Applied Mathematics, 2018, 9, 17-27
http://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

Generating Sets of the Complete Semigroups of Binary Relations Defined by Semilattices of the Class $\Sigma_2(X,4)$

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How to cite this paper: Albayrak, B., Givradze, O. and Partenadze, G. (2018) Generating Sets of the Complete Semigroups of Binary Relations Defined by Semilattices of the Class $\Sigma_2(X,4)$. *Applied Mathematics*, **9**, 17-27.
https://doi.org/10.4236/am.2018.91002

Received: December 11, 2017
Accepted: January 16, 2018
Published: January 19, 2018

Abstract

In this paper, we have studied generating sets of the complete semigroups defined by X-semilattices of the class $\Sigma_2(X,4)$.

Keywords

Semigroup, Semilattice, Binary Relations, Idempotent Elements

1. Introduction

Let X be an arbitrary nonempty set and D be a nonempty set of subsets of the set X. If D is closed under the union, then D is called a *complete X-semilattice of unions*. The union of all elements of the set D is denoted by the symbol $\breve{D}$.

Let B_X be the set of all binary relations on X. It is well known that B_X is a semigroup.

Let f be an arbitrary mapping from X into D. Then we denote a binary relation $\alpha_f = \bigcup_{x\in X}(\{x\}\times f(x))$ for each f. The set of all such binary relations is denoted by $B_X(D)$. It is easy to prove that $B_X(D)$ is a semigroup with respect to the product operation of binary relations. This semigroup $B_X(D)$ is called a complete semigroup of binary relations defined by an X-semilattice of unions D. This structure was comprehensively investigated in Diasamidze [1] and [2]. We assume that $t, y \in X$, $Y \subseteq X$, $\alpha \in B_X$, $T \in D$ and $\varnothing \neq D' \subseteq D$. Then we denote following sets

$$y\alpha = \{x \in X \mid y\alpha x\},\ Y\alpha = \bigcup_{y\in Y} y\alpha,$$

$$V(D,\alpha)=\{Y\alpha \mid Y\in D\},\ X^*=\{Y \mid \varnothing\neq Y\subseteq X\}$$
$$Y_T^{\alpha}=\{y\in X \mid y\alpha=T\},\ V(X^*,\alpha)=\{Y\alpha \mid \varnothing\neq Y\subseteq X\}$$
$$D_t=\{Z'\in D \mid t\in Z'\},\ B_0=\{\alpha\in B_X(D) \mid V(X^*,\alpha)=D\}$$

Let $D=\{\breve{D},Z_1,Z_2,\cdots,Z_{m-1}\}$ be finite X-semilattice of unions and $C(D)=\{P_0,P_1,P_2,\cdots,P_{m-1}\}$ be the family of pairwise nonintersecting subsets of X. If $\varphi=\begin{pmatrix}\breve{D} & Z_1 & \cdots & Z_{m-1}\\ P_0 & P_1 & \cdots & P_{m-1}\end{pmatrix}$ is a mapping from D on $C(D)$, then the equalities $\breve{D}=P_0\cup P_1\cup P_2\cup\cdots\cup P_{m-1}$ and $Z_i=P_0\cup\bigcup_{T\in D\setminus D_Z}\varphi(T)$ are valid. These equalities are called formal.

Let D be a complete X-semilattice of unions $\alpha\in B_X$. Then a representation of a binary relation α of the form $\alpha=\bigcup_{T\in V(X^*,\alpha)}(Y_T^{\alpha}\times T)$ is called quasinormal.

Let $P_0,P_1,P_2,\cdots,P_{m-1}$ be parameters in the formal equalities, $\beta\in B_X(D)$, $\bar{\beta}_2$ be mapping from $X\setminus\breve{D}$ to D. Then $\bar{\beta}=\bigcup_{i=0}^{m-1}\left(P_i\times\bigcup_{t\in P_i}t\beta\right)\cup\bigcup_{t'\in X\setminus\breve{D}}\left(\{t'\}\times\bar{\beta}_2(t')\right)$ is called subquasinormal represantation of β. It can be easily seen that the following statements are true.

a) $\bar{\beta}\in B_X(D)$.

b) $\bigcup_{i=0}^{m-1}\left(P_i\times\bigcup_{t\in P_i}t\beta\right)\subseteq\beta$ and $\beta=\bar{\beta}$ for some $\bar{\beta}_2$.

c) Subquasinormal represantation of β is quasinormal.

d) $\bar{\beta}_1=\begin{pmatrix}P_0 & P_1 & \cdots & P_{m-1}\\ P_0\bar{\beta} & P_1\bar{\beta} & \cdots & P_{m-1}\bar{\beta}\end{pmatrix}$ is mapping from $C(D)$ on $D\cup\{\varnothing\}$.

$\bar{\beta}_1$ and $\bar{\beta}_2$ are respectively called normal and complement mappings for β.

Let $\alpha\in B_X(D)$. If $\alpha\neq\delta\circ\beta$ for all $\delta,\beta\in B_X(D)\setminus\{\alpha\}$ then α is called external element. Every element of the set $B_0=\{\alpha\in B_X(D) \mid V(X^*,\alpha)=D\}$ is an external element of $B_X(D)$.

Theorem 1. [1] *Let X be a finite set and $\alpha,\beta\in B_X(D)$. If $\bar{\beta}$ is subquasinormal representation of β then $\alpha\circ\beta=\alpha\circ\bar{\beta}$.*

Corollary 1. [1] *Let $\tilde{B}'\subseteq\tilde{B}\subseteq B_X(D)$. If $\alpha\neq\delta\circ\bar{\beta}$ for $\alpha\in\tilde{B}'$, $\delta\in\tilde{B}\setminus\{\alpha\}$, $\bar{\beta}\in\tilde{B}\setminus\{\alpha\}$ and subquasinormal representation of $\beta\in\tilde{B}\setminus\{\alpha\}$ then $\alpha\neq\delta\circ\beta$.*

It is known that the set of all external elements is subset of any generating set of $B_X(D)$ in [3].

2. Results

In this work by symbol $\Sigma_{2.2}(X,4)$ we denote all semilattices $D=\{Z_3,Z_2,Z_1,\breve{D}\}$ of the class $\Sigma_2(X,4)$ which the intersection of minimal elements $Z_3\cap Z_2=\varnothing$. This semilattices graphic is given in **Figure 1**. By using formal equalities, we have $Z_3\cap Z_2=P_0=\varnothing$. So, the formal equalities of the semilattice D has a form

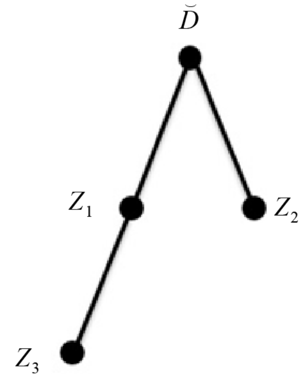

Figure 1. Graphic of semilattice $D = \{Z_3, Z_2, Z_1, D\}$ which the intersection of minimal elements $Z_3 \cap Z_2 = \varnothing$.

$$\begin{aligned} \breve{D} &= P_1 \cup P_2 \cup P_3 \\ Z_1 &= P_2 \cup P_3 \\ Z_2 &= P_1 \cup P_3 \\ Z_3 &= P_2 \end{aligned} \tag{1}$$

Let $\delta, \overline{\beta} \in B_X(D)$. If quasinormal representation of binary relation δ has a form $\delta = (Y_3^\delta \times Z_3) \cup (Y_2^\delta \times Z_2) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \breve{D})$ then

$$\delta \circ \overline{\beta} = (Y_3^\delta \times Z_3\overline{\beta}) \cup (Y_2^\delta \times Z_2\overline{\beta}) \cup (Y_1^\delta \times Z_1\overline{\beta}) \cup (Y_0^\delta \times \breve{D}\overline{\beta})$$

We denote the set

$$B_{32} = \left\{\alpha \in B_X(D) \mid V(X^*, \alpha) = \{Z_3, Z_2, \breve{D}\}\right\}$$

$$B_{21} = \left\{\alpha \in B_X(D) \mid V(X^*, \alpha) = \{Z_2, Z_1, \breve{D}\}\right\}$$

$$B_{31} = \left\{\alpha \in B_X(D) \mid V(X^*, \alpha) = \{Z_3, Z_1\}\right\}$$

$$\tilde{B}_{32} = \left\{\alpha \in B_{32} \mid \alpha = (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2), Y_3^\alpha \cup Y_2^\alpha = X, Y_3^\alpha \cap Y_2^\alpha = \varnothing\right\}$$

$$\tilde{B}_{21} = \left\{\alpha \in B_{21} \mid \alpha = (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1), Y_2^\alpha \cup Y_1^\alpha = X, Y_2^\alpha \cap Y_1^\alpha = \varnothing\right\}$$

It is easy to see that

$$B_0 \cap B_{32} = B_0 \cap B_{21} = B_0 \cap B_{31} = B_{21} \cap B_{32} = B_{31} \cap B_{32} = B_{21} \cap B_{31} = \varnothing.$$

Lemma 2. *Let $D = \{Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_{2.2}(X, 4)$. Then following statements are true for the sets $B_0, B_{32}, \tilde{B}_{32}$.*

a) If $\alpha = (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D})$ for some $Y_3^\alpha, Y_1^\alpha, Y_0^\alpha \notin \varnothing$, then α is product of some elements of the set B_0.

b) If $\beta_0 = (Z_3 \times Z_3) \cup ((X \setminus Z_3) \times Z_2)$, then $(B_0 \circ \beta_0) \cup \tilde{B}_{32} = B_{32}$.

c) If $\sigma_1 = (Z_2 \times Z_2) \cup ((X \setminus Z_2) \times Z_1)$, then $(B_0 \circ \sigma_1) \cup \tilde{B}_{21} = B_{21}$.

d) If $\sigma_1 = (Z_2 \times Z_2) \cup ((X \setminus Z_2) \times Z_1)$, then $B_{32} \circ \sigma_1 = B_{21}$.

e) If $\sigma_0 = (Z_3 \times Z_3) \cup ((X \setminus Z_3) \times Z_1)$, then $B_{32} \circ \sigma_0 = B_{31}$.

f) Every element of the set B_{32} is product of elements of the set $B_0 \cup \tilde{B}_{32}$.

g) Every element of the set B_{21} is product of elements of the set

$B_0 \cup \tilde{B}_{32} \cup \{\sigma_1\}$.

Proof. It will be enough to show only *a*, *b* and *g*. The rest can be similarly seen.

a. Let $\alpha = (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D})$ for some $Y_3^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$, $\delta, \overline{\beta} \in B_0$. Then quasinormal representation of δ has a form

$$\delta = (Y_3^\delta \times Z_3) \cup (Y_2^\delta \times Z_2) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \breve{D})$$

where $Y_3^\delta, Y_1^\delta, Y_0^\delta \notin \{\varnothing\}$. We suppose that

$$\overline{\beta} = (P_2 \times Z_3) \cup (P_1 \times Z_2) \cup (P_3 \times Z_1) \cup \bigcup_{t' \in X \setminus \breve{D}} (\{t'\} \times \overline{\beta}_2(t'))$$

where $\overline{\beta}_1 = \begin{pmatrix} \varnothing & P_1 & P_2 & P_3 \\ \varnothing & Z_2 & Z_3 & Z_1 \end{pmatrix}$ is normal mapping for $\overline{\beta}$ and $\overline{\beta}_2$ is complement mapping of the set $X \setminus \breve{D}$ on the set $\breve{D}$. So, $\overline{\beta} \in B_0$ since $V(X^*, \overline{\beta}) = D$. From the equalities (2.1) and definition of $\overline{\beta}$

$$Z_3\overline{\beta} = P_2\overline{\beta} = Z_3$$
$$Z_2\overline{\beta} = (P_1 \cup P_3)\overline{\beta} = P_1\overline{\beta} \cup P_3\overline{\beta} = Z_2 \cup Z_1 = \breve{D}$$
$$Z_1\overline{\beta} = (P_2 \cup P_3)\overline{\beta} = P_2\overline{\beta} \cup P_3\overline{\beta} = Z_3 \cup Z_1 = Z_1$$
$$\breve{D}\overline{\beta} = (P_1 \cup P_2 \cup P_3)\overline{\beta} = P_1\overline{\beta} \cup P_2\overline{\beta} \cup P_3\overline{\beta} = Z_2 \cup \breve{D} \cup Z_1 = \breve{D}$$

$$\begin{aligned} \delta \circ \overline{\beta} &= (Y_3^\delta \times Z_3\overline{\beta}) \cup (Y_2^\delta \times Z_2\overline{\beta}) \cup (Y_1^\delta \times Z_1\overline{\beta}) \cup (Y_0^\delta \times \breve{D}\overline{\beta}) \\ &= (Y_3^\delta \times \breve{D}) \cup (Y_2^\delta \times \breve{D}) \cup (Y_1^\delta \times \breve{D}) \cup (Y_0^\delta \times \breve{D}) \\ &= (Y_3^\delta \times \breve{D}) \cup (Y_1^\delta \times \breve{D}) \cup ((Y_2^\delta \cup Y_0^\delta) \times \breve{D}) = \alpha. \end{aligned}$$

b. Let $\alpha \in B_0 \circ \beta_0 \cup \tilde{B}_{32}$. Then $\alpha \in B_0 \circ \beta_0$ or $\alpha \in \tilde{B}_{32}$. If $\alpha \in B_0 \circ \beta_0$ then $\alpha = \delta \circ \beta_0$ for some $\delta \in B_0$. In this case we have

$$\delta = (Y_3^\delta \times Z_3) \cup (Y_2^\delta \times Z_2) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \breve{D})$$

where $Y_3^\delta, Y_1^\delta, Y_0^\delta \notin \{\varnothing\}$. Also

$$\begin{aligned} \alpha = \delta \circ \beta_0 &= (Y_3^\delta \times Z_3\beta_0) \cup (Y_2^\delta \times Z_2\beta_0) \cup (Y_1^\delta \times Z_1\beta_0) \cup (Y_0^\delta \times \breve{D}\beta_0) \\ &= (Y_3^\delta \times Z_3) \cup (Y_2^\delta \times Z_2) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \breve{D}) \\ &= (Y_3^\delta \times Z_3) \cup (Y_2^\delta \times Z_2) \cup ((Y_1^\delta \cup Y_0^\delta) \times \breve{D}) \in B_{32} \setminus \tilde{B}_{32} \end{aligned}$$

is satisfied. So, we have $(B_0 \circ \beta_0) \cup \tilde{B}_{32} \subseteq B_{32}$. On the other hand, if $\alpha \in \tilde{B}_{32} \subseteq B_{32}$ then $(B_0 \circ \beta_0) \cup \tilde{B}_{32} \subseteq B_{32}$ is satisfied. Conversely, if $\alpha \in B_{32}$ then quasinormal representation of α has a form

$$\alpha = (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2) \cup (Y_0^\alpha \times \breve{D})$$

where $Y_3^\alpha, Y_2^\alpha, Y_0^\alpha \notin \{\varnothing\}$ or $Y_3^\alpha, Y_2^\alpha \notin \{\varnothing\}$ and $Y_0^\alpha = \varnothing$. We suppose that $Y_3^\alpha, Y_2^\alpha \notin \{\varnothing\}$. In this case, we have

$$\begin{aligned} \delta \circ \beta_0 &= (Y_3^\delta \times Z_3\beta_0) \cup (Y_2^\delta \times Z_2\beta_0) \cup (Y_0^\delta \times Z_1\beta_0) \\ &= (Y_3^\delta \times Z_3) \cup (Y_2^\delta \times Z_2)(Y_0^\delta \times \breve{D}) = \alpha \end{aligned}$$

for $\delta = (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2) \cup (Y_0^\alpha \times Z_1) \in B_0$. So, we have $B_{32} \subseteq (B_0 \circ \beta_0) \cup \tilde{B}_{32}$. Now suppose that $Y_3^\alpha, Y_2^\alpha \notin \{\varnothing\}$ and $Y_0^\alpha = \varnothing$. In this case, we have

$\alpha \in \tilde{B}_{32} \subseteq (B_0 \circ \beta_0) \cup \tilde{B}_{32}$. So, $(B_0 \circ \beta_0) \cup \tilde{B}_{32} = B_{32}$.

g. From the statement *c*, we have that $(B_0 \circ \beta_0) \cup \tilde{B}_{32} = B_{32}$ where $\beta_0 \in \tilde{B}_{32}$ by definition of β_0. Thus, every element of the set B_{32} is product of elements of the set $B_0 \cup \tilde{B}_{32}$.

Lemma 3. *Let* $D = \{Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_{2.2}(X,4)$. *If* $|X \setminus \breve{D}| \geq 1$ *then the following statements are true.*

a) If $\alpha = X \times \breve{D}$ then α is product of elements of the set B_0.

b) If $\alpha = X \times Z_1$ then α is product of elements of the set B_0.

c) If $\alpha = (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1)$ for some $Y_3^\alpha, Y_1^\alpha \notin \varnothing$, then α is product of elements of the B_0.

d) If $\alpha = (Y_3^\alpha \times Z_3) \cup (Y_0^\alpha \times \breve{D})$ for some $Y_3^\alpha, Y_0^\alpha \notin \varnothing$, then α is product of elements of the B_0.

e) If $\alpha = (Y_2^\alpha \times Z_2) \cup (Y_0^\alpha \times \breve{D})$ for some $Y_2^\alpha, Y_0^\alpha \notin \varnothing$, then α is product of elements of the B_0.

f) If $\alpha = (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D})$ for some $Y_1^\alpha, Y_0^\alpha \notin \varnothing$, then α is product of elements of the B_0.

Proof. c. Let quasinormal representation of α has a form $\alpha = (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1)$ where $Y_3^\delta, Y_1^\delta \notin \{\varnothing\}$. By definition of the semilattice *D*, $|X| \geq 3$. We suppose that $|Y_3^\alpha| \geq 1$ and $|Y_1^\alpha| \geq 2$. In this case, we suppose that

$$\overline{\beta} = (P_2 \times Z_3) \cup ((P_1 \cup P_3) \times Z_1) \cup \bigcup_{t' \in X \setminus \breve{D}} (\{t'\} \times \overline{\beta}_2(t'))$$

where $\overline{\beta}_1 = \begin{pmatrix} \varnothing & P_1 & P_2 & P_3 \\ \varnothing & Z_1 & Z_3 & Z_1 \end{pmatrix}$ is normal mapping for $\overline{\beta}$ and $\overline{\beta}_2$ is complement mapping of the set $X \times \breve{D}$ on the set $\tilde{D} \setminus \{Z_3, Z_1\} = \{Z_2\}$ (by suppose $|X \setminus \breve{D}| \geq 1$). So, $\overline{\beta} \in B_0$ since $V(X^*, \overline{\beta}) = D$. Also, $Y_3^\delta = Y_3^\alpha$ and $Y_2^\delta \cup Y_1^\delta \cup Y_0^\delta = Y_1^\delta$ since $|Y_3^\delta| \geq 1, |Y_2^\delta| \geq 1, |Y_1^\delta| \geq 1, |Y_0^\delta| \geq 0$. From the equalities (2.1) and definition of $\overline{\beta}$ we obtain that

$$\begin{aligned}
&Z_3\overline{\beta} = P_2\overline{\beta} = Z_3\\
&Z_2\overline{\beta} = (P_1 \cup P_3)\overline{\beta} = P_1\overline{\beta} \cup P_3\overline{\beta} = Z_1 \cup Z_1 = Z_1\\
&Z_1\overline{\beta} = (P_2 \cup P_3)\overline{\beta} = P_2\overline{\beta} \cup P_3\overline{\beta} = Z_3 \cup Z_1 = Z_1\\
&\breve{D}\overline{\beta} = (P_1 \cup P_2 \cup P_3)\overline{\beta} = P_1\overline{\beta} \cup P_2\overline{\beta} \cup P_3\overline{\beta} = Z_1 \cup Z_3 \cup Z_1 = Z_1\\
&\delta \circ \overline{\beta} = (Y_3^\delta \times Z_3\overline{\beta}) \cup (Y_2^\delta \times Z_2\overline{\beta}) \cup (Y_1^\delta \times Z_1\overline{\beta}) \cup (Y_0^\delta \times \breve{D}\overline{\beta})\\
&\quad = (Y_3^\delta \times Z_3) \cup (Y_2^\delta \times Z_1) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times Z_1)\\
&\quad = (Y_3^\delta \times Z_3) \cup ((Y_2^\delta \cup Y_1^\delta \cup Y_0^\delta) \times Z_1) = \alpha
\end{aligned}$$

Now, we suppose that $|Y_3^\alpha| \geq 2$ and $|Y_1^\alpha| \geq 1$. In this case, we suppose that

$$\overline{\beta} = ((P_2 \cup P_3) \times Z_3) \cup (P_1 \times Z_1) \cup \bigcup_{t' \in X \setminus \breve{D}} (\{t'\} \times \overline{\beta}_2(t'))$$

where $\overline{\beta}_1 = \begin{pmatrix} \varnothing & P_1 & P_2 & P_3 \\ \varnothing & Z_1 & Z_3 & Z_3 \end{pmatrix}$ is normal mapping for $\overline{\beta}$ and $\overline{\beta}_2$ is complement mapping of the set $X \times \breve{D}$ on the set $\tilde{D} \setminus \{Z_3, Z_1\} = \{Z_2\}$ (by suppose

$\left|X \setminus \breve{D}\right| \geq 1$). So, $\overline{\beta} \in B_0$ since $V\left(X^*, \overline{\beta}\right) = D$. Also, $Y_3^\delta \cup Y_1^\delta = Y_3^\alpha$ and $Y_2^\delta \cup Y_0^\delta = Y_1^\alpha$ since $\left|Y_3^\delta\right| \geq 1, \left|Y_2^\delta\right| \geq 1, \left|Y_1^\delta\right| \geq 1, \left|Y_0^\delta\right| \geq 0$. From the equalities (2.1) and definition of $\overline{\beta}$ we obtain that

$$
\begin{aligned}
&Z_3\overline{\beta} = P_2\overline{\beta} = Z_3 \\
&Z_2\overline{\beta} = \left(P_1 \cup P_3\right)\overline{\beta} = P_1\overline{\beta} \cup P_3\overline{\beta} = Z_1 \cup Z_3 = Z_1 \\
&Z_1\overline{\beta} = \left(P_2 \cup P_3\right)\overline{\beta} = P_2\overline{\beta} \cup P_3\overline{\beta} = Z_3 \cup Z_3 = Z_3 \\
&\breve{D}\overline{\beta} = \left(P_1 \cup P_2 \cup P_3\right)\overline{\beta} = P_1\overline{\beta} \cup P_2\overline{\beta} \cup P_3\overline{\beta} = Z_1 \cup Z_3 \cup Z_3 = Z_1 \\
&\delta \circ \overline{\beta} = \left(Y_3^\delta \times Z_3\overline{\beta}\right) \cup \left(Y_2^\delta \times Z_2\overline{\beta}\right) \cup \left(Y_1^\delta \times Z_1\overline{\beta}\right) \cup \left(Y_0^\delta \times \breve{D}\overline{\beta}\right) \\
&\qquad = \left(Y_3^\delta \times Z_3\right) \cup \left(Y_2^\delta \times Z_1\right) \cup \left(Y_1^\delta \times Z_3\right) \cup \left(Y_0^\delta \times Z_1\right) \\
&\qquad = \left(\left(Y_3^\delta \cup Y_1^\delta\right) \times Z_3\right) \cup \left(\left(Y_2^\delta \cup Y_0^\delta\right) \times Z_1\right) = \alpha
\end{aligned}
$$

Lemma 4. *Let* $D = \{Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_{2.2}(X, 4)$, $\sigma_0 = (Z_3 \times Z_3) \cup ((X \setminus Z_3) \times Z_1)$ *and* $\sigma_1 = (Z_2 \times Z_2) \cup ((X \setminus Z_2) \times Z_1)$. *If* $X = \breve{D}$ *then the following statements are true*

a) If $\alpha = \left(Y_3^\alpha \times Z_3\right) \cup \left(Y_0^\alpha \times \breve{D}\right)$ for some $Y_3^\alpha, Y_0^\alpha \notin \{\varnothing\}$, then α is product of elements of the $B_0 \cup B_{32}$.

b) If $\alpha = \left(Y_2^\alpha \times Z_2\right) \cup \left(Y_0^\alpha \times \breve{D}\right)$ for some $Y_2^\alpha, Y_0^\alpha \notin \{\varnothing\}$, then α is product of elements of the $B_{32} \cup \{\sigma_1\}$.

c) If $\alpha = \left(Y_1^\alpha \times Z_1\right) \cup \left(Y_0^\alpha \times \breve{D}\right)$ for some $Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$, then α is product of elements of the $B_{32} \cup \{\sigma_0, \sigma_1\}$.

Proof. First, remark that $Z_3\sigma_0 = Z_3$, $Z_2\sigma_0 = \breve{D}\sigma_0 = Z_1$, $Z_3\sigma_1 = Z_1$, $Z_2\sigma_1 = Z_2$, $\breve{D}\sigma_1 = \breve{D}$.

a. Let $\alpha = \left(Y_3^\alpha \times Z_3\right) \cup \left(Y_0^\alpha \times \breve{D}\right)$ for some $Y_3^\alpha, Y_0^\alpha \notin \varnothing$. In this case, we suppose that

$$\delta = \left(Y_3^\delta \times Z_3\right) \cup \left(Y_2^\delta \times Z_2\right) \cup \left(Y_0^\delta \times \breve{D}\right)$$

and

$$\beta_1 = (Z_3 \times Z_3) \cup ((Z_1 \setminus Z_3) \times Z_1) \cup ((X \setminus Z_1) \times \breve{D})$$

where $Y_3^\delta, Y_2^\delta \notin \{\varnothing\}$. It is easy to see that $\delta \in B_{32}$ and β_1 is generating by elements of the B_0 by statement b of Lemma 2. Also, $Y_3^\delta = Y_3^\alpha$ and $Y_2^\delta \cup Y_0^\delta = Y_0^\alpha$ since $Z_3\overline{\beta} = Z_3$, $Z_2\overline{\beta} = \breve{D}\overline{\beta} = \breve{D}$ and $\left|Y_3^\delta\right| \geq 1, \left|Y_2^\delta\right| \geq 1, \left|Y_0^\delta\right| \geq 0$. So, α is product of elements of the $B_0 \cup B_{32}$. □

Lemma 5. *Let*

$$D = \{Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_{2.2}(X, 4)$$

and

$$\sigma_1 = (Z_2 \times Z_2) \cup ((X \setminus Z_2) \times Z_1).$$

If $\left|X \setminus \breve{D}\right| \geq 1$ *then* $S_1 = B_0 \cup \tilde{B}_{32} \cup \{\sigma_1\}$ *is an irreducible generating set for the semigroup* $B_X(D)$.

Proof. First, we must prove that every element of $B_X(D)$ is product of elements of S_1. Let $\alpha \in B_X(D)$ and

$$\alpha = \left(Y_3^\alpha \times Z_3\right) \cup \left(Y_2^\alpha \times Z_2\right) \cup \left(Y_1^\alpha \times Z_1\right) \cup \left(Y_0^\alpha \times \breve{D}\right)$$

where $Y_3^\alpha \cup Y_2^\alpha \cup Y_1^\alpha \cup Y_0^\alpha = X$ and $Y_3^\alpha \cap Y_2^\alpha = \varnothing$, $(0 \le i \ne j \le 3)$. We suppose that $\left|V\left(X^*, \alpha\right)\right| = 1$. Then we have $V\left(X^*, \alpha\right) \in \left\{\{Z_3\}, \{Z_2\}, \{Z_1\}, \{\breve{D}\}\right\}$. If $V\left(X^*, \alpha\right) \in \left\{\{Z_3\}, \{Z_2\}, \{Z_1\}\right\}$ then $\alpha = X \times Z_3$ or $\alpha = X \times Z_2$ or $\alpha = X \times Z_1$. Quasinormal representations of δ, β_1, β_2 and β_3 has form

$$\delta = \left(Y_3^\delta \times Z_3\right) \cup \left(Y_2^\delta \times Z_2\right) \cup \left(Y_1^\delta \times Z_1\right) \cup \left(Y_0^\delta \times \breve{D}\right)$$

$$\beta_1 = \left(\breve{D} \times Z_3\right) \cup \left(\left(X \setminus \breve{D}\right) \times Z_2\right)$$

$$\beta_2 = \left(\breve{D} \times Z_2\right) \cup \left(\left(X \setminus \breve{D}\right) \times Z_1\right)$$

$$\beta_3 = \left(\breve{D} \times Z_1\right) \cup \left(\left(X \setminus \breve{D}\right) \times Z_2\right)$$

where $Y_3^\delta, Y_2^\delta, Y_1^\delta \notin \{\varnothing\}$. So, $\delta \in B_0$, $\beta_1 \in \tilde{B}_{32}$ and $\beta_2, \beta_3 \in B_{21}$ since $\left|X \setminus \breve{D}\right| \ge 1$. From the definition of δ, β_1, β_2 and β_3 we obtain that

$$\begin{aligned}
\delta \circ \beta_1 &= \left(Y_3^\delta \times Z_3\beta_1\right) \cup \left(Y_2^\delta \times Z_2\beta_1\right) \cup \left(Y_1^\delta \times Z_1\beta_1\right) \cup \left(Y_0^\delta \times \breve{D}\beta_1\right) \\
&= \left(Y_3^\delta \times Z_3\right) \cup \left(Y_2^\delta \times Z_3\right) \cup \left(Y_1^\delta \times Z_3\right) \cup \left(Y_0^\delta \times Z_3\right) \\
&= \left(Y_3^\delta \cup Y_2^\delta \cup Y_1^\delta \cup Y_0^\delta\right) \times Z_3 = X \times Z_3
\end{aligned}$$

$$\begin{aligned}
\delta \circ \beta_2 &= \left(Y_3^\delta \times Z_3\beta_2\right) \cup \left(Y_2^\delta \times Z_2\beta_2\right) \cup \left(Y_1^\delta \times Z_1\beta_2\right) \cup \left(Y_0^\delta \times \breve{D}\beta_2\right) \\
&= \left(Y_3^\delta \times Z_2\right) \cup \left(Y_2^\delta \times Z_2\right) \cup \left(Y_1^\delta \times Z_2\right) \cup \left(Y_0^\delta \times Z_2\right) \\
&= \left(Y_3^\delta \cup Y_2^\delta \cup Y_1^\delta \cup Y_0^\delta\right) \times Z_2 = X \times Z_2
\end{aligned}$$

$$\begin{aligned}
\delta \circ \beta_3 &= \left(Y_3^\delta \times Z_3\beta_3\right) \cup \left(Y_2^\delta \times Z_2\beta_3\right) \cup \left(Y_1^\delta \times Z_1\beta_3\right) \cup \left(Y_0^\delta \times \breve{D}\beta_3\right) \\
&= \left(Y_3^\delta \times Z_1\right) \cup \left(Y_2^\delta \times Z_1\right) \cup \left(Y_1^\delta \times Z_1\right) \cup \left(Y_0^\delta \times Z_1\right) \\
&= \left(Y_3^\delta \cup Y_2^\delta \cup Y_1^\delta \cup Y_0^\delta\right) \times Z_1 = X \times Z_1
\end{aligned}$$

That means, $X \times Z_1, X \times Z_2$ and $X \times Z_3$ are generated by $B_0 \cup \tilde{B}_{32}$, $B_0 \cup B_{21}$ and $B_0 \cup B_{21}$ respectively. By using statement g and h of Lemma 3, we have $X \times Z_1, X \times Z_2$ and $X \times Z_3$ are generated by $B_0 \cup \tilde{B}_{32} \cup \{\sigma_1\}$. On the other hand, if $V\left(X^*, \alpha\right) = \{\breve{D}\}$ then $\alpha = X \times \breve{D}$ By using statement a of Lemma 3, we have α is product of some elemets of B_0.

So, S_1 is generating set for the semigroup $B_X(D)$. Now, we must prove that $S_1 = B_0 \cup \tilde{B}_{32} \cup \{\sigma_1\}$ is irreducible. Let $\alpha \in S_1$.

If $\alpha \in B_0$ then $\alpha \ne \sigma \circ \tau$ for all $\sigma, \tau \in B_X(D) \setminus \{\alpha\}$ from Lemma 2. So, $\alpha \ne \sigma \circ \tau$ for all $\sigma, \tau \in S_1 \setminus \{\alpha\}$. That means, $\alpha \notin B_0$.

If $\alpha \in \tilde{B}_{32}$ then the quasinormal representation of α has form $\alpha = \left(Y_3^\alpha \times Z_3\right) \cup \left(Y_2^\alpha \times Z_2\right)$ for some $Y_3^\alpha, Y_2^\alpha \notin \varnothing$. Let $\alpha = \delta \circ \beta$ for some $\delta, \beta \in S_1 \setminus \{\alpha\}$.

We suppose that $\delta \in B_0 \setminus \{\alpha\}$ and $\beta \in S_1 \setminus \{\alpha\}$. By definition of B_0, quasinormal representation of δ has form

$$\delta = \left(Y_3^\delta \times Z_3\right) \cup \left(Y_2^\delta \times Z_2\right) \cup \left(Y_1^\delta \times Z_1\right) \cup \left(Y_0^\delta \times \breve{D}\right)$$

where $Y_3^\delta, Y_2^\delta, Y_1^\delta \notin \{\varnothing\}$. By using $Z_3 \subset Z_1 \subset \breve{D}$ and $Z_2 \subset \breve{D}$ we have $Z_3\beta$ and $Z_2\beta$ are minimal elements of the semilattice $\{Z_3\beta, Z_2\beta, Z_1\beta, \breve{D}\beta\}$. Also, we have

$$\begin{aligned}\left(Y_3^\alpha \times Z_3\right) \cup \left(Y_2^\alpha \times Z_2\right) &= \alpha = \delta \circ \beta \\ &= \left(Y_3^\delta \times Z_3\beta\right) \cup \left(Y_2^\delta \times Z_2\beta\right) \cup \left(Y_1^\delta \times Z_1\beta\right) \cup \left(Y_0^\delta \times \breve{D}\beta\right)\end{aligned}$$

Since Z_3 and Z_2 are minimal elements of the semilattice $\{Z_3, Z_2, \breve{D}\}$, this equality is possible only if $Z_3 = Z_3\beta$, $Z_2 = Z_2\beta$ or $Z_3 = Z_2\beta$, $Z_2 = Z_3\beta$. By using formal equalities and $P_3\beta, P_2\beta, P_1\beta \in D$, we obtain

$$\begin{aligned}&Z_3 = Z_3\beta = P_2\beta \quad \text{and} \quad Z_2 = Z_2\beta = P_1\beta = P_3\beta \\ &Z_2 = Z_3\beta = P_2\beta \quad \text{and} \quad Z_3 = Z_2\beta = P_1\beta = P_3\beta\end{aligned}$$

respectively. Let $Z_3 = P_2\beta$ and $Z_2 = P_1\beta = P_3\beta$. If $\overline{\beta}$ is sub-quasinormal representation of β then $\delta \circ \beta = \delta \circ \overline{\beta}$ and

$$\overline{\beta} = \left(\left(P_1 \cup P_3\right) \times Z_2\right) \cup \left(P_2 \times Z_3\right) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times \overline{\beta}_2(t')\right)$$

where $\overline{\beta}_1 = \begin{pmatrix} \varnothing & P_1 & P_2 & P_3 \\ \varnothing & Z_2 & Z_3 & Z_2 \end{pmatrix}$ is normal mapping for $\overline{\beta}$ and $\overline{\beta}_2$ is complement mapping of the set $X \times \breve{D}$ on the set $\tilde{D} = \{Z_3, Z_2, Z_1\}$. From formal equalities, we obtain

$$\overline{\beta} = \left(Z_2 \times Z_2\right) \cup \left(Z_3 \times Z_3\right) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times \overline{\beta}_2(t')\right) \in S_1 \setminus \{\alpha\}$$

and by using $Z_1 \cap Z_2 \neq \varnothing, Z_3 \cup Z_2 = D$ and $\left|Y_1^\delta \cup Y_0^\delta\right| \geq 1$, we have

$$\begin{aligned}\delta \circ \overline{\beta} &= \left(Y_3^\delta \times Z_3\overline{\beta}\right) \cup \left(Y_2^\delta \times Z_2\overline{\beta}\right) \cup \left(Y_1^\delta \times Z_1\overline{\beta}\right) \cup \left(Y_0^\delta \times \breve{D}\overline{\beta}\right) \\ &= \left(Y_3^\delta \times Z_3\right) \cup \left(Y_2^\delta \times Z_2\right) \cup \left(Y_1^\delta \times \breve{D}\right) \cup \left(Y_0^\delta \times \breve{D}\right) \\ &= \left(Y_3^\delta \times Z_3\right) \cup \left(Y_2^\delta \times Z_2\right) \cup \left(\left(Y_1^\delta \cup Y_0^\delta\right) \times \breve{D}\right) \neq \alpha\end{aligned}$$

This contradicts with $\alpha = \delta \circ \beta$. So, $\delta \notin B_0 \setminus \{\alpha\}$.

Now, we suppose that $\delta \in \tilde{B}_{32} \setminus \{\alpha\}$ and $\beta \in S_1 \setminus \{\alpha\}$. Similar operations are applied as above, we obtain $\delta \notin \tilde{B}_{32} \setminus \{\alpha\}$.

Now, we suppose that $\delta = \sigma_1$ and $\beta \in S_1 \setminus \{\alpha\}$. Similar operations are applied as above, we obtain $\delta \neq \sigma_1$.

That means $\alpha \neq \delta \circ \beta$ for any $\alpha \in \tilde{B}_{32}$ and $\delta, \beta \in S_1 \setminus \{\alpha\}$.

If $\alpha = \sigma_1$, then by the definition of σ_1, quasinormal representation of α has a form $\alpha = \left(Z_2 \times Z_2\right) \cup \left(\left(X \setminus Z_2\right) \times Z_1\right)$. Let $\alpha = \delta \circ \beta$ for some $\delta, \beta \in S_1 \setminus \{\sigma_1\}$.

We suppose that $\delta \in B_0 \setminus \{\sigma_1\}$ and $\beta \in S_1 \setminus \{\sigma_1\}$. By definition of B_0, quasinormal representation of δ has form

$$\delta = \left(Y_3^\delta \times Z_3\right) \cup \left(Y_2^\delta \times Z_2\right) \cup \left(Y_1^\delta \times Z_1\right) \cup \left(Y_0^\delta \times \breve{D}\right)$$

where $Y_3^\delta, Y_2^\delta, Y_1^\delta \notin \{\varnothing\}$. By using $Z_3 \subset Z_1 \subset \breve{D}$ and $Z_2 \subset \breve{D}$ we have $Z_3\beta$ and $Z_2\beta$ are minimal elements of the semilattice $\{Z_3\beta, Z_2\beta, Z_1\beta, \breve{D}\beta\}$. Also, we have

$$\begin{aligned}(Z_2\times Z_2)\cup((X\setminus Z_2)\times Z_1)&=\alpha=\delta\circ\beta\\ &=(Y_3^\delta\times Z_3\beta)\cup(Y_2^\delta\times Z_2\beta)\cup(Y_1^\delta\times Z_1\beta)\cup(Y_0^\delta\times\breve{D}\beta)\end{aligned}$$

From Z_2 and Z_1 are minimal elements of the semilattice $\{Z_2,Z_1,\breve{D}\}$, this equality is possible only if $Z_2=Z_3\beta$, $Z_1=Z_2\beta$ or $Z_2=Z_2\beta$, $Z_1=Z_3\beta$. By using formal equalities, we obtain

$$Z_2=Z_3\beta=P_2\beta \quad\text{and}\quad Z_1=Z_2\beta=P_1\beta\cup P_3\beta$$
$$Z_1=Z_3\beta=P_2\beta \quad\text{and}\quad Z_2=Z_2\beta=P_1\beta=P_3\beta$$

respectively. Let $Z_2=P_2\beta$ and $Z_1=P_1\beta\cup P_3\beta$ where $P_1\beta,P_3\beta\in\{Z_3,Z_1\}$. Then subquasinormal representation of β has one of the form

$$\bar{\beta}^1=(P_1\times Z_3)\cup(P_2\times Z_2)\cup(P_3\times Z_1)\cup\bigcup_{t'\in X\setminus\breve{D}}(\{t'\}\times\bar{\beta}_2(t'))$$

$$\bar{\beta}^2=(P_3\times Z_3)\cup(P_2\times Z_2)\cup(P_1\times Z_1)\cup\bigcup_{t'\in X\setminus\breve{D}}(\{t'\}\times\bar{\beta}_2(t'))$$

$$\bar{\beta}^3=(P_2\times Z_2)\cup((P_1\cup P_3)\times Z_1)\cup\bigcup_{t'\in X\setminus\breve{D}}(\{t'\}\times\bar{\beta}_2(t'))$$

where

$$\bar{\beta}_1^1=\begin{pmatrix}\varnothing & P_1 & P_2 & P_3\\ \varnothing & Z_3 & Z_2 & Z_1\end{pmatrix},\ \bar{\beta}_1^2=\begin{pmatrix}\varnothing & P_1 & P_2 & P_3\\ \varnothing & Z_1 & Z_2 & Z_3\end{pmatrix},\ \bar{\beta}_1^3=\begin{pmatrix}\varnothing & P_1 & P_2 & P_3\\ \varnothing & Z_1 & Z_2 & Z_1\end{pmatrix}$$

are normal mapping for $\bar{\beta}$, $\bar{\beta}_2$ is complement mapping of the set $X\times\breve{D}$ on the set $\tilde{D}=\{Z_3,Z_2,Z_1\}$ and $\delta\circ\beta=\delta\circ\bar{\beta}_i$. From formal equalities, we obtain

$$\bar{\beta}^1=((Z_2\setminus Z_1)\times Z_3)\cup((Z_1\setminus Z_2)\times Z_2)\cup((Z_2\setminus Z_1)\times Z_1)\cup\bigcup_{t'\in X\setminus\breve{D}}(\{t'\}\times\bar{\beta}_2(t'))$$

$$\bar{\beta}^2=((Z_2\cap Z_1)\times Z_3)\cup((Z_1\setminus Z_2)\times Z_2)\cup((Z_2\setminus Z_1)\times Z_1)\cup\bigcup_{t'\in X\setminus\breve{D}}(\{t'\}\times\bar{\beta}_2(t'))$$

$$\bar{\beta}^3=((Z_1\setminus Z_2)\times Z_2)\cup(Z_2\times Z_1)\cup\bigcup_{t'\in X\setminus\breve{D}}(\{t'\}\times\bar{\beta}_2(t'))$$

and by using $|Y_1^\delta\cup Y_0^\delta|\geq 1$, we have

$$\begin{aligned}\delta\circ\bar{\beta}^1&=\delta\circ\bar{\beta}^2=\delta\circ\bar{\beta}^3\\ &=(Y_3^\delta\times Z_3\bar{\beta}^1)\cup(Y_2^\delta\times Z_2\bar{\beta}^1)\cup(Y_1^\delta\times Z_1\bar{\beta}^1)\cup(Y_0^\delta\times\breve{D}\bar{\beta}^1)\\ &=(Y_3^\delta\times Z_2)\cup(Y_2^\delta\times Z_1)\cup(Y_1^\delta\times\breve{D})\cup(Y_0^\delta\times\breve{D})\\ &=(Y_3^\delta\times Z_2)\cup(Y_2^\delta\times Z_1)\cup((Y_1^\delta\cup Y_0^\delta)\times\breve{D})\neq\alpha\end{aligned}$$

This contradicts with $\alpha=\delta\circ\beta$. So, $\delta\notin B_0\setminus\{\sigma_1\}$.

Now, we suppose that $\delta\in\tilde{B}_{32}\setminus\{\sigma_1\}$ and $\beta\in S_1\setminus\{\sigma_1\}$. Similar operations are applied as above, we obtain $\delta\notin\tilde{B}_{32}\setminus\{\sigma_1\}$.

That means $\alpha\neq\delta\circ\beta$ for any $\alpha\in\tilde{B}_{32}$ and $\delta,\beta\in S_1\setminus\{\alpha\}$. □

Lemma 6. *Let* $D=\{Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_{2.2}(X,4)$, $\sigma_0=(Z_3\times Z_3)\cup((X\setminus Z_3)\times Z_1)$ *and* $\sigma_1=(Z_2\times Z_2)\cup((X\setminus Z_2)\times Z_1)$. *If* $X=\breve{D}$ *then* $S_2=B_0\cup\tilde{B}_{32}\cup\{\sigma_0,\sigma_1\}$ *is irreducible generating set for the semigroup* $B_X(D)$.

Theorem 7. *Let* $D=\{Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_{2.2}(X,4)$, $\sigma_0=(Z_3\times Z_3)\cup((X\setminus Z_3)\times Z_1)$ *and* $\sigma_1=(Z_2\times Z_2)\cup((X\setminus Z_2)\times Z_1)$. *If* X *is a finite set and* $|X|=n$ *then the following statements are true*

a) If $|X\setminus\breve{D}|\geq 1$ then $|B_0\cup\tilde{B}_{32}\cup\{\sigma_1\}|=4^n-3^{n+1}+2^{n+2}-2$

b) If $X=\breve{D}$ then $|B_0\cup\tilde{B}_{32}\cup\{\sigma_0,\sigma_1\}|=4^n-3^{n+1}+2^{n+2}-1$

Proof. Let

$$S_n=\{\varphi_i\mid\varphi_i:M=\{1,2,\cdots,n\}\to M=\{1,2,\cdots,n\},\text{one to one mapping}\}$$

be a group, $\varphi_{i_1},\varphi_{i_2},\cdots,\varphi_{i_m}\in S_n$ $(m\leq n)$ and $Y_{\varphi_1},Y_{\varphi_2},\cdots,Y_{\varphi_m}$ be partitioning of *X*. It is well known that $k_n^m=\left|\{Y_{\varphi_1},Y_{\varphi_2},\cdots,Y_{\varphi_m}\}\right|=\sum_{i=1}^{m}\frac{(-1)^{m+i}}{(i-1)!(m-i)!}$. If $m=2,3,4$ then we have

$$k_n^2=2^{n-1}-1$$
$$k_n^3=\frac{1}{2}\cdot 3^{n-1}-2^{n-1}+\frac{1}{2}$$
$$k_n^4=\frac{1}{6}\cdot 4^{n-1}-\frac{1}{2}\cdot 3^{n-1}+\frac{1}{2}\cdot 2^{n-1}-\frac{1}{6}$$

If $Y_{\varphi_1},Y_{\varphi_2}$ are any two elements of partitioning of *X* and $\overline{\beta}=(Y_{\varphi_1}\times T_1)\cup(Y_{\varphi_2}\times T_2)$ where $T_1,T_2\in D$ and $T_1\neq T_2$, then the number of different binary relations $\overline{\beta}$ of semigroup $B_X(D)$ is equal to

$$2\cdot k_n^2=2^n-2 \tag{2}$$

If $Y_{\varphi_1},Y_{\varphi_2},Y_{\varphi_3}$ are any three elements of partitioning of X and $\overline{\beta}=(Y_{\varphi_1}\times T_1)\cup(Y_{\varphi_2}\times T_2)\cup(Y_{\varphi_3}\times T_3)$ where T_1,T_2,T_3 are pairwise different elements of *D*, then the number of different binary relations $\overline{\beta}$ of semigroup $B_X(D)$ is equal to

$$6\cdot k_n^3=3^n-3\cdot 2^n+3 \tag{3}$$

If $Y_{\varphi_1},Y_{\varphi_2},Y_{\varphi_3},Y_{\varphi_4}$ are any four elements of partitioning of X and $\overline{\beta}=(Y_{\varphi_1}\times T_1)\cup(Y_{\varphi_2}\times T_2)\cup(Y_{\varphi_3}\times T_3)\cup(Y_{\varphi_4}\times T_4)$ where T_1,T_2,T_3,T_4 are pairwise different elements of *D*, then the number of different binary relations $\overline{\beta}$ of semigroup $B_X(D)$ is equal to

$$24\cdot k_n^4=4^n-4\cdot 3^n+3\cdot 2^n-4 \tag{4}$$

Let $\alpha\in B_0$. Quasinormal represantation of α has form

$$\alpha=(Y_3^\alpha\times Z_3)\cup(Y_2^\alpha\times Z_2)\cup(Y_1^\alpha\times Z_1)\cup(Y_0^\alpha\times\breve{D})$$

where $Y_3^\alpha,Y_2^\alpha,Y_1^\alpha\notin\{\varnothing\}$. Also, $Y_3^\alpha,Y_2^\alpha,Y_1^\alpha$ or $Y_3^\alpha,Y_2^\alpha,Y_1^\alpha,Y_0^\alpha$ are partitioning of *X* for $|X|\geq 4$. By using Equations (2.3) and (2.4) we obtain

$$|B_0|=4^n-3^{n+1}+3\cdot 2^n-1$$

Let $\alpha\in\tilde{B}_{32}$. Quasinormal represantation of α has form $\alpha=(Y_3^\alpha\times Z_3)\cup(Y_2^\alpha\times Z_2)$ where $Y_3^\alpha,Y_2^\alpha\notin\{\varnothing\}$. Also, Y_3^α,Y_2^α are partitioning of *X*. By using (2.2) we obtain

$$\left|\tilde{B}_{32}\right| = 2^n - 2$$

So, we have

$$\left|B_0 \cup \tilde{B}_{32} \cup \{\sigma_1\}\right| = 4^n - 3^{n+1} + 2^{n+2} - 2$$

$$\left|B_0 \cup \tilde{B}_{32} \cup \{\sigma_0, \sigma_1\}\right| = 4^n - 3^{n+1} + 2^{n+2} - 1$$

since $B_0 \cap \tilde{B}_{32} = B_0 \cap \{\sigma_0, \sigma_1\} = \tilde{B}_{32} \cap \{\sigma_0, \sigma_1\} = \varnothing$. □

Acknowledgements

Sincere thanks to Prof. Dr. Neşet AYDIN for his valuable suggestions.

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