Open Journal of Discrete Mathematics, 2018, 8, 65-72
http://www.scirp.org/journal/ojdm
ISSN Online: 2161-7643
ISSN Print: 2161-7635

Cyclically Interval Total Coloring of the One Point Union of Cycles

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How to cite this paper: Su, S.J., Zhao, W.W. and Zhao, Y.Q. (2018) Cyclically Interval Total Coloring of the One Point Union of Cycles. *Open Journal of Discrete Mathematics*, **8**, 65-72.
https://doi.org/10.4236/ojdm.2018.83006

Received: March 26, 2018
Accepted: May 15, 2018
Published: May 18, 2018

Abstract

A total coloring of a graph G with colors $1,2,\cdots,t$ is called a cyclically interval total t-coloring if all colors are used, and the edges incident to each vertex $v\in V(G)$ together with v are colored by $(d_G(v)+1)$ consecutive colors modulo t, where $d_G(v)$ is the degree of the vertex v in G. The one point union $C_n^{(k)}$ of k-copies of cycle C_n is the graph obtained by taking v as a common vertex such that any two distinct cycles C_n' and C_n'' are edge disjoint and do not have any vertex in common except v. In this paper, we study the cyclically interval total colorings of $C_n^{(k)}$, where $n\geq 3$ and $k\geq 2$.

Keywords

Total Coloring, Interval Total Coloring, Cyclically Interval Total Coloring, Cycle, One Point Union of Cycles

1. Introduction

We denote the sets of vertices and edges in a graph G by $V(G)$ and $E(G)$, respectively. For a vertex $x\in V(G)$, we denote the degree of x in G by $d_G(x)$, and we use $\Delta(G)$ to denote the maximum degree of vertices of G.

For an arbitrary finite set A, we denote the number of elements of A by $|A|$. We use $\mathbb{N}$ to denote the set of positive integers. An arbitrary nonempty subset of consecutive integers is called an interval. An interval with the minimum element p and the maximum element q is denoted by $[p,q]$. An interval D is called a h-interval if $|D|=h$.

A total coloring of a graph G is a function mapping $E(G)\cup V(G)$ to $\mathbb{N}$

such that no adjacent vertices, edges, and no incident vertices and edges obtain the same color. The concept of total coloring was introduced by V. Vizing [1] and independently by M. Behzad [2]. The total chromatic number $\chi''(G)$ is the smallest number of colors needed for total coloring of *G*. For a total coloring α of a graph G and for any $v \in V(G)$, let $S[\alpha, v] = \{\alpha(v)\} \cup \{\alpha(e) \mid e \text{ is incident to } v\}$.

An interval total *t*-coloring of a graph G is a total coloring of G with colors $1, 2, \cdots, t$ such that at least one vertex or edge of G is colored by $i, i = 1, 2, \cdots, t$, and for any $x \in V(G)$, the set $S[\alpha, x]$ is a $(d_G(x)+1)$-interval. A graph G is interval total colorable if it has an interval total *t*-coloring for some positive integer *t*. The concept of interval total coloring was first introduced by Petrosyan [3].

Recently, Zhao and Su [4] generalized the concept interval total coloring to the cyclically interval total coloring as follow. A total *t*-coloring α of a graph G is called a cyclically interval total *t*-coloring of *G*, if for any $x \in V(G)$, $S[\alpha, x]$ is a $(d_G(x)+1)$-interval, or $[1, t] \setminus S[\alpha, x]$ is a $(t - d_G(x) - 1)$-interval. A graph G is cyclically interval total colorable if it has a cyclically interval total *t*-coloring for some positive integer *t*.

For any $t \in \mathbb{N}$, we denote by $\mathfrak{F}_t$ the set of graphs for which there exists a cyclically interval total *t*-coloring. Let $\mathfrak{F} = \bigcup_{t \ge 1} \mathfrak{F}_t$. For any graph $G \in \mathfrak{F}$, the minimum and the maximum values of *t* for which G has a cyclically interval total *t*-coloring are denoted by $w_\tau^c(G)$ and $W_\tau^c(G)$, respectively.
It is clear that for any $G \in \mathfrak{F}$, the following inequality is true:

$$\chi''(G) \le w_\tau^c(G) \le W_\tau^c(G) \le |V(G)| + |E(G)|.$$

The one point union $C_n^{(k)}$ of *k*-copies of cycle C_n is the graph obtained by taking v as a common vertex such that any two distinct cycles C_n' and C_n'' are edge disjoint and do not have any vertex in common except *v*. In this paper, we study the cyclically interval total colorability of $C_n^{(k)}$. Let $V(C_n^{(k)}) = \bigcup_{i=1}^{k} V(C_n^i)$ and $V(C_n^i) = \{v_1^i, v_2^i, \cdots, v_n^i\}$, where C_n^i is the *i*-th copy of C_n and $i \in [1, k]$. Without loss of generality, we may assume that the common vertex v of the *k*-copies of cycle C_n is the first vertex in each cycle, *i.e.*, $v = v_1^1 = v_1^2 = \cdots = v_1^k$. For example, the graphs in **Figure 1** are all $C_6^{(3)}$. Note that in the paper we always use the kind of diagram like (b) in **Figure 1** to denote $C_n^{(k)}$.

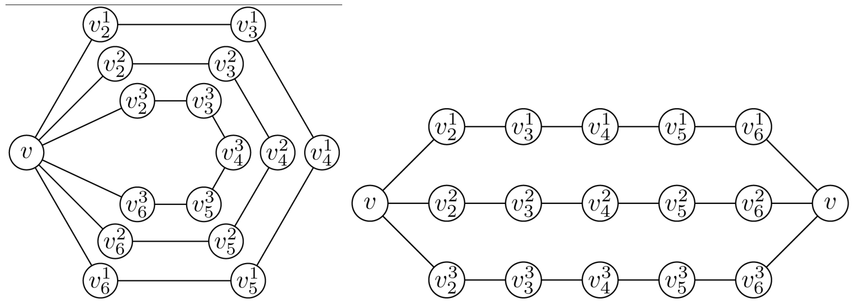
All graphs considered in this paper are finite undirected simple graphs.

2. Main Results

Vaidya and Isaac [5] studied the total coloring of $C_n^{(k)}$ and got the following result.

Theorem 1 (Vaidya and Isaac) *For any integers* $n \ge 3$ *and* $k \ge 2$, $\chi''(C_n^{(k)}) = 2k + 1$.

Now we consider the cyclically interval total colorings of $C_n^{(k)}$, show that $C_n^{(k)} \in \mathfrak{F}$, get the exact values of $w_\tau^c(C_n^{(k)})$, and provide a lower bound of



(a) (b)

Figure 1. (a) $C_6^{(3)}$; (b) Another diagram of $C_6^{(3)}$.

$W_\tau^c\left(C_n^{(k)}\right)$.

Theorem 2 *For any integers* $n \ge 3$ *and* $k \ge 2$, $w_\tau^c\left(C_n^{(k)}\right) = 2k+1$.

Proof. Suppose that $n \ge 3$ and $k \ge 2$. Let $V\left(C_n^{(k)}\right) = \bigcup_{i=1}^k C_n^i$, where C_n^i is the *i*-th copy of C_n. Let $V\left(C_n^i\right) = \left\{v_1^i, v_2^i, \cdots, v_n^i\right\}$, where $i \in [1,k]$. Without loss of generality, we may assume that the common vertex v of the *k*-copies of cycle C_n is the first vertex in each cycle, *i.e.*, $v = v_1^1 = v_1^2 = \cdots = v_1^k$. Now we define a total $(2k+1)$-coloring α of the graph $C_n^{(k)}$ as follows:

Case 1. $n \equiv 0 \pmod 3$.

Let

$$\alpha(v) = 1,$$

$$\alpha\left(v_i^j\right) = \begin{cases} 2j-1, & i \in [2,n], j \in [1,k] \text{ and } i \equiv 1 \pmod 3; \\ 2j, & i \in [2,n], j \in [1,k] \text{ and } i \equiv 0 \pmod 3; \\ 2j+1, & i \in [2,n], j \in [1,k] \text{ and } i \equiv 2 \pmod 3, \end{cases}$$

$$\alpha\left(v_i^j v_{i+1}^j\right) = \begin{cases} 2j-1, & i \in [1,n], j \in [1,k] \text{ and } i \equiv 2 \pmod 3; \\ 2j, & i \in [1,n], j \in [1,k] \text{ and } i \equiv 1 \pmod 3; \\ 2j+1, & i \in [1,n], j \in [1,k] \text{ and } i \equiv 0 \pmod 3, \end{cases}$$

where $v_{n+1}^j = v_1^j = v$ for any $j \in [1,k]$. See **Figure 2** for an example.

By the definition of α, we have

$$S[\alpha, v] = [1, 2k+1];$$

$$S\left[\alpha, v_i^j\right] = [2j-1, 2j+1], i \in [2,n] \text{ and } j \in [1,k].$$

This shows that α is a cyclically interval total $(2k+1)$-coloring of $C_n^{(k)}$.

Case 2. $n \equiv 1 \pmod 3$.

Let

$$\alpha(v) = 1,$$

$$\alpha\left(v_i^j\right) = \begin{cases} 2j-1, & i \in [2,n-3], j \in [1,k] \text{ and } i \equiv 1 \pmod 3; \\ 2j, & i \in [2,n-3], j \in [1,k] \text{ and } i \equiv 0 \pmod 3; \\ 2j+1, & i \in [2,n-3], j \in [1,k] \text{ and } i \equiv 2 \pmod 3; \\ 2j+2, & i \in \{n-2,n\} \text{ and } j \in [1,k]; \\ 2j-1, & i = n-1 \text{ and } j \in [1,k], \end{cases}$$

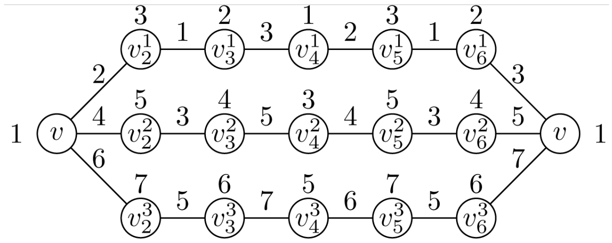


Figure 2. 7-total coloring of $C_6^{(3)}$.

$$\alpha\left(v_i^j v_{i+1}^j\right)=\begin{cases}2j-1, & i\in[1,n-3], j\in[1,k] \text{ and } i\equiv 2(\operatorname{mod}3);\\ 2j, & i\in[1,n-3], j\in[1,k] \text{ and } i\equiv 1(\operatorname{mod}3);\\ 2j+1, & i\in[1,n-3], j\in[1,k] \text{ and } i\equiv 0(\operatorname{mod}3),\\ 2j+1, & i\in\{n-2,n\} \text{ and } j\in[1,k];\\ 2j, & i=n-1 \text{ and } j\in[1,k],\end{cases}$$

where $v_{n+1}^j=v_1^j=v$ for any $j\in[1,k]$. Recolor $v_{n-2}^k, v_{n-1}^k, v_n^k$ and $v_{n-2}^k v_{n-1}^k$ as $\alpha\left(v_{n-2}^k\right)=2k-2, \alpha\left(v_{n-1}^k\right)=2k+1, \alpha\left(v_n^k\right)=2k-1$ and $\alpha\left(v_{n-2}^k v_{n-1}^k\right)=2k-1$. See **Figure 3** for an example.

By the definition of α, we have

$$S[\alpha,v]=[1,2k+1];$$

$$S\left[\alpha,v_i^j\right]=[2j-1,2j+1], i\in[2,n-3]\cup\{n-1\} \text{ and } j\in[1,k];$$

$$S\left[\alpha,v_i^j\right]=[2j,2j+2], i\in\{n-2,n\} \text{ and } j\in[1,k-1];$$

$$S\left[\alpha,v_{n-2}^k\right]=[2k-2,2k];$$

$$S\left[\alpha,v_n^k\right]=[2k-1,2k+1].$$

This shows that α is a cyclically interval total $(2k+1)$-coloring of $C_n^{(k)}$.

Case 3. $n\equiv 2(\operatorname{mod}3)$.

Let

$$\alpha(v)=1,$$

$$\alpha\left(v_i^j\right)=\begin{cases}2j-1, & i\in[2,n-1], j\in[1,k] \text{ and } i\equiv 1(\operatorname{mod}3);\\ 2j, & i\in[2,n-1], j\in[1,k] \text{ and } i\equiv 0(\operatorname{mod}3);\\ 2j+1, & i\in[2,n-1], j\in[1,k] \text{ and } i\equiv 2(\operatorname{mod}3);\\ 2j+2, & i=n \text{ and } j\in[1,k],\end{cases}$$

$$\alpha\left(v_i^j v_{i+1}^j\right)=\begin{cases}2j-1, & i\in[1,n-4]\cup[n-2,n-1], j\in[1,k] \text{ and } i\equiv 2(\operatorname{mod}3);\\ 2j, & i\in[1,n-4]\cup[n-2,n-1], j\in[1,k] \text{ and } i\equiv 1(\operatorname{mod}3);\\ 2j+1, & i\in[1,n-4]\cup[n-2,n-1], j\in[1,k] \text{ and } i\equiv 0(\operatorname{mod}3);\\ 2j+2, & i=n-3 \text{ and } j\in[1,k];\\ 2j+1, & i=n \text{ and } j\in[1,k],\end{cases}$$

where $v_{n+1}^j=v_1^j=v$ for any $j\in[1,k]$. Recolor $v_{n-2}^k, v_{n-1}^k, v_n^k, v_{n-3}^k v_{n-2}^k, v_{n-2}^k v_{n-1}^k$ and $v_{n-1}^k v_n^k$ as $\alpha\left(v_{n-2}^k\right)=2k-2$, $\alpha\left(v_{n-1}^k\right)=2k+1$, $\alpha\left(v_n^k\right)=2k$, $\alpha\left(v_{n-3}^k v_{n-2}^k\right)=2k-1$, $\alpha\left(v_{n-2}^k v_{n-1}^k\right)=2k$ and $\alpha\left(v_{n-1}^k v_n^k\right)=2k-1$. See **Figure 4** for

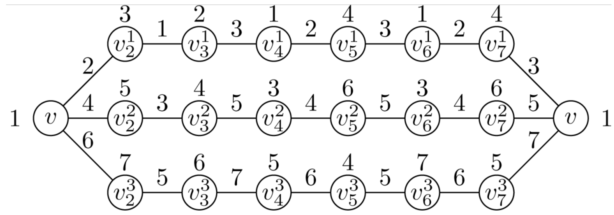


Figure 3. 7-total coloring of $C_7^{(3)}$.

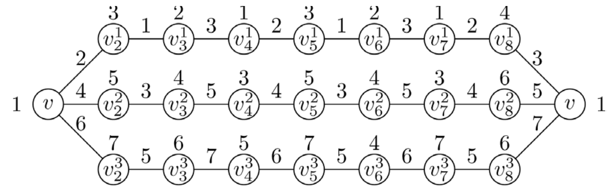


Figure 4. 7-total coloring of $C_8^{(3)}$.

an example.

By the definition of α, we have

$$S[\alpha,v]=[1,2k+1];$$

$$S\left[\alpha,v_i^j\right]=[2j-1,2j+1], i\in[2,n-4]\cup\{n-1\} \text{ and } j\in[1,k];$$

$$S\left[\alpha,v_i^j\right]=[2j,2j+2], i\in\{n-3,n-2,n\} \text{ and } j\in[1,k-1];$$

$$S\left[\alpha,v_{n-3}^k\right]=[2k-1,2k+1];$$

$$S\left[\alpha,v_{n-2}^k\right]=[2k-2,2k];$$

$$S\left[\alpha,v_n^k\right]=[2k-1,2k+1].$$

This shows that α is a cyclically interval total $(2k+1)$-coloring of $C_n^{(k)}$.

Combining Cases 1-3, we have $w_\tau^c\left(C_n^{(k)}\right)\le 2k+1$. On the other hand, by Theorem 1, $w_\tau^c\left(C_n^{(k)}\right)\ge \chi''\left(C_n^{(k)}\right)=2k+1$. So we have $w_\tau^c\left(C_n^{(k)}\right)=2k+1$.

Theorem 3 *For any integers* $n\ge 3$ *and* $k\ge 2$,

$$W_\tau^c\left(C_n^{(k)}\right)\ge\begin{cases}2n+k-1, & k\le 2n-2;\\ (2n-2)\left\lfloor\dfrac{k}{2n-2}\right\rfloor+2n+k-1, & k\ge 2n-1.\end{cases}$$

Proof. Suppose that $n\ge 3$ and $k\ge 2$. We consider the following two cases.

Case 1. $k\le 2n-2$.

Now we define a total $(2n+k-1)$-coloring α of the graph $C_n^{(k)}$ as follows: Let

$$\alpha(v)=1,$$

$$\alpha\left(v_i^j\right)=2i+j-2, i\in[2,n] \text{ and } j\in[1,k];$$

$$\alpha\left(v_i^jv_{i+1}^j\right)=2i+j-1, i\in[1,n] \text{ and } j\in[1,k],$$

where $v_{n+1}^j=v_1^j=v$ for any $j\in[1,k]$. See **Figure 5** for an example.

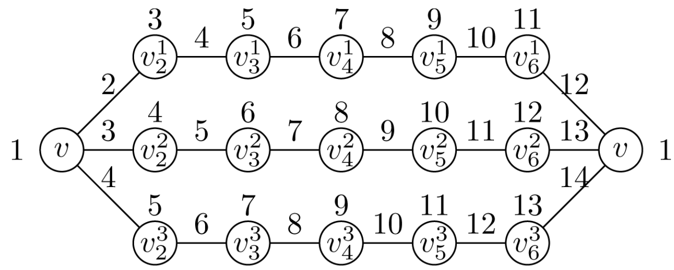


Figure 5. 20-total coloring of $C_4^{(7)}$.

By the definition of α, we have

$$S[\alpha,v]=[1,k+1]\cup[2n,2n+k-1];$$

$$S\left[\alpha,v_i^j\right]=[2i+j-3,2i+j-1],\ i\in[2,n]\text{ and } j\in[1,k].$$

This shows that α is a cyclically interval total $(2n+k-1)$-coloring of $C_n^{(k)}$. So we have $W_\tau^c\left(C_n^{(k)}\right)\ge 2n+k-1$ if $k\le 2n-2$.

Case 2. $k\ge 2n-1$.

Let $s_j=\left\lfloor \frac{j}{2n-2}\right\rfloor$ and $t_j=j-(2n-2)s_j$. Now we define a total $(2n+k-1)$-coloring α of the graph $C_n^{(k)}$ as follows:

Let

$$\alpha(v)=1,$$

$$\begin{aligned}\alpha\left(v_i^j\right)&=(4n-4)s_j+2i+t_j-2\\&=(2n-2)\left\lfloor\frac{j}{2n-2}\right\rfloor+2i+j-2,\ i\in[2,n]\text{ and } j\in[1,k];\end{aligned}$$

$$\begin{aligned}\alpha\left(v_i^jv_{i+1}^j\right)&=(4n-4)s_j+2i+t_j-1\\&=(2n-2)\left\lfloor\frac{j}{2n-2}\right\rfloor+2i+j-1,\ i\in[1,n]\text{ and } j\in[1,k],\end{aligned}$$

where $v_{n+1}^j=v_1^j=v$ for any $j\in[1,k]$. See **Figure 6** for an example.

By the definition of α, we have

$$\begin{aligned}S[\alpha,v]&=[1,(4n-4)s_k+t_k+1]\cup[(4n-4)s_k+2n,(4n-4)s_k+2n+t-1]\\&=\left[1,(2n-2)\left\lfloor\frac{k}{2n-2}\right\rfloor+k+1\right]\\&\quad\cup\left[(4n-4)\left\lfloor\frac{k}{2n-2}\right\rfloor+2n,(2n-2)\left\lfloor\frac{k}{2n-2}\right\rfloor+2n+k-1\right];\end{aligned}$$

$$\begin{aligned}S\left[\alpha,v_i^j\right]&=\left[(4n-4)s_j+2i+t_j-3,(4n-4)s_j+2i+t_j-1\right]\\&=\left[(2n-2)\left\lfloor\frac{j}{2n-2}\right\rfloor+2i+j-3,(2n-2)\left\lfloor\frac{j}{2n-2}\right\rfloor+2i+j-1\right],\\&\quad i\in[2,n]\text{ and } j\in[1,k].\end{aligned}$$

This shows that α is a cyclically interval total $\left((2n-2)\left\lfloor\frac{k}{2n-2}\right\rfloor+2n+k-1\right)$

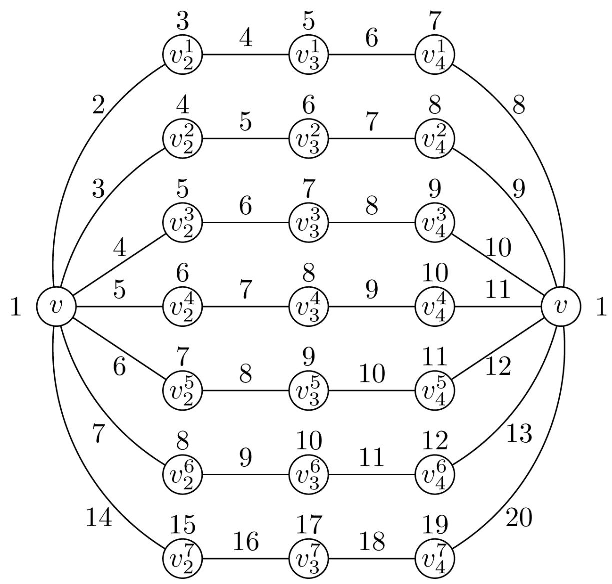


Figure 6. 20-total coloring of $C_4^{(7)}$.

-coloring of $C_n^{(k)}$. So we have $W_\tau^c\left(C_n^{(k)}\right)\ge(2n-2)\left\lfloor\dfrac{k}{2n-2}\right\rfloor+2n+k-1$ for any $k\ge 2n-1$.

3. Generalization

The one point of union $C^{(k)}$ of any k cycles $C_{n_1}^1,C_{n_2}^2,\cdots,C_{n_k}^k$ is the graph obtained by taking v as a common vertex such that any two distinct cycles $C_{n_i}^i$ and $C_{n_j}^j$ are edge disjoint and do not have any vertex in common except v.

By the proof of Theorem 2, the following definitions are well defined.

Definition 4 *A partial* $(i,i+1)$*-total coloring of* C_n $(n\ge 3)$ *is a coloring* $\alpha:V(C_n)\cup E(C_n)\setminus\{v_1\}\to[i-1,i+2]$ *such that* $\alpha(v_1v_2)=i$, $\alpha(v_nv_1)=i+1$ *and* $S[\alpha,v_j]$ *is an interval for each* $j\in[2,n]$. *A partial* $(i,i+1)'$*-total coloring of* C_n $(n\ge 3)$ *is a coloring* $\alpha':V(C_n)\cup E(C_n)\setminus\{v_1,v_n\}\to[i-2,i+1]$ *such that* $\alpha'(v_1v_2)=i$, $\alpha'(v_nv_1)=i+1$ *and* $S[\alpha',v_j]$ *is an interval for each* $j\in[2,n]$.

Now we consider the cyclically interval total colorings of $C^{(k)}$.

Theorem 5 *For any integer* $k\ge 2$, $w_\tau^c\left(C^{(k)}\right)=2k+1$.

Proof. Suppose that graph $C^{(k)}$ is the one point of union of cycles $C_{n_1}^1,C_{n_2}^2,\cdots,C_{n_k}^k$. Let $V\left(C_{n_i}^i\right)=\left\{v_1^i,v_2^i,\cdots,v_{n_i}^i\right\}$, where $i\in[1,k]$. Without loss of generality, we may assume that the common vertex v of the k cycles $C_{n_i}^i$ is the first vertex in each cycle, *i.e.*, $v=v_1^1=v_1^2=\cdots=v_1^k$. Now we define a total $(2k+1)$-coloring α of the graph $C^{(k)}$ as follows: Let $\alpha(v)=1$, $\alpha|_{C_{n_i}^i}$ be a partial $(2i,2i+1)$-total coloring of $C_{n_i}^i$ for each $i\in[1,k-1]$, and $\alpha|_{C_{n_k}^k}$ be a partial $(2k,2k+1)'$-total coloring of $C_{n_k}^k$, respectively. By the definition of α,

$2k+1$ is the largest color used in coloring α, and $S[\alpha,v]=[1,2k+1]$. By Definition 4, $S\left[\alpha,v_{n_i}^{i}\right]$ is an interval for each $i\in[1,k]$. So we have $w_{\tau}^{c}\left(C^{(k)}\right)\le 2k+1$. On the other hand, since $\Delta\left(C^{(k)}\right)=2k$ and $w_{\tau}^{c}\left(C^{(k)}\right)\ge \Delta\left(C^{(k)}\right)+1=2k+1$, then $w_{\tau}^{c}\left(C^{(k)}\right)=2k+1$.

In this section, we consider the one point of union $C^{(k)}$ of k cycles with different length, show that $C^{(k)}\in\mathfrak{F}$, get the exact values of $w_{\tau}^{c}\left(C^{(k)}\right)$, and the further research maybe more interesting.

Acknowledgements

We thank the editor and the referee for their valuable comments. The work was supported in part by the Natural Science Foundation of Hebei Province of China under Grant A2015106045, and in part by the Institute of Applied Mathematics of Shijiazhuang University.

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