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Hausdorff Dimension and Fractal Dimension of the Global Attractor for the Higher-Order Coupled Kirchhoff-Type Equations

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Abstract

This paper mainly deals with the higher-order coupled Kirchhoff-type equations with nonlinear strong damped and source terms in a bounded domain. We obtain some results that are estimation of the upper bounds of Hausdorff dimension and Fractal dimension of the global attractor.

Keywords

Higher-Order Coupled Kirchhoff-Type Equations, Source Term, Hausdorff Dimension, Fractal Dimension, Nonlinear Dissipation

1. Introduction

Guoguang Lin and Sanmei Yang [1] had studied the existence and uniqueness of the solution and global attractors for the higher-order coupled Kirchhoff-type equations. Furthermore, we consider the Hausdorff dimension and Fractal dimension of the global attractor for the following Hinder-order coupled Kirchhoff equations:

$$u_{tt}+M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)(-\Delta)^m u+\beta(-\Delta)^m u_t+g_1(u,v)=f_1(x), \tag{1.1}$$

$$v_{tt}+M\left(\|\nabla u\|^2+\|\nabla v\|^2\right)(-\Delta)^m v+\beta(-\Delta)^m v_t+g_2(u,v)=f_2(x), \tag{1.2}$$

$$u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),\ x\in\Omega, \tag{1.3}$$

$$v(x,0)=v_0(x),\ v_t(x,0)=v_1(x),\ x\in\Omega, \tag{1.4}$$

$$u|_{\partial\Omega}=0,\ \frac{\partial^i u}{\partial\mu^i}=0,\ i=1,2,3,\cdots,m-1, \tag{1.5}$$

$$v|_{\partial\Omega} = 0,\ \frac{\partial^i v}{\partial v^i} = 0,\ i = 1, 2, 3, \cdots, m-1, \tag{1.6}$$

where $m > 1$ is an integer constant and Ω is a bounded domain of R^n with a smooth Dirichlet boundary $\partial\Omega$ and initial value. μ_i and v_i are the unit outward normal on $\partial\Omega$, $M(s)$ is a nonnegative C^1 function, $(-\Delta)^m u_t$ and $(-\Delta)^m v_t$ are strongly damping, $g_1(u,v)$ and $g_2(u,v)$ are nonlinear source terms, $f_1(x)$ and $f_2(x)$ are given forcing function.

When considering single Higher-order Kirchhoff-type equation, $\beta = \sigma\left(\|\nabla^m u\|^2\right)$, $g(u,v) = 0$ and $m = 1$ in $M(s)$, becomes following Higher-order Kirchhoff-type wave equation with nonlinear strongly damping:

$$u_{tt} + \sigma\left(\|\nabla^m u\|^2\right)(-\Delta)^m u_t + \phi\left(\|\nabla^m u\|^2\right)(-\Delta)^m u = f(x), \tag{1.7}$$

$$u(x,t) = 0, \frac{\partial^i u}{\partial v^i} = 0, i = 1, \cdots, m-1, x \in \partial\Omega, t \in (0, +\infty), \tag{1.8}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega. \tag{1.9}$$

This equation had been studied some main results that are existence and uniqueness of the solution in $H^{2m}(\Omega) \times H_0^m(\Omega)$ and global attractors by Yuting Sun, Yunlong Gao and Guoguang Lin, see [2].

In case of $\beta = 1$ and $m = 1$ in $M(s)$, the Equation (1.1) becomes a Higher-order Kirchhoff-type equation with nonlinear strongly dissipation and source term:

$$u_{tt} + (-\Delta)^m u_t + M\left(\left\|(-\Delta)^{\frac{m}{2}} u\right\|^2\right)(-\Delta)^m u + g(u) = f(x), x \in \Omega, t > 0, m > 1, \tag{1.10}$$

$$u(x,t) = 0, \frac{\partial^i u}{\partial v^i} = 0, i = 1, \cdots, m-1, x \in \partial\Omega, x \in \Omega, t > 0, \tag{1.11}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x). \tag{1.12}$$

This equation had been investigated the existence and uniqueness of the solution, global attractors and estimation Hausdorff and fractal dimensions of the global attractor by Chen, Wei Wang and Guoguang Lin, see [3]. As for the study of estimation Hausdorff dimension of the global attractor, we applied different method from theirs.

Under the situation of $\beta = 1$ and $M(s) = \left(\alpha + \beta\|\nabla^m u\|^2\right)^q$, the problem (1.1) becomes a class of strongly damped Higher-order Kirchhoff-type equation:

$$u_{tt} + (-\Delta)^m u_t + \left(\alpha + \beta\|\nabla^m u\|^2\right)^q (-\Delta)^m u + g(u) = f(x), (x,t) \in \Omega \times [0, +\infty), \tag{1.13}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \tag{1.14}$$

$$u(x,t) = 0, \frac{\partial^i u}{\partial v^i} = 0, i = 1, \cdots, m-1, x \in \partial\Omega, t \in (0, +\infty). \tag{1.15}$$

This equation had been studied the existence and uniqueness of the solution, global attractors and estimation of the upper bounds of Hausdorff for the global attractors and the existence of a fractal exponential attractor with non-supercritical and critical cases by Guoguang Lin and Yunlong Gao, see [4]. Their novelty is that it overcomes $\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^q$ by using generalized Gronwall's inequality in Lemma 2.

Next, the main purpose of this paper is to study a precise estimation of upper bounds of Hausdorff dimension and Fractal dimension of the global attractor.

2. Preliminaries

To better carry out our work, We denote the some simple symbol, $\|\cdot\|$ represents norm, (,) stands for inner product and $H^m(\Omega)=H^m(\Omega)$, $H_0^m(\Omega)=H^m(\Omega)\cap H_0^1(\Omega)$, $H_0^{2m}(\Omega)=H^{2m}(\Omega)\cap H_0^1(\Omega)$, $f_i=f_i(x)(i=1,2)$, $H(\Omega)=L^2(\Omega)$, $\|\cdot\|=\|\cdot\|_{L^2}$, $\|\cdot\|_\infty=\|\cdot\|_{L^\infty}$, $\upsilon=\|\nabla u\|^2+\|\nabla v\|^2$. $c_i(i=1,\cdots)$, $\mu_i(i=0,1)$ are constants. λ_1^m is the first eigenvalue of the operator $(-\Delta)^m$.

Next, we give some assumptions needed for problem (1.1)-(1.6).

(H1) $M(s)\in C^2(\Omega), M'(s)\ge S_0\ge 0.$ (2.1)

(H2) $$0\le\mu_0\le M(\upsilon)\le\mu_1,\quad \mu=\begin{cases}\mu_0, & \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\nabla^m\theta\right\|^2+\left\|\nabla^m\sigma\right\|^2\right)\ge 0,\\ \mu_1, & \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\nabla^m\theta\right\|^2+\left\|\nabla^m\sigma\right\|^2\right)<0,\end{cases}\tag{2.2}$$

(H3) $\forall M>0$, there exists k_1,k_2, such that

$$\left\|g_{iu}(\tilde{u},\tilde{v})-g_{iu}(u,v)\right\|_{L^\infty}\le k_1\left(\|\tilde{u}-u\|^k+\|\tilde{v}-v\|^{k+1}\right).\tag{2.3}$$

$$\left\|g_{iv}(\tilde{u},\tilde{v})-g_{iv}(u,v)\right\|_{L^\infty}\le k_2\left(\|\tilde{u}-u\|^{k+1}+\|\tilde{v}-v\|^{k}\right).\tag{2.4}$$

$$g_i(u,v)\in C^1(\Omega).\tag{2.5}$$

Lemma 2.1. (Young's inequality [5]) For any $\varepsilon>0$ and $a,b\ge 0$, then

$$ab\le\frac{\varepsilon^p}{p}a^p+\frac{b^q}{q\varepsilon^q},\tag{2.6}$$

where $\dfrac{1}{p}+\dfrac{1}{q}=1, p>1, q>1.$

Lemma 2.2. (Gronwall's inequality [5]) If $\forall t\in[t_0,+\infty)$, $y(t)\ge 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}t}+gy\le h$, such that

$$y(t)\le y(t_0)\mathrm{e}^{-g(t-t_0)}+\frac{h}{g}, t\ge t_0,\tag{2.7}$$

where $g\ge 0, h\ge 0$ are constants.

Lemma 2.3. (Sobolev-Poincare inequality [6]) Let's be a number with $2\le s<+\infty, n\le 2m$ and $2\le s<\dfrac{2m}{n-2m}, n>2m$. Then there is a constant k depending on Ω and s such that

$$\|u\|_s \le k\left\|(-\Delta)^{\frac{m}{2}} u\right\|, \forall u \in H_0^m(\Omega). \tag{2.8}$$

3. Hausdorff Dimensions and Fractal Dimension for the Global Attractor

3.1. Differentiability of the Semigroup

We denote $E_0 = H^m \times H^m \times L^2 \times L^2$. The inner product and the norm in E_0 space are defined as follows:

$\forall \varphi_i = (u_i, v_i, p_i, q_i) \in E_0\ (i = 1, 2)$, we can get

$$(\varphi_1, \varphi_2)_{E_0} = (\nabla^m u_1, \nabla^m u_2) + (\nabla^m v_1, \nabla^m v_2) + (p_1, p_2) + (q_1, q_2). \tag{3.1}$$

$$\begin{aligned}\|\varphi_1\|_{E_0}^2 &= (\varphi_1, \varphi_1)_{E_0} \\ &= \|\nabla^m u_1\|^2 + \|\nabla^m v_1\|^2 + \|p_1\|^2 + \|q_1\|^2.\end{aligned} \tag{3.2}$$

Setting $\forall \varphi = (u, v, p, q)^{\mathrm{T}} \in E_0,\ p = u_t + \varepsilon u,\ q = v_t + \varepsilon v,\ \varepsilon > 0$, the equations (1.1) and (1.2) are equivalent to

$$\varphi_t + H(\varphi) = F(\varphi), \tag{3.3}$$

where

$$H(\varphi) = \begin{pmatrix} \varepsilon u - p \\ \varepsilon v - q \\ -\varepsilon p + \beta(-\Delta)^m p + \varepsilon^2 u + (1 - \varepsilon\beta)(-\Delta)^m u \\ -\varepsilon q + \beta(-\Delta)^m q + \varepsilon^2 v + (1 - \varepsilon\beta)(-\Delta)^m v \end{pmatrix}, \tag{3.4}$$

$$F(\varphi) = \begin{pmatrix} 0 \\ 0 \\ (I - M(\upsilon))(-\Delta)^m u - g_1(u, v) + f_1(x) \\ (I - M(\upsilon))(-\Delta)^m v - g_2(u, v) + f_2(x) \end{pmatrix}. \tag{3.5}$$

Lemma 3.1.1. $\forall \varphi = (u, v, p, q)^{\mathrm{T}} \in E_0$, we can get

$$(H(\varphi), \varphi)_{E_0} \ge \frac{\varepsilon}{4}\|\varphi\|_{E_0}^2 + \frac{\beta}{4}\|\nabla^m p\|^2 + \frac{\beta}{4}\|\nabla^m q\|^2. \tag{3.6}$$

Proof. According to (3.1)-(3.5), Holder inequality, Young's inequality and Poincare inequality, we can obtain

$$\begin{aligned}&(H(\varphi), \varphi)_{E_0} \\ &= (\varepsilon\nabla^m u - \nabla^m p, \nabla^m u) + (\varepsilon\nabla^m v - \nabla^m q, \nabla^m v) + \big(-\varepsilon p + \beta(-\Delta)^m p \\ &\quad + \varepsilon^2 u + (1 - \varepsilon\beta)(-\Delta)^m u, p\big) + \big(-\varepsilon q + \beta(-\Delta)^m q + \varepsilon^2 v + (1 - \varepsilon\beta)(-\Delta)^m v, q\big) \\ &= \varepsilon\|\nabla^m u\|^2 + \varepsilon\|\nabla^m v\|^2 - \varepsilon\|p\|^2 - \varepsilon\|q\|^2 + \beta\|\nabla^m p\|^2 + \beta\|\nabla^m q\|^2 \\ &\quad + \varepsilon^2(u, p) + \varepsilon^2(v, q) - \varepsilon\beta(\nabla^m u, \nabla^m p) - \varepsilon\beta(\nabla^m v, \nabla^m q)\end{aligned}$$

$$
\begin{aligned}
&\geq \varepsilon\left\|\nabla^m u\right\|^2+\varepsilon\left\|\nabla^m v\right\|^2-\varepsilon\|p\|^2-\varepsilon\|q\|^2+\beta\left\|\nabla^m p\right\|^2+\beta\left\|\nabla^m q\right\|^2 \\
&\quad-\frac{\varepsilon^2}{2\lambda_1^m}\left\|\nabla^m u\right\|^2-\frac{\varepsilon^2}{2}\|p\|^2-\frac{\varepsilon^2}{2\lambda_1^m}\left\|\nabla^m v\right\|^2-\frac{\varepsilon^2}{2}\|q\|^2-\frac{\beta\varepsilon^2}{2}\left\|\nabla^m u\right\|^2 \\
&\quad-\frac{\beta}{2}\left\|\nabla^m p\right\|^2-\frac{\beta\varepsilon^2}{2}\left\|\nabla^m v\right\|^2-\frac{\beta}{2}\left\|\nabla^m q\right\|^2 \\
&\geq\left(\varepsilon-\frac{\varepsilon^2}{2\lambda_1^m}-\frac{\beta\varepsilon^2}{2}\right)\left\|\nabla^m u\right\|^2+\left(\varepsilon-\frac{\varepsilon^2}{2\lambda_1^m}-\frac{\beta\varepsilon^2}{2}\right)\left\|\nabla^m v\right\|^2+\frac{\beta}{4}\left\|\nabla^m p\right\|^2 \\
&\quad+\frac{\beta}{4}\lambda_1^m\|p\|^2+\left(-\varepsilon-\frac{\beta\varepsilon^2}{2}\right)\|p\|^2+\frac{\beta}{4}\left\|\nabla^m q\right\|^2+\frac{\beta}{4}\lambda_1^m\|q\|^2+\left(-\varepsilon-\frac{\beta\varepsilon^2}{2}\right)\|q\|^2 \\
&\geq \frac{\varepsilon}{4}\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^m v\right\|^2+\|p\|^2+\|q\|^2\right)+\frac{\beta}{4}\left\|\nabla^m p\right\|^2+\frac{\beta}{4}\left\|\nabla^m q\right\|^2 \\
&\geq \frac{\varepsilon}{4}\|\varphi\|_{E_0}^2+\frac{\beta}{4}\left\|\nabla^m p\right\|^2+\frac{\beta}{4}\left\|\nabla^m q\right\|^2 .
\end{aligned} \tag{3.7}
$$

The proof of Lemma 3.1.1. is completed.

The linearized equations of (1.1)-(1.6), the above equations as follows:

$$
\begin{aligned}
&U_{tt}+M(\upsilon)(-\Delta)^m U+2M'(\upsilon)(\nabla U,\nabla u)(-\Delta)^m u \\
&+2M'(\upsilon)(\nabla V,\nabla v)(-\Delta)^m u+\beta(-\Delta)^m U_t+g_{1u}(u,v)U+g_{1v}(u,v)V=0,
\end{aligned} \tag{3.8}
$$

$$
\begin{aligned}
&V_{tt}+M(\upsilon)(-\Delta)^m V+2M'(\upsilon)(\nabla U,\nabla u)(-\Delta)^m v \\
&+2M'(\upsilon)(\nabla V,\nabla v)(-\Delta)^m v+\beta(-\Delta)^m V_t+g_{2u}(u,v)U+g_{2v}(u,v)V=0,
\end{aligned} \tag{3.9}
$$

$$
U(x,t)\big|_{x\in\partial\Omega}=0, U(x,0)=\xi_1, U_t(x,0)=\xi_2, \tag{3.10}
$$

$$
V(x,t)\big|_{x\in\partial\Omega}=0, V(x,0)=\eta_1, V_t(x,0)=\eta_2, \tag{3.11}
$$

where $(\xi_1,\eta_1,\xi_2,\eta_2)\in E_0,(u,v,u_t,v_t)=S(t)(u_0,v_0,u_1,v_1)$ is the solution of (1.1)-(1.6) with $(u_0,v_0,u_1,v_1)\in \mathrm{A}$.

Given $(u_0,v_0,u_1,v_1)\in \mathrm{A}$ and $S(t):E_0\to E_0$, the solution $S(t)(u_0,v_0,u_1,v_1)\in E_0$ by stand methods we can show that for any $(\xi_1,\eta_1,\xi_2,\eta_2)\in E_0$, the linear initial boundary value problem (3.8)-(3.11) possess a unique solution:

$$
\left(U(t),V(t),U_t(t),V_t(t)\right)\in L^\infty\left((0,+\infty);E_0\right). \tag{3.12}
$$

Lemma 3.1.2. For any $t>0,R>0$, the mapping $S(t):E_0\to E_0$ is Frechet differentiable on. It is differential at $\varphi_0=(u_0,v_0,u_1,v_1)$ is the linear operator on $F:(\xi_1,\eta_1,\xi_2,\eta_2)^{\mathrm{T}}\to\left(U(t),V(t),P(t),Q(t)\right)^{\mathrm{T}}$, where $U(t)$ and $V(t)$ are solutions of (3.8)-(3.11)

Proof. Let $\varphi_0=(u_0,v_0,u_1,v_1)^{\mathrm{T}}\in E_0$, $\tilde{\varphi}_0=(u_0+\xi_1,v_0+\eta_1,u_1+\xi_2,v_1+\eta_2)^{\mathrm{T}}\in E_0$ with $\|\varphi\|_{E_0}\leq R,\|\tilde{\varphi}\|_{E_0}\leq R$, we define $(u,v,u_t,v_t)=S(t)\varphi_0,(\tilde{u},\tilde{v},\tilde{u}_t,\tilde{v}_t)=S(t)\tilde{\varphi}_0$. We can obtain the Lipchitz property of $S(t)$ on the bounded sets of E_0, that is

$$
\left\|S(t)\varphi_0-S(t)\tilde{\varphi}_0\right\|_{E_0}^2\leq \mathrm{e}^{C_{11}t}\left\|(\xi_1,\eta_1,\xi_2,\eta_2)\right\|_{E_0}^2. \tag{3.13}
$$

Let $\theta=\tilde{u}-u-U$ and $\sigma=\tilde{v}-v-V$ are solutions of problem

$$\theta_{tt}+M(\upsilon)A^m\theta+\beta A^m\theta_t=h_1, \tag{3.14}$$

$$\sigma_{tt}+M(\upsilon)A^m\sigma+\beta A^m\sigma_t=h_2, \tag{3.15}$$

$$\theta(0)=\theta_t(0)=0, \tag{3.16}$$

$$\sigma(0)=\sigma_t(0)=0, \tag{3.17}$$

with

$$h_1=\left(M(\upsilon)-M(\tilde{\upsilon})\right)(-\Delta)^m\tilde{u}+2M'(\upsilon)(\nabla U,\nabla u)(-\Delta)^m u$$
$$+2M'(\upsilon)(\nabla V,\nabla v)(-\Delta)^m u+g_1(u,v)-g_1(\tilde{u},\tilde{v})+g_{1u}(u,v)U+g_{1v}(u,v)V, \tag{3.18}$$

$$h_2=\left(M(\upsilon)-M(\tilde{\upsilon})\right)(-\Delta)^m\tilde{v}+2M'(\upsilon)(\nabla U,\nabla u)(-\Delta)^m v$$
$$+2M'(\upsilon)(\nabla V,\nabla v)(-\Delta)^m v+g_2(u,v)-g_2(\tilde{u},\tilde{v})+g_{2u}(u,v)U+g_{2v}(u,v)V, \tag{3.19}$$

where $\upsilon=\|\nabla u\|^2+\|\nabla v\|^2,\tilde{\upsilon}=\|\nabla\tilde{u}\|^2+\|\nabla\tilde{v}\|^2$.

Taking the scalar product of each side of (3.14) with θ_t. Because of

$$(\nabla u,\nabla u)=(\nabla\tilde{u}-\nabla u-\nabla\tilde{u},-\nabla u)=-(\nabla\tilde{u}-\nabla u,\nabla u)+(\nabla\tilde{u},\nabla u). \tag{3.20}$$

$$(\nabla\tilde{u},\nabla\tilde{u})=(\nabla\tilde{u}-\nabla u+\nabla u,\nabla\tilde{u})=(\nabla\tilde{u}-\nabla u,\nabla\tilde{u})+(\nabla u,\nabla\tilde{u}). \tag{3.21}$$

So

$$(\nabla u,\nabla u)-(\nabla\tilde{u},\nabla\tilde{u})=(\nabla u-\nabla\tilde{u},\nabla u+\nabla\tilde{u}). \tag{3.22}$$

$$\Big(\left(M(\upsilon)-M(\tilde{\upsilon})\right)(-\Delta)^m\tilde{u}+2M'(\upsilon)(\nabla U,\nabla u)(-\Delta)^m u$$
$$+2M'(\upsilon)(\nabla V,\nabla v)(-\Delta)^m u,\theta_t\Big)$$
$$=\Big(M'(\theta\upsilon+(1-\theta)\tilde{\upsilon})(\upsilon-\tilde{\upsilon})(-\Delta)^m\tilde{u}+2M'(\upsilon)\left((\nabla U,\nabla u)+(\nabla V,\nabla v)\right)(-\Delta)^m u,\theta_t\Big)$$
$$=M'(\theta\upsilon+(1-\theta)\tilde{\upsilon})\left((\nabla u,\nabla u)-(\nabla\tilde{u},\nabla\tilde{u})+(\nabla v,\nabla v)-(\nabla\tilde{v},\nabla\tilde{v})\right)\left((-\Delta)^m\tilde{u},\theta_t\right)$$
$$+2M'(\upsilon)\left((\nabla U,\nabla u)+(\nabla V,\nabla v)\right)\left((-\Delta)^m u,\theta_t\right)$$
$$=M'(\theta\upsilon+(1-\theta)\tilde{\upsilon})\left((\nabla\hat{u},\nabla u+\nabla\tilde{u})+(\nabla\hat{v},\nabla v+\nabla\tilde{v})\right)\left((-\Delta)^m\tilde{u},\theta_t\right)$$
$$+2M'(\upsilon)\left((\nabla U,\nabla u)+(\nabla V,\nabla v)\right)\left((-\Delta)^m u,\theta_t\right)$$
$$=\Big[M'(\theta\upsilon+(1-\theta)\tilde{\upsilon})\left((\nabla\hat{u},\nabla u+\nabla\tilde{u})+(\nabla\hat{v},\nabla v+\nabla\tilde{v})\right)\left((-\Delta)^m\tilde{u},\theta_t\right)$$
$$-M'(\upsilon)\left((\nabla\hat{u},\nabla u+\nabla\tilde{u})+(\nabla\hat{v},\nabla v+\nabla\tilde{v})\right)\left((-\Delta)^m\tilde{u},\theta_t\right)\Big]$$
$$+\Big[M'(\upsilon)\left((\nabla\hat{u},\nabla u+\nabla\tilde{u})+(\nabla\hat{v},\nabla v+\nabla\tilde{v})\right)\left((-\Delta)^m\tilde{u},\theta_t\right)$$
$$-2M'(\upsilon)\left((\nabla\hat{u}+\nabla\theta,\nabla u)+(\nabla\hat{v}+\nabla\sigma,\nabla v)\right)\left((-\Delta)^m\tilde{u},\theta_t\right)\Big]$$
$$+\Big[-2M'(\upsilon)\left((\nabla U,\nabla u)+(\nabla V,\nabla v)\right)\left((-\Delta)^m\tilde{u},\theta_t\right)$$
$$+2M'(\upsilon)\left((\nabla U,\nabla u)+(\nabla V,\nabla v)\right)\left((-\Delta)^m u,\theta_t\right)\Big]$$
$$=I_1+I_2+I_3,$$

$$I_1=\left(M'\left(\theta\upsilon+(1-\theta)\tilde{\upsilon}\right)-M'(\upsilon)\right)\left(\left(\nabla\hat{u},\nabla u+\nabla\tilde{u}\right)+\left(\nabla\hat{v},\nabla v+\nabla\tilde{v}\right)\right)\left((-\Delta)^m\tilde{u},\theta_t\right)$$
$$=M''(\zeta)(\tilde{\upsilon}-\upsilon)\left(\left(\nabla\hat{u},\nabla u+\nabla\tilde{u}\right)+\left(\nabla\hat{v},\nabla v+\nabla\tilde{v}\right)\right)\left((-\Delta)^m\tilde{u},\theta_t\right)$$
$$\le M''(\zeta)\left(\|\nabla\hat{u}\|\|\nabla u+\nabla\tilde{u}\|+\|\nabla\hat{v}\|\|\nabla v+\nabla\tilde{v}\|\right)^2\|\nabla^m\tilde{u}\|\|\nabla^m\theta_t\|$$
$$\le 2M''(\zeta)\left(\|\nabla\hat{u}\|^2\|\nabla u+\nabla\tilde{u}\|^2+\|\nabla\hat{v}\|^2\|\nabla v+\nabla\tilde{v}\|^2\right)\|\nabla^m\tilde{u}\|\|\nabla^m\theta_t\|$$
$$\le 2\lambda_1^{-m+1}M''(\zeta)\left(\|\nabla^m\hat{u}\|^2\left(\|\nabla u\|+\|\nabla\tilde{u}\|\right)^2+\|\nabla^m\hat{v}\|^2\left(\|\nabla v\|+\|\nabla\tilde{v}\|\right)^2\right)\|\nabla^m\tilde{u}\|\|\nabla^m\theta_t\|, \tag{3.23}$$

$$I_2=M'(\upsilon)\left(\left(\nabla\hat{u},\nabla u+\nabla\tilde{u}\right)+\left(\nabla\hat{v},\nabla v+\nabla\tilde{v}\right)\right)\left((-\Delta)^m\tilde{u},\theta_t\right)$$
$$-2M'(\upsilon)\left(\left(\nabla\hat{u}+\nabla\theta,\nabla u\right)+\left(\nabla\hat{v}+\nabla\sigma,\nabla v\right)\right)\left((-\Delta)^m\tilde{u},\theta_t\right)$$
$$=M'(\upsilon)\left(\left(\nabla\hat{u},\nabla\hat{u}\right)+\left(\nabla\hat{v},\nabla\hat{v}\right)\right)\left((-\Delta)^m\tilde{u},\theta_t\right)$$
$$-2M'(\upsilon)\left(\left(\nabla\theta,\nabla u\right)+\left(\nabla\sigma,\nabla v\right)\right)\left((-\Delta)^m\tilde{u},\theta_t\right)$$
$$\le M'(\upsilon)\left(\|\nabla\hat{u}\|^2+\|\nabla\hat{v}\|^2\right)\|\nabla^m\tilde{u}\|\|\nabla^m\theta_t\|$$
$$-2M'(\upsilon)\left(\left(\nabla\theta,\nabla u\right)+\left(\nabla\sigma,\nabla v\right)\right)\left((-\Delta)^m\tilde{u},\theta_t\right)$$
$$\le M'(\upsilon)\left(\|\nabla\hat{u}\|^2+\|\nabla\hat{v}\|^2\right)\|\nabla^m\tilde{u}\|\|\nabla^m\theta_t\|$$
$$+2\lambda_1^{\frac{-m+1}{2}}M'(\upsilon)\left(\|\nabla^m\theta\|\|\nabla u\|+\|\nabla^m\sigma\|\|\nabla v\|\right)\|\nabla^m\tilde{u}\|\|\nabla^m\theta_t\|, \tag{3.24}$$

$$I_3=2M'(\upsilon)\left(\left(\nabla U,\nabla u\right)+\left(\nabla V,\nabla v\right)\right)\left((-\Delta)^m\hat{u},\theta_t\right)$$
$$=2M'(\upsilon)\left(\left(\nabla\tilde{u}-\nabla u-\nabla\theta,\nabla u\right)+\left(\nabla\tilde{v}-\nabla v-\nabla\sigma,\nabla v\right)\right)\left(\nabla^m\hat{u},\nabla^m\theta_t\right)$$
$$\le 2M'(\upsilon)\left(\|\nabla\hat{u}\|\|\nabla u\|+\|\nabla\theta\|\|\nabla u\|+\|\nabla\hat{v}\|\|\nabla v\|+\|\nabla\sigma\|\|\nabla v\|\right)\|\nabla^m\hat{u}\|\|\nabla^m\theta_t\|$$
$$\le 2\lambda_1^{\frac{-m+1}{2}}M'(\upsilon)\left(\|\nabla^m\hat{u}\|\|\nabla u\|+\|\nabla^m\theta\|\|\nabla u\|+\|\nabla^m\hat{v}\|\|\nabla v\|+\|\nabla^m\sigma\|\|\nabla v\|\right)\|\nabla^m\hat{u}\|\|\nabla^m\theta_t\|$$
$$\le 2\lambda_1^{\frac{-m+1}{2}}M'(\upsilon)\Big(\|\nabla^m\hat{u}\|^2\|\nabla u\|\|\nabla^m\theta_t\|+\|\nabla^m\theta\|\|\nabla u\|\left(\|\nabla^m u\|+\|\nabla^m\tilde{u}\|\right)\|\nabla^m\theta_t\|$$
$$+\left(\|\nabla^m\hat{v}\|^2+\|\nabla^m\hat{u}\|^2\right)\|\nabla v\|\|\nabla^m\theta_t\|+\|\nabla^m\sigma\|\|\nabla v\|\left(\|\nabla^m u\|+\|\nabla^m\tilde{u}\|\right)\|\nabla^m\theta_t\|\Big), \tag{3.25}$$

(3.26)

where $\hat{u}=u-\tilde{u},\hat{v}=v-\tilde{v}$.

By (H3) and Young's inequality, we can get

$$\left(g_1(u,v)-g_1(\tilde{u},\tilde{v})+g_{1u}(u,v)U+g_{1v}(u,v)V,\theta_t\right)$$
$$=\left(g_1(u,v)-g_1(\tilde{u},\tilde{v})+g_{1u}(u,v)(\tilde{u}-u-\theta)+g_{1v}(u,v)(\tilde{v}-v-\sigma),\theta_t\right)$$
$$=\left(g_1(u,v)-g_1(\tilde{u},\tilde{v})+g_{1u}(u,v)\hat{u}-g_{1u}(u,v)\theta+g_{1v}(u,v)\hat{v}-g_{1v}(u,v)\sigma,\theta_t\right)$$
$$=\left(g_1(u,v)-g_1(\tilde{u},v)+g_1(\tilde{u},v)-g_1(\tilde{u},\tilde{v})+g_{1u}(u,v)\hat{u}+g_{1v}(u,v)\hat{v},\theta_t\right)$$
$$+\left(-g_{1u}(u,v)\theta-g_{1v}(u,v)\sigma,\theta_t\right)$$

$$\begin{aligned}
&=\left(\left(g_{1u}(u,v)-g_{1u}(\xi,v)\right)\hat{u}+\left(g_{1v}(u,v)-g_{1v}(\tilde{u},\eta)\right)\hat{v},\theta_t\right) \\
&\quad+\left(-g_{1u}(u,v)\theta-g_{1v}(u,v)\sigma,\theta_t\right) \\
&\le\left\|g_{1u}(u,v)-g_{1u}(\xi,v)\right\|_{L^\infty}\|\hat{u}\|\|\theta_t\|+\left\|g_{1v}(u,v)-g_{1v}(\tilde{u},\eta)\right\|_{L^\infty}\|\hat{v}\|\|\theta_t\| \\
&\quad+\left\|g_{1u}(u,v)\right\|_{L^\infty}\|\theta\|\|\theta_t\|+\left\|g_{1v}(u,v)\right\|_{L^\infty}\|\sigma\|\|\theta_t\| \\
&\le k_1\|\hat{u}\|^{k+1}\|\theta_t\|+k_2\left(\|\hat{u}\|^{k+1}\left(\|v\|+\|\tilde{v}\|\right)+\|\hat{v}\|^{k+1}\right)\|\theta_t\|+C_{16}\|\theta\|^2+\frac{2}{4}\|\theta_t\|^2+C_{17}\|\sigma\|^2 .
\end{aligned} \tag{3.27}$$

By using Young inequality, we can get

$$\begin{aligned}
h_1\le{}&\frac{C_1}{\varepsilon}\left\|\nabla^m u-\nabla^m\tilde{u}\right\|^4+\frac{C_2}{\varepsilon}\left\|\nabla^m v-\nabla^m\tilde{v}\right\|^4+\frac{C_3}{\varepsilon}\left\|\nabla^m\theta\right\|^2+\frac{C_4}{\varepsilon}\left\|\nabla^m\sigma\right\|^2 \\
&+\frac{11\varepsilon}{4}\left\|\nabla^m\theta_t\right\|^2+C_5\left\|\nabla^m\hat{u}\right\|^{2k+2}+\frac{5}{4}\|\theta_t\|^2+C_6\left\|\nabla^m\hat{v}\right\|^{2k+2}+C_7\|\theta\|^2+C_8\|\sigma\|^2 .
\end{aligned} \tag{3.28}$$

From (3.23)-(3.28), we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_t\|^2+M(\upsilon)\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^m\theta\right\|^2\le{}&C_9\left(\left\|\nabla^m\hat{u}\right\|^4+\left\|\nabla^m\theta\right\|^2+\left\|\nabla^m\hat{v}\right\|^4+\left\|\nabla^m\sigma\right\|^2\right) \\
&+C_{10}\left(\left\|\nabla^m\hat{u}\right\|^{2k+2}+\left\|\nabla^m\hat{v}\right\|^{2k+2}+\|\theta_t\|^2\right).
\end{aligned} \tag{3.29}$$

Take the scalar product of each side of (3.15) with σ_t. Because of

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\sigma_t\|^2+M(\upsilon)\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^m\sigma\right\|^2\le{}&C_{11}\left(\left\|\nabla^m\hat{u}\right\|^4+\left\|\nabla^m\theta\right\|^2+\left\|\nabla^m\hat{v}\right\|^4+\left\|\nabla^m\sigma\right\|^2\right) \\
&+C_{12}\left(\left\|\nabla^m\hat{u}\right\|^{2k+2}+\left\|\nabla^m\hat{v}\right\|^{2k+2}+\|\theta_t\|^2\right).
\end{aligned} \tag{3.30}$$

Summing up (3.29) and (3.30), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\theta_t\|^2+\|\sigma_t\|^2+\mu\left(\left\|\nabla^m\theta\right\|^2+\left\|\nabla^m\sigma\right\|^2\right)\right) \\
&\le C_{13}\left(\|\theta_t\|^2+\|\sigma_t\|^2+\mu\left(\left\|\nabla^m\theta\right\|^2+\left\|\nabla^m\sigma\right\|^2\right)\right) \\
&\quad+C_{14}\left(\left\|\nabla^m\hat{u}\right\|^4+\left\|\nabla^m\hat{v}\right\|^4+\left\|\nabla^m\hat{u}\right\|^{2k+2}+\left\|\nabla^m\hat{v}\right\|^{2k+2}\right).
\end{aligned} \tag{3.31}$$

By using Gronwall inequality, we can get

$$\begin{aligned}
&\|\theta_t\|^2+\|\sigma_t\|^2+\mu\left(\left\|\nabla^m\theta\right\|^2+\left\|\nabla^m\sigma\right\|^2\right) \\
&\le C_{15}\mathrm{e}^{C_{16}t}\int_0^t\left(\left\|\nabla^m\hat{u}\right\|^4+\left\|\nabla^m\hat{v}\right\|^4+\left\|\nabla^m\hat{u}\right\|^{2k+2}+\left\|\nabla^m\hat{v}\right\|^{2k+2}\right)\mathrm{d}\tau \\
&\le C_{17}\mathrm{e}^{C_{18}t}\left[\left(\left\|\nabla^m\xi_1\right\|^2+\left\|\nabla^m\eta_1\right\|^2+\|\xi_2\|^2+\|\eta_2\|^2\right)^2\right. \\
&\quad\left.+\left(\left\|\nabla^m\xi_1\right\|^2+\left\|\nabla^m\eta_1\right\|^2+\|\xi_2\|^2+\|\eta_2\|^2\right)^{k+1}\right] \\
&\le C_{20}\mathrm{e}^{C_{19}t}\left(\left\|(\xi_1,\eta_1,\xi_2,\eta_2)\right\|_{E_0}^4+\left\|(\xi_1,\eta_1,\xi_2,\eta_2)\right\|_{E_0}^{2k+2}\right),
\end{aligned} \tag{3.32}$$

where $C_{17},C_{18},C_{19},C_{20}>0$.

From (3.32), we can get

$$\begin{aligned}
&\frac{\left\|\tilde{\varphi}(t)-\varphi(t)-B(t)\right\|_{E_0}^2}{\left\|\left(\xi_1,\eta_1,\xi_2,\eta_2\right)^{\mathrm{T}}\right\|_{E_0}^2} \\
&\le \frac{C_{20}\mathrm{e}^{C_{19}t}\left(\left\|\left(\xi_1,\eta_1,\xi_2,\eta_2\right)^{\mathrm{T}}\right\|_{E_0}^4+\left\|\left(\xi_1,\eta_1,\xi_2,\eta_2\right)^{\mathrm{T}}\right\|_{E_0}^{2k+2}\right)}{\left\|\left(\xi_1,\eta_1,\xi_2,\eta_2\right)^{\mathrm{T}}\right\|_{E_0}^2} \\
&\le C_{20}\mathrm{e}^{C_{19}t}\left(\left\|\left(\xi_1,\eta_1,\xi_2,\eta_2\right)^{\mathrm{T}}\right\|_{E_0}^2+\left\|\left(\xi_1,\eta_1,\xi_2,\eta_2\right)^{\mathrm{T}}\right\|_{E_0}^{2k}\right)\to 0,
\end{aligned} \tag{3.33}$$

here

$$B(t)=\left(U(t),V(t),U_t(t),V_t(t)\right). \tag{3.34}$$

As $\left(\xi_1,\eta_1,\xi_2,\eta_2\right)^{\mathrm{T}}\to 0$ in E_0. The lemma 3.1.2 is completed.

3.2. The Upper Bounds of Hausdorff Dimension and Fractal Dimension for the Global Attractor

Consider the first variation of (3.3) with initial condition;

$$\Psi_t'+p(\varphi)\Psi=\Gamma_1(\varphi)\Psi+\Gamma_2\Psi^{\mathrm{T}},\ \Psi(0)=\left(\xi_1,\eta_1,\xi_2,\eta_2\right)^{\mathrm{T}}\in E_0,\ t>0, \tag{3.35}$$

where $\Psi=(U,V,P,Q)^{\mathrm{T}}\in E_0,\ P=U_t+\varepsilon U,\ Q=V_t+\varepsilon V$ and $\varphi=(u,v,p,q)^{\mathrm{T}}\in E_0$ are solutions of (3.35),

$$p(\varphi)=\begin{pmatrix}\varepsilon I & 0 & -I & 0\\ 0 & \varepsilon I & 0 & -I\\ \zeta_1 & 0 & \zeta_2 & 0\\ 0 & \zeta_1 & 0 & \zeta_2\end{pmatrix}, \tag{3.36}$$

$$\Gamma_1=\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -g_{1u}(u,v) & -g_{1v}(u,v) & 0 & 0\\ -g_{2u}(u,v) & -g_{2u}(u,v) & 0 & 0\end{pmatrix}. \tag{3. 37}$$

$$\Gamma_2=\begin{pmatrix}0 & 0 & \zeta_3 & \zeta_4\end{pmatrix}^{\mathrm{T}}, \tag{3.38}$$

where $\zeta_1=\varepsilon^2+(I-\beta\varepsilon)(-\Delta)^m,\zeta_2=-\varepsilon I+\beta(-\Delta)^m$,

$\zeta_3=\left[1-M(\upsilon)\right](-\Delta)^m U-2M'(\upsilon)(\nabla U,\nabla u)(-\Delta)^m u-2M'(\upsilon)(\nabla V,\nabla v)(-\Delta)^m u,$

$\zeta_4=\left[1-M(\upsilon)\right](-\Delta)^m V-2M'(\upsilon)(\nabla U,\nabla u)(-\Delta)^m v-2M'(\upsilon)(\nabla V,\nabla v)(-\Delta)^m v.$

It is easy to show from lemma 3.1.2. that (3.35) is a well-posed problem in E_0, the mapping $S_\varepsilon(\tau):\{u_0,v_0,p_1,q_1\}\to\{u,v,p,q\}$, where $p_1=u_1+\varepsilon u_0$, $q_1=v_1+\varepsilon v_0$,

$p=u_t+\varepsilon u$, $q=v_t+\varepsilon v$.

$\varphi(\tau)=\{u(\tau),v(\tau),p(\tau)=u_t(\tau)+\varepsilon u(\tau),q(\tau)=v_t(\tau)+\varepsilon v(\tau)\}$ is Frechet differentiable on E_0 for any $t>0$, its differential at $\varphi_0=\{u_0,v_0,p_1,q_1\}^{\mathrm{T}}$ is the linear operator on E_0, $\left(\xi_1,\eta_1,\xi_2,\eta_2\right)^{\mathrm{T}}\to\left(U(t),V(t),P(t),Q(t)\right)^{\mathrm{T}}$, where $\left(U(t),V(t),P(t),Q(t)\right)^{\mathrm{T}}$ is the solution of (3.35).

Lemma 3.2.1. For any orthonormal family of elements of $\left(E_0,\|\|_{E_0}\right)$, $\left(\xi_{1j},\xi_{2j}\right)^{\mathrm{T}}$, $\left(\eta_{1j},\eta_{2j}\right)^{\mathrm{T}}$, $j=1,2,\cdots,n_1$, we have

$$\sum_{j=1}^{n_1}\left\|A^{\frac{1}{2}-v}\xi_{1j}\right\|^2\le 2\sum_{j=1}^{n_1}\mu_j^{v-1},\ v\in[0,1), \tag{3.39}$$

$$\sum_{j=1}^{n_1}\left\|A^{\frac{1}{2}-v}\eta_{1j}\right\|^2\le 2\sum_{j=1}^{n_1}\mu_j^{v-1},\ v\in[0,1), \tag{3.40}$$

where $\left\{\mu_j\right\}_{j=1}^{+\infty}$ is the eigenvalue of $(-\Delta)^m$.

Proof. This is a direct consequence of lemma VI 6.3 of [7]

Theorem 3.2.2. If we take proper μ_0,s satisfy $\frac{1-\mu_0}{2}+2K\lambda_1^{-m+1}R_1^2\le\frac{\varepsilon}{8}$ and (H1)-(H3) hold, then there exists $\rho(R_1)>0$, such that the Hausdorff dimension and Fractal dimension of global attractor A in E_0 satisfies

$$d_H(\mathrm{A})\le\min\left\{n_1\left|n_1\in N,\frac{1}{n_1}\sum_{j=1}^{n_1}\mu_j^{\delta-1}<\frac{\varepsilon}{4\rho}\right.\right\}, \tag{3.41}$$

$$d_F(\mathrm{A})\le 2n_1, \tag{3.42}$$

where R_0 is as in [1], and

$$\delta=\begin{cases}\frac{(n-2)(p-1)-2}{2},\ \frac{n}{n-2m}\le r_i\le\frac{n+2m}{n-2m},\ n\ge 2m,\\ 0,\ n<2m\text{ or }0\le r_i\le\frac{n}{n-2m},\ n\ge 2m,\end{cases} \tag{3.43}$$

here $i=1,2$.

Proof. Let $n_1\in N$ be fixed. Consider n_1 solutions $\Psi_1,\Psi_2,\cdots,\Psi_{n_1}$ of (3.35). At given time τ, let $Q_{n_1}(\tau)$ define the orthogonal projection in E_0 onto $span\left\{\Psi_1,\Psi_2,\cdots,\Psi_{n_1}\right\}$. Let $y_j(s)=\left(\xi_{1j},\eta_{1j},\xi_{2j},\eta_{2j}\right)^{\mathrm{T}}\in E_0,\ j=1,2,\cdots,n_1$ be an Orthonormal basis of

$$Q_{n_1}E_0=span\left\{\Psi_1,\Psi_2,\cdots,\Psi_{n_1}\right\}, \tag{3.44}$$

with respect to the inner product $(,)_{E_0}$ and norm $\|\cdot\|_{E_0}$.

Suppose

$$\varphi(\tau)=\left(u(\tau),v(\tau),p(\tau),q(\tau)\right)^{\mathrm{T}}\in\mathrm{A}, \tag{3.45}$$

then $\|\varphi(\tau)\|_{E_0}\le M_0,\forall s>\tau$. By $\left\|y_j\right\|_{E_0}=1$ and Lemma 4.1.1, we can get

$$-\left(p(\varphi(s))y_j(s),y_j(s)\right)_{E_0}\le-\frac{\varepsilon}{4}-\frac{\beta}{4}\left\|\nabla^m\xi_{2j}\right\|^2-\frac{\beta}{4}\left\|\nabla^m\eta_{2j}\right\|^2, \tag{3.46}$$

$$\begin{aligned}&\left(\Gamma_1(\varphi(s))y_j(s),y_j(s)\right)_{E_0}\\&=\left(-g_{1u}(u,v)\xi_{1j}-g_{1v}(u,v)\eta_{1j},\xi_{2j}\right)+\left(-g_{2u}(u,v)\xi_{1j}-g_{2v}(u,v)\eta_{1j},\eta_{2j}\right)\\&=\left(-g_{1u}(u,v)\xi_{1j},\xi_{2j}\right)+\left(-g_{1v}(u,v)\eta_{1j},\xi_{2j}\right)\\&\quad+\left(-g_{2u}(u,v)\xi_{1j},\eta_{2j}\right)+\left(-g_{2v}(u,v)\eta_{1j},\eta_{2j}\right)\end{aligned}$$

$$
\begin{aligned}
&\le \left\|A^{-\frac{m}{2}}\left(g_{1u}(u,v)\xi_{1j}\right)\right\|\left\|\nabla^m\xi_{2j}\right\|+\left\|A^{-\frac{m}{2}}\left(g_{1v}(u,v)\eta_{1j}\right)\right\|\left\|\nabla^m\xi_{2j}\right\| \\
&\quad+\left\|A^{-\frac{m}{2}}\left(g_{2u}(u,v)\xi_{1j}\right)\right\|\left\|\nabla^m\eta_{2j}\right\|+\left\|A^{-\frac{m}{2}}\left(g_{2u}(u,v)\eta_{1j}\right)\right\|\left\|\nabla^m\eta_{2j}\right\|.
\end{aligned} \tag{3.47}
$$

where $A=-\Delta$.

By the hypothesis (H4) in [1], the mean value theorem and Sobolev embedding theorem:

$$H_0^{m\kappa}(\Omega)\subset D\left(A^{\frac{m}{2}\kappa}\right)\subset H^{m\kappa}(\Omega)\subset L^q(\Omega)\subset L^2(\Omega)\subset L^{q'}(\Omega)\subset H^{-m\kappa}(\Omega), \tag{3.48}$$

where $\frac{1}{q}=\frac{1}{2}-\frac{m\kappa}{n},\frac{1}{q}+\frac{1}{q'}=1,\kappa\in[0,1)$.

Thus, by Lemma 2.4. in [1] and (3.47), for $n=1$,

$$H_0^{m\kappa}(\Omega)\subset L^\infty(\Omega)\subset L^1(\Omega)\subset H^{-m}(\Omega)\subset\left(H_0^m(\Omega)\right)'. \tag{3.49}$$

There exists $C_{21}(R_0),C_{23}(R_0),C_{25}(R_0),C_{27}(R_0)>0$, such that

$$\left\|A^{-\frac{m}{2}}\left(g_{1u}(u,v)\xi_{1j}\right)\right\|\le C_{21}\left|g_{1u}(u,v)\xi_{1j}\right|_{L^1}\le C_{22}(R_0)\left\|\xi_{1j}\right\|, \tag{3.50}$$

$$\left\|A^{-\frac{m}{2}}\left(g_{1v}(u,v)\eta_{1j}\right)\right\|\le C_{23}\left|g_{1v}(u,v)\eta_{1j}\right|_{L^1}\le C_{24}(R_0)\left\|\eta_{1j}\right\|, \tag{3.51}$$

$$\left\|A^{-\frac{m}{2}}\left(g_{2u}(u,v)\xi_{1j}\right)\right\|\le C_{25}\left|g_{2u}(u,v)\xi_{1j}\right|_{L^1}\le C_{26}(R_0)\left\|\xi_{1j}\right\|, \tag{3.52}$$

$$\left\|A^{-\frac{m}{2}}\left(g_{2v}(u,v)\eta_{1j}\right)\right\|\le C_{27}\left|g_{2v}(u,v)\eta_{1j}\right|_{L^1}\le C_{28}(R_0)\left\|\eta_{1j}\right\|. \tag{3.53}$$

For $1<n<2m$, by $H_0^m(\Omega)\subset L^q(\Omega)\subset H^{-m}(\Omega)\subset\left(H_0^m(\Omega)\right)',q>0$, there exists $C_{30}(R_0),C_{32}(R_0),C_{34}(R_0),C_{36}(R_0)>0$, such that

$$\left\|A^{-\frac{m}{2}}\left(g_{1u}(u,v)\xi_{1j}\right)\right\|\le C_{29}\left\|g_{1u}(u,v)\xi_{1j}\right\|_{L^{\frac{3}{2}}}\le C_{30}(R_0)\left\|\xi_{1j}\right\|, \tag{3.54}$$

$$\left\|A^{-\frac{m}{2}}\left(g_{1v}(u,v)\eta_{1j}\right)\right\|\le C_{31}\left\|g_{1v}(u,v)\eta_{1j}\right\|_{L^{\frac{3}{2}}}\le C_{32}(R_0)\left\|\eta_{1j}\right\|, \tag{3.55}$$

$$\left\|A^{-\frac{m}{2}}\left(g_{2u}(u,v)\xi_{1j}\right)\right\|\le C_{33}\left\|g_{2u}(u,v)\xi_{1j}\right\|_{L^{\frac{3}{2}}}\le C_{34}(R_0)\left\|\xi_{1j}\right\|, \tag{3.56}$$

$$\left\|A^{-\frac{m}{2}}\left(g_{2v}(u,v)\eta_{1j}\right)\right\|\le C_{35}\left\|g_{2v}(u,v)\eta_{1j}\right\|_{L^{\frac{3}{2}}}\le C_{36}(R_0)\left\|\eta_{1j}\right\|. \tag{3.57}$$

For $n>2m$, by (H4) in [1] there exists $C_{37}(R_1),C_{38}(R_1),C_{39}(R_1),C_{40}(R_1)>0$, such that

$$\left\|A^{-\frac{m}{2}}\left(g_{1u}(u,v)\xi_{1j}\right)\right\|\le\left\|g_{1u}(u,v)\xi_{1j}\right\|_{L^{\frac{2n}{n+2m}}}\le C_{37}(R_0)\left\|A^{\frac{m}{2}\delta}\xi_{1j}\right\|, \tag{3.58}$$

$$\left\|A^{-\frac{m}{2}}\left(g_{1v}(u,v)\eta_{1j}\right)\right\| \le \left\|g_{1v}(u,v)\eta_{1j}\right\|_{L^{\frac{2n}{n+2m1}}} \le C_{38}(R_0)\left\|A^{\frac{m}{2}\delta}\eta_{1j}\right\|, \tag{3.59}$$

$$\left\|A^{-\frac{m}{2}}\left(g_{2u}(u,v)\xi_{1j}\right)\right\| \le \left\|g_{2u}(u,v)\xi_{1j}\right\|_{L^{\frac{2n}{n+2m1}}} \le C_{39}(R_0)\left\|A^{\frac{m}{2}\delta}\xi_{1j}\right\|, \tag{3.60}$$

$$\left\|A^{-\frac{m}{2}}\left(g_{2v}(u,v)\eta_{1j}\right)\right\| \le \left\|g_{2v}(u,v)\eta_{1j}\right\|_{L^{\frac{2n}{n+2m}}} \le C_{40}(R_0)\left\|A^{\frac{m}{2}\delta}\eta_{1j}\right\|. \tag{3.61}$$

From above, we have

$$\begin{aligned}
&\left(\Gamma_1(\varphi(s))y_j(s), y_j(s)\right)_{E_0} \\
&\le \frac{C_{41}}{2}\left\|A^{\frac{m}{2}\delta}\xi_{1j}\right\|\left\|A^{\frac{m}{2}}\xi_{2j}\right\| + \frac{C_{42}}{2}\left\|A^{\frac{m}{2}\delta}\eta_{1j}\right\|\left\|A^{\frac{m}{2}}\xi_{2j}\right\| \\
&\quad + \frac{C_{43}}{2}\left\|A^{\frac{m}{2}\delta}\xi_{1j}\right\|\left\|A^{\frac{m}{2}}\eta_{2j}\right\| + \frac{C_{44}}{2}\left\|A^{\frac{m}{2}\delta}\eta_{1j}\right\|\left\|A^{\frac{m}{2}}\eta_{2j}\right\| \\
&\le C_{45}\left(\left\|A^{\frac{m}{2}\delta}\xi_{1j}\right\|\left\|A^{\frac{m}{2}}\xi_{2j}\right\| + \left\|A^{\frac{m}{2}\delta}\eta_{1j}\right\|\left\|A^{\frac{m}{2}}\xi_{2j}\right\|\right. \\
&\quad \left. + \left\|A^{\frac{m}{2}\delta}\xi_{1j}\right\|\left\|A^{\frac{m}{2}}\eta_{2j}\right\| + \left\|A^{\frac{m}{2}\delta}\eta_{1j}\right\|\left\|A^{\frac{m}{2}}\eta_{2j}\right\|\right),
\end{aligned} \tag{3.62}$$

where $C_{41} = \max\{C_{22}, C_{37}\}$, $C_{42} = \max\{C_{24}, C_{38}\}$, $C_{43} = \max\{C_{26}, C_{39}\}$, $C_{44} = \max\{C_{28}, C_{40}\}$,

$$C_{45} = \max\{C_{41}, C_{42}, C_{43}, C_{44}\}.$$

According to Lemma 2.4. in [1], we can get

$$\begin{aligned}
&\left(\Gamma_2(\varphi(s))y_j(s), y_j(s)\right) \\
&= (1-M(\upsilon))\left(\nabla^m\xi_{1j}, \nabla^m\xi_{2j}\right) - 2M'(\upsilon)\left(\left(\nabla\xi_{1j}, \nabla u\right) + \left(\nabla\eta_{1j}, \nabla v\right)\right)\left(\nabla^m u, \nabla^m\xi_{2j}\right) \\
&\quad + (1-M(\upsilon))\lambda_j^{\frac{m}{2}}\left(\nabla^m\eta_{1j}, \nabla^m\eta_{2j}\right) \\
&\quad - 2M'(\upsilon)\lambda_j^{\frac{m}{2}}\left(\left(\nabla\xi_{1j}, \nabla u\right) + \left(\nabla\eta_{1j}, \nabla v\right)\right)\left(\nabla^m v, \nabla^m\eta_{2j}\right) \\
&\le (1-\mu_0)\lambda_j^{\frac{m}{2}}\left\|\nabla^m\xi_{1j}\right\|\left\|\xi_{2j}\right\| + 2\lambda_1^{-m+1}R_0^2M'(\upsilon)\lambda_j^{\frac{m}{2}}\left(\left\|\nabla^m\xi_{1j}\right\|\left\|\xi_{2j}\right\| + \left\|\nabla^m\eta_{1j}\right\|\left\|\xi_{2j}\right\|\right) \\
&\quad + (1-\mu_0)\lambda_j^{\frac{m}{2}}\left\|\nabla^m\eta_{1j}\right\|\left\|\eta_{2j}\right\| + 2\lambda_1^{-m+1}R_0^2M'(\upsilon)\lambda_j^{\frac{m}{2}}\left(\left\|\nabla^m\xi_{1j}\right\|\left\|\eta_{2j}\right\| + \left\|\nabla^m\eta_{1j}\right\|\left\|\eta_{2j}\right\|\right) \\
&\le \frac{1-\mu_0}{2}\lambda_j^{\frac{m}{2}}\left(\left\|\nabla^m\xi_{1j}\right\|^2 + \left\|\nabla^m\eta_{1j}\right\|^2 + \left\|\xi_{2j}\right\|^2 + \left\|\eta_{2j}\right\|^2\right) \\
&\quad + \lambda_1^{-m+1}R_0^2S_0\lambda_j^{\frac{m}{2}}\left(2\left\|\nabla^m\xi_{1j}\right\|^2 + 2\left\|\xi_{2j}\right\|^2 + 2\left\|\eta_{2j}\right\|^2 + 2\left\|\nabla^m\eta_{1j}\right\|^2\right) \\
&= \frac{1-\mu_0}{2}\lambda_j^{\frac{m}{2}} + 2\lambda_1^{-m+1}R_0^2S_0\lambda_j^{\frac{m}{2}},
\end{aligned} \tag{3.63}$$

where $\mu_0 \in [0,1]$.

If $\|\xi_{2j}\|^2+\|\eta_{2j}\|^2=0$, then $P_{n_1}\le -\frac{\varepsilon}{4}n_1$, $q_{n_1}<0$.

By Lemma VI 6.3 of [8], Young's inequality and existing μ_0, s_0 satisfying $\frac{1-\mu_0+4S_0\lambda_1^{-m+1}R_0^2}{2}\lambda_j^{\frac{m}{2}}-\frac{\beta}{8}r\lambda_j^m-\frac{\varepsilon}{8}\le 0$, we obtain

if $\|\xi_{2j}\|^2+\|\eta_{2j}\|^2=r\ne 0$, then

$$\begin{aligned}
p_{n_1}(s)&=\sum_{j=1}^{n_1}\left(-p(\varphi(s))+\Gamma_1(\varphi(s))+\Gamma_2(\varphi(s))y_j(s),y_j(s)\right)_{E_0}\\
&\le\sum_{j=1}^{n_1}\left(\left(-\frac{\varepsilon}{4}-\frac{\beta}{4}\|\nabla^m\xi_{2j}\|^2-\frac{\beta}{4}\|\nabla^m\eta_{2j}\|^2+\left(\frac{8C_{45}^2}{\beta}\left\|A^{\frac{m}{2}\delta}\xi_{1j}\right\|^2+\frac{\beta}{8}\|\nabla^m\xi_{2j}\|^2\right.\right.\right.\\
&\quad\left.\left.+\frac{8C_{45}^2}{\beta}\left\|A^{\frac{m}{2}\delta}\eta_{1j}\right\|^2+\frac{\beta}{8}\|\nabla^m\eta_{2j}\|^2\right)+\frac{1-\mu_0}{2}\lambda_j^{\frac{m}{2}}+2\lambda_1^{-m+1}R_0^2S_0\lambda_j^{\frac{m}{2}}\right)\\
&=\sum_{j=1}^{n_1}\left(\frac{1-\mu_0}{2}\lambda_j^{\frac{m}{2}}+2\lambda_1^{-m+1}R_0^2S_0\lambda_j^{\frac{m}{2}}-\frac{\beta}{8}\lambda_j^{\frac{m}{2}}\left(\|\xi_{2j}\|^2+\|\eta_{2j}\|^2\right)-\frac{\varepsilon}{4}\right)+\frac{32C_{45}^2}{\beta}\sum_{j=1}^{n_1}\mu_j^{\delta-1}\\
&=\sum_{j=1}^{n_1}\left(\frac{1-\mu_0}{2}\lambda_j^{\frac{m}{2}}+2\lambda_1^{-m+1}R_0^2S_0\lambda_j^{\frac{m}{2}}-\frac{\beta}{8}\lambda_j^{\frac{m}{2}}r-\frac{\varepsilon}{4}\right)+\frac{32C_{45}^2}{\beta}\sum_{j=1}^{n_1}\mu_j^{\delta-1}\\
&\le-\frac{\varepsilon}{8}n_1+\frac{\rho}{2}\sum_{j=1}^{n_1}\mu_j^{\delta-1},
\end{aligned}\tag{3.64}$$

where $0<r\le 1,\rho=\frac{64C_{45}^2}{\beta}$.

If $\frac{\varepsilon}{4\rho}\ge\frac{1}{n_1}\sum_{j=1}^{n_1}\lambda_j^{\delta-1}$ hold, then

$$q_{n_1}=\liminf_{t\to\infty}\sup_{\tau\in R}\sup_{\Phi\subset E_0}\sup_{\varphi(\tau)\in A}\frac{1}{t}\int_\tau^{\tau+t}p_{n_1}(s)\,\mathrm{d}s\le-\rho n_1\left(\frac{\varepsilon}{4\rho}-\frac{1}{n_1}\sum_{j=1}^{n_1}\lambda_j^{\delta-1}\right)<0.\tag{3.65}$$

Thus, by Lemma 4 of (S. Zhou, 1999 [9]), we obtain (3.39) and (3.40). The proof is completed.

4. Conclusion

In this paper, we study estimation of the upper bounds of Hausdorff dimension and Fractal dimension of the global attractor for a class of Higher-order coupled Kirchhoff-type equations. In the process of research, we have avoided further restriction by taking the overall treatment of $M(s)$ item, thus making the application of this model more extensive. Although the theoretical derivation of the beam vibration model is not combined with the application in real life, it is necessary to combine with practical application to further study.

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