, a contradiction. Hence $a = e_a \vee a \in B(A)$, that is, $A' \subseteq B(A)$. Then $A' = B(A)$, hence $Q(A) = B(A)$.

6. Localization of Commutative Bounded BCK-ALgebras

In [4], G. Georgescu exhibited the localization lattice L_F of a distributive lattice L with respect to a topology F on L in a similar way as for rings (see [16]) or monoids (see [17]). The aim of this section is to define the notion of *localization BCK-algebra* A_F of a commutative bounded BCK-algebra A with respect to a topology F on A. In the last part of this section is proved that the maximal commutative BCK-algebra of quotients (defined in Section 5) and the commutative BCK-algebra of fractions relative to a $\vee$-closed system (defined in Section 3) are BCK-algebras of localization.

In this section A will be a bounded commutative BCK-algebra and F a topological system on A.

Definition 12 A non-empty family F of elements on $T(A)$ will be called a *topological system* on A if the following properties hold:

(t_1) If $D_1 \in F, D_2 \in T(A)$ and $D_1 \subseteq D_2$, then $D_2 \in F$ (hence $A \in F$);

(t_2) If $D_1, D_2 \in F$, then $D_1 \cap D_2 \in F$.

Example 3 If $D \in T(A)$, then the set $F_D = \{D' \in T(A) : D \subseteq D'\}$ is a topological system on A.

Example 4 We recall that by $R(A)$ we denote the set of all regular subsets of A (see Definition 8). Then $F = T(A) \cap R(A)$ is a topological system on A (see Lemma 19).

Example 5 Let $S \subseteq A$ a $\vee$-closed subset of A (see Definition 4). If we denote by $F_S = \{D \in T(A) : D \cap S \cap B(A) \neq \varnothing\}$, then F_S is a topological system on A.

If F is a topological system on A, let us consider the relation θ_F of A defined by: $(x, y) \in \theta_F \Leftrightarrow$ there exists $D \in F$ such that $t \vee x = t \vee y$ for any $t \in D \cap B(A)$. As in the case of θ_S (see Proposition 11), we deduce that θ_F is a congruence on A.

We shall denote by x/θ_F the congruence class of an element $x \in A$ and by $p_F : A \to A/\theta_F$ the canonical morphism of BCK-algebras.

Remark 11 Clearly, if $a \in B(A) \Rightarrow a/\theta_F \in B(A/\theta_F)$.

Definition 13 A F*-multiplier* on A is a mapping $f : D \to A/\theta_F$ where $D \in F$ such that for every $a \in B(A)$ and $x \in D$,

(m_1) $f(a \vee x) = a/\theta_F \vee f(x)$;

(m_2) $x/\theta_F \leq f(x)$.

If $F = \{A\}$, then a F-multiplier is a function $f : A \to A$ which verify only the conditions sm_1 and sm_2 from Definition 7. The maps **0**,**1**: $A \to A/\theta_F$, defined by $\mathbf{0}(x) = x/\theta_F$ and $\mathbf{1}(x) = 1/\theta_F$ for every $x \in A$ are F-multipliers. Also, for $a \in B(A)$, $f_a : D \to A/\theta_F$ defined by $f_a(x) = a/\theta_F \vee x/\theta_F$ for every $x \in D$, is a F-multiplier (where $D \in F$).

For $D \in F$, we shall denote by $M(D, A/\theta_F)$ the set of all the F-multipliers having the domain D. If $D_1, D_2 \in F$, $D_1 \subseteq D_2$ we have a canonical mapping $\phi_{D_1,D_2} : M(D_2, A/\theta_F) \to M(D_1, A/\theta_F)$ defined by $\phi_{D_1,D_2}(f) = f_{|D_1}$ for $f \in M(D_2, A/\theta_F)$. Let us consider the directed system of sets $< \{M(D, A/\theta_F)\}_{D \in F}$, $\{\phi_{D_1,D_2}\}_{D_1,D_2 \in F, D_1 \subseteq D_2} >$ and denote by A_F the inductive limit (in the category of sets): $A_F = \varinjlim_{D \in F} M(D, A/\theta_F)$.

For any F-multiplier $f : D \to A/\theta_F$ we shall denote by $\overline{(D, f)}$ the equivalence class of f in A_F.

Remark 12 We recall that if $f_i : D_i \to A/\theta_F$, $i = 1,2$ are F-multipliers, then $\overline{(D_1, f_1)} = \overline{(D_2, f_2)}$ (in A_F) iff there exists $D \in F, D \subseteq D_1 \cap D_2$ such that $f_{1|D} = f_{2|D}$.

Let $f_i : D_i \to A/\theta_F$, (with $D_i \in F, i = 1,2$), F-multipliers. Let us consider the mapping $f_1 \to f_2 : D_1 \cap D_2 \to A/\theta_F$, defined by $(f_1 \to f_2)(x) = f_1(x) \to f_2(x)$, for any $x \in D_1 \cap D_2$, and let

$$\overline{(D_1, f_1)} \to \overline{(D_2, f_2)} = \overline{(D_1 \cap D_2, f_1 \to f_2)}.$$

This definition is correct. Indeed, let $f_i' : D_i' \to A/\theta_F$, with $D_i' \in F, i = 1,2$ such that $\overline{(D_i, f_i)} = \overline{(D_i', f_i')}$, $i = 1,2$. Then there exist $D_1'', D_2'' \in F$ such that $D_1'' \subseteq D_1 \cap D_1', D_2'' \subseteq D_2 \cap D_2'$ and $f_{1|D''1} = f'_{1|D''1}$, $f_{2|D''2} = f'_{2|D''2}$. If we set $D'' = D_1'' \cap D_2'' \subseteq D_1 \cap D_2 \cap D_1' \cap D_2'$, then $D'' \in F$ and clearly $(f_1 \to f_2)_{|D''} = (f_1' \to f_2')_{|D''}$,

hence $\overline{(D_1 \cap D_2, f_1 \to f_2)} = \overline{(D_1' \cap D_2', f_1' \to f'2)}$.

Lemma 30 $f_1 \to f_2 \in M(D_1 \cap D_2, A/\theta_F)$.

Proof. If $x \in D_1 \cap D_2$ and $a \in B(A)$, then

$$(f_1 \to f_2)(a \vee x) = f_1(a \vee x) \to f_2(a \vee x)$$
$$= (a/\theta_F / \vee f_1(x)) \to (a/\theta_F \vee f_2(x))$$
$$\overset{(c_{21})}{=} a/\theta_F \vee (f_1 \to f_2)(x)$$

and $(f_1 \to f_2)(x) = f_1(x) \to f_2(x) \geq f_2(x) \overset{(m_2)}{\geq} x/\theta_F$. □

Corollary 31 $(A_F, \to, \overline{0}, \overline{1})$ is a bounded commutative BCK-algebra (where $\overline{0} = \overline{(A,0)}$ and $1 = \overline{(A,1)}$) (see Corollary 17).

Definition 14 A_F will be called the *localization BCK-algebra of A with respect to the topology* F.

Lemma 32 The mapping $v_F : B(A) \to A_F$ defined by $v_F(a) = \overline{(A, \overline{f_a})}$ for every $a \in B(A)$ is a morphism of BCK-algebras and $v_F(B(A))$ is a regular subset of A_F.

Proof. If $a, b \in B(A)$ then

$$v_F(a) \to v_F(b) = \overline{(A, \overline{f_a})} \to \overline{(A, \overline{f_b})}$$
$$= \overline{(A, \overline{f_a} \to \overline{f_b})} = \overline{(A, \overline{f_{a \to b}})} = v_F(a \to b).$$

To prove that $v_F(A)$ is a regular subset of A_F, let $\overline{(D_i, f_i)} \in A_F$, $D_i \in F, i = 1,2$, such that

$\overline{(A, \overline{f_a})} \vee \overline{(D_1, f_1)} = \overline{(A, \overline{f_a})} \vee \overline{(D_2, f_2)}$ for every $a \in B(A)$.

Then $\overline{(D_1, f_a \vee f_1)} = \overline{(D_2, f_a \vee f_2)} \Leftrightarrow$ there exists $D \in F, D \subseteq D_1 \cap D_2$ such that

$$(f_a \vee f_1)_{|D} = (f_a \vee f_2)_{|D}$$
$$\Leftrightarrow (a \vee x)/\theta_F \vee f_1(x) = (a \vee x)/\theta_F \vee f_2(x),$$

for every $x \in D$ and $a \in B(A)$. If in this last equivalence we choose $a = 0 \in B(A)$, then we obtain that

$$x/\theta_F \vee f_1(x) = x/\theta_F \vee f_2(x)$$
$$\Leftrightarrow f_1(x) = f_2(x) \Leftrightarrow \overline{(D_1, f_1)} = \overline{(D_2, f_2)},$$

hence $v_F(B(A))$ is a regular subset of A_F. □

7. Applications

In that follows we describe the localization BCK-algebra A_F in some special instances.

1) If $D \in T(A)$ and F is the topological system $F_D = \{D' \in T(A) : D \subseteq D'\}$ (see Example 3), then $A_F \subseteq M(D, A/\theta_F)$ and $v_F : B(A) \to A_F$ is defined by $v_F(a) = \overline{(D, f_{a|D})}$ for any $a \in B(A)$. For $x, y \in B(A)$ we have $(x,y) \in \theta_F \Leftrightarrow$ for every $t \in D, t \vee x = t \vee y \Leftrightarrow f_{x|D} = f_{y|D} \Leftrightarrow v_F(x) = v_F(y)$ then there exists an injective morphism of BCK-algebras $\varphi : A/\theta_F \to A_F$, $\varphi(x/\theta_F) = v_F(x)$ such that $\varphi \circ v_F = p_F$.

2) To obtain the maximal BCK-algebra of quotients $Q(A)$ as a localization relative to a topological system F we will develop another theory of F-multipliers (meaning we add new axioms for F-multipliers).

Definition 15 Let F be a topological system on A. A *strong-F-multiplier* is a mapping $f : D \to A/\theta_F$ (where $D \in F$) which verifies the axioms m_1 and m_2 and

(m_3) If $e \in D \cap B(A)$, then $f(e) \in B(A/\theta_F)$;

(m_4) $(x/\theta_F) \vee f(a) = (a/\theta_F) \vee f(x)$, for every $a \in D \cap B(A)$ and $x \in D$.

If $F = \{A\}$, then θ_F is the identity congruence of A so a strong F-multiplier is a strong total multiplier (in sense of Definition 7).

Remark 13 If A is a BCK-algebra, the maps **0**,**1**: $A \to A/\theta_F$ defined by $\mathbf{0}(x) = x/\theta_F$ and $\mathbf{1}(x) = 1/\theta_F$ for every $x \in A$ are strong F-multipliers. If $f_i : D_i \to A/\theta_F$, (with $D_i \in F, i = 1,2$) are strong F-multipliers, the mapping $f_1 \to f_2 : D_1 \cap D_2 \to A/\theta_F$ defined by $(f_1 \to f_2)(x) = f_1(x) \to f_2(x)$, for any $x \in D_1 \cap D_2$ is also a strong-F-multiplier.

Remark 14 Analogous as in the case of F-multipliers if we work with strong-F-multipliers we obtain a BCK-subalgebra of A_F denoted by $s - A_F$ which will be called the *strong localization BCK-algebra of A with respect to the topological system F*.

If $F = T(A) \cap R(A)$, then θ_F is the identity congruence of A and we obtain the definition for strong multipliers on A, so $A_F = \varinjlim_{D \in F} M(D, A)$. In this situation it is easy to see that v_F is injective, so we have:

Proposition 33 In the case $F = I(A) \cap R(A)$, $s - A_F$ is exactly the maximal commutative BCK-algebra of quotients $Q(A)$ of A (see Section 5, Theorem 29).

3. Let S be a $\vee$-closed system of A. We recall (see Proposition 11) that on A we have the congruence θ_S defined by: $(x,y) \in \theta_S$ iff there is $s \in S \cap B(A)$ such that $s \vee x = s \vee y$ and $A[S] = A/\theta_S$ is called the (*commutative*) *BCK-algebra of fractions of A relative to the* $\vee$*-closed system S* (see Remark 5 from Section 3). In this case we have the topological system F_S associated with S, $F_S = \{D \in T(A) : D \cap S \cap B(A) \neq \emptyset\}$.

Lemma 34 $\theta_{F_S} = \theta_S$.

Proof. For $x, y \in A$, if $(x, y) \in \theta_{F_S}$ then there exists $D \in F_S$ such that $s \vee x = s \vee y$ for every $s \in S \cap B(A)$. Since $D \in F_S$, $D \cap S \cap B(A) \neq \emptyset$, so there exists $s_0 \in D \cap S \cap B(A)$; in particular we obtain $s_0 \vee x = s_0 \vee y$, hence $(x, y) \in \theta_S$, that is, $\theta_{F_S} \subseteq \theta_S$.

If $(x, y) \in \theta_S$, then $s_0 \vee x = s_0 \vee y$, for some $s_0 \in S \cap B(A)$. If consider $D = [s_0) = \{a \in A : s_0 \leq a\}$ (the principal deductive system generated by s_0, see Corollary 8, 1)), then $D \in F_S$ (since $s_0 \in D \cap S \cap B(A)$). If $s \in S \cap B(A)$ then $s_0 \leq s \Rightarrow s = s \vee s_0$ hence

$$s \vee x = (s \vee s_0) \vee x = s \vee (s_0 \vee x)$$
$$= s \vee (s_0 \vee y) = (s \vee s_0) \vee y = s \vee y \qquad \square$$
$$\Rightarrow (x, y) \in \theta_{F_S} \Rightarrow \theta_S \subseteq \theta_{F_S} \Rightarrow \theta_{F_S} = \theta_S.$$

Proposition 35 If F_S is the topological system on A associated with a $\vee$-closed subset S of A, then $s - A_{F_S}$ is isomorphic with $B(A[S])$.

Proof. Following Lemma 34, $\theta_{F_S} = \theta_S$, therefore a F_S-multiplier can be considered in this case as a mapping $f : D \to A[S]$ ($D \in F_S$) having for $x \in D$ and $a \in D \cap B(A)$ the properties

$$f(a \vee x) = a/\theta_S \vee f(x)$$
$$= x/\theta_S \vee f(a), x/\theta_S \leq f(x),$$
$$f(a) \in B(A[S]).$$

If $\overline{(D_1, f_1)}, \overline{(D_2, f_2)} \in s - A_{F_S} = \varinjlim_{D \in F_S} M(D, A[S])$, and $\overline{(D_1, f_1)} = \overline{(D_2, f_2)}$ then there exists $D \in F_S$ such that $D \subseteq D_1 \cap D_2$ and $f_{1|D} = f_{2|D}$. Since $D, D_1, D_2 \in F_S$, then

$$D \cap S \cap B(A), D_1 \cap S \cap B(A), D_2 \cap S \cap B(A)$$

are nonempty, hence there exist $s \in D \cap S \cap B(A)$, $s_1 \in D_1 \cap S \cap B(A)$ and $s_2 \in D_2 \cap S \cap B(A)$.

We shall prove that $f_1(s_1) = f_2(s_2)$. Indeed, if consider $t = s \vee s_1 \vee s_2 \in D \cap S \cap B(A)$, then $f_1(t) = s/\theta_S \vee s_2/\theta_S \vee f_1(s_1) = f_1(s_1)$ (since $s/\theta_S = s_2/\theta_S = \mathbf{0}$) and analogously $f_2(t) = f_2(s_2) \Rightarrow f_1(s_1) = f_2(s_2)$. In a similar way we can show that $f_1(t_1) = f_2(t_2)$ for any $t_1, t_2 \in D \cap S \cap B(A)$. In accordance with these considerations we can define the mapping $\alpha : s - A_{F_S} = \varinjlim_{D \in F_S} M(D, A[S]) \to B(A[S])$ by putting $\alpha\overline{(D, f)} = f(s)$, where $s \in D \cap S \cap B(A)$. It is easy to prove that α is a morphism of BCK-algebras. We shall prove that α is injective and surjective. To prove the injectivity of α let $\overline{(D_1, f_1)}, \overline{(D_2, f_2)} \in s - A_{F_S}$ such that $\alpha(\overline{(D_1, f_1)}) = \alpha(\overline{(D_2, f_2)})$. Then for any $s_1 \in D_1 \cap S \cap B(A)$, $s_2 \in D_2 \cap S \cap B(A)$ we have $f_1(s_1) = f_2(s_2)$. For two fixed elements s_1, s_2 with $s_i \in D_i \cap S \cap B(A)$, $i = 1, 2$, we consider the element $s = s_1 \vee s_2 \in (D_1 \cap D_2) \cap S \cap B(A)$. We have $f_1(s) = s_2/\theta_S \vee f_1(s_1) = 0 \vee f_1(s_1) = f_1(s_1)$ and $f_2(s) = s_1/\theta_S \vee f_2(s_2) = 0 \vee f_2(s_2) = f_2(s_2)$, hence $f_1(s) = f_2(s)$. Now let $D_s = [s) \cap D_1 \cap D_2 = \{s' \in D_1 \cap D_2 : s \leq s'\}$. Since $s \in D_s$ we deduce that $D_s \neq \emptyset$. If $a \in A$ and $s' \in D_s$ then $s \leq s' \leq a \vee s' \Rightarrow a \vee s' \in D_s \Rightarrow D_s \in S(A)$. Since $s \in D_s \cap D \Rightarrow D_s \in F_S$. If $s' \in D_s$, then

$$s \vee s' = s' \Rightarrow f_1(s') = f_1(s \vee s')$$
$$= s'/\theta_S \vee f_1(s) = 0 \vee f_1(s) = f_1(s)$$

and analogously,

$$f_2(s') = f_2(s) \Rightarrow f_1(s') = f_2(s')$$
$$\Rightarrow f_{1|Ds} = f_{2|Ds} \Rightarrow \overline{(D_1, f_1)} = \overline{(D_2, f_2)},$$

that is, α is injective. To prove the surjectivity of α, let $a/\theta_S \in B(A[S])$ with $a \in A$.

For one fixed element $s \in S$, we consider $D = [s) = \{x \in A : s \leq x\}$. Clearly $D \in F_S$. We define $f_a : D \to A[S]$ by putting $f_a(x) = (a \vee x)/\theta_S$, for every $x \in D$. Clearly, f_a is a strong F_S-multiplier (clearly (m_3) is verified since if $e \in D \cap B(A)$, then $f_a(e) = (a \vee e)/\theta_S = a/\theta_S \vee e/\theta_S \in B(A[S])$. From

$$(a \vee s) \vee s = a \vee s \Rightarrow (a \vee s)/\theta_S = a/\theta_S$$
$$\Rightarrow f_a(s) = a/\theta_S \Rightarrow \alpha(\overline{(D, f_a)}) = a/\theta_S,$$

that is, α is surjective, hence bijective. $\square$

8. References

[1] D. Piciu and C. Busneag, "The Localization of Commutative (Unbounded) Hilbert Algebras," *Mathematical Reports*, Vol. 12, No. 3, 2010, pp. 285-300.

[2] Y. Imai and K. Iséki, "On Axiom Systems of Propositional Calculi XIV," *Proceedings of the Japan Academy*, Vol. 42, No. 1, 1966, pp. 19-22. doi:10.3792/pja/1195522169

[3] A. N. Prior, "Formal logic, 2nd Ed," Oxford, 1962.

[4] G. Georgescu, "F-Multipliers and the Localization of Distributive Lattices," *Algebra Universalis*, Vol. 21, No. 2-3, 1985, pp. 181-197. doi:10.1007/BF01188055

[5] S. Rudeanu, "Localizations and Fractions in Algebra of Logic," *Journal of Multi-Valued Logic & Soft Computing*, Vol. 16, No. 3-5, 2010, pp. 465-467.

[6] R. Cignoli and A. Torens, "Glivenko Like Theorems in

Natural Expansions of BCK-Logic," *Mathematical Logic Quarterly*, Vol. 50, No. 2, 2004, pp. 111-125. doi:10.1002/malq.200310082

[7] J. Gispert and A. Torrens, "Boolean Representation of Bounded BCK-Algebras," *Soft Computing*, Vol. 12, No. 10, 2008, pp. 941-954. doi:10.1007/s00500-007-0261-0

[8] A. Iorgulescu, "Algebras of Logic as BCK-Algebras," ASE, Bucharest, 2008.

[9] K. Iséki and S. Tanaka, "An Introduction to the Theory of BCK-Algebras," *Mathematica Japonica*, Vol. 23, No. 1, 1978, pp. 1-25.

[10] F. M. G. Olmedo and A. J. R. Salas, "Negation and BCK-Algebras," *Mathematical Logic Quarterly*, Vol. 49, No. 4, 2003, pp. 336-346. doi:10.1002/malq.200310035

[11] J. Schmid, "Multipliers on Distributive Lattices and Rings of Quotients," *Houston Journal of Mathematics*, Vol. 6, No. 3, 1980, pp. 401-425.

[12] J. Schmid, "Distributive Lattices and Rings of Quotients," *Colloquia Mathematica Societatis János Bolyai*, Vol. 33, Szeged, Hungary, 1980.

[13] J. Lambek, "Lectures on Rings and Modules," Blaisdell Publishing Company, New York, 1966.

[14] H. Cornish, "The Multiplier Extension of a Distributive Lattice," *Journal of Algebra*, Vol. 32, No. 2, 1974, pp. 339-355. doi:10.1016/0021-8693(74)90143-4

[15] R. Balbes and Ph. Dwinger, "Distributive Lattices," University of Missouri Press, Columbia, 1974.

[16] N. Popescu, "Abelian Categories with Applications to Rings and Modules," Academic Press, New York, 1973.

[17] B. Strenström, "Platnes and Localization over Monoids," *Mathematische Nachrichten*, Vol. 48, No. 1-6, 1971, pp. 315-334. doi:10.1002/mana.19710480124