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Asymptotic Formulas of the Solutions and the Trace Formulas for the Polynomial Pencil of the Sturm-Liouville Operators

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Abstract

This work studies the asymptotic formulas for the solutions of the Sturm-Liouville equation with the polynomial dependence in the spectral parameter. Using these asymptotic formulas it is proved some trace formulas for the eigenvalues of a simple boundary problem generated in a finite interval by the considered Sturm-Liouville equation.

Keywords

Sturm-Liouville Equation, Asymptotic Formulas for Solutions, Spectral Parameter, Eigenvalue, Boundary Value Problem, Trace Formula, Fractional Integrals and Derivatives

1. Introduction

Consider the differential equation

$$-y'' + \sum_{k=0}^{n-1} \lambda^k q_k(x) y = \lambda^{2n} y, \quad 0 \le x \le a \tag{1}$$

where $n > 1, q_0(x) \in C[0,\pi], q_n(x) \in C^1[0,\pi]\left(k = \overline{1, n-1}\right)$ are complex valued functions and λ is a complex parameter.

Differential equations of type (1) often appear in connection with some spectral problems and nonlinear evolution equations (see [1] [2] [3]). In the case $n = 1$ the equation is the classical Sturm-Liouville equation and in this case there are a wide class of spectral problems and inverse spectral problems which were investigated by constructing integral representations for the independent solutions of the Sturm-Liouville equation (see [4]). We studied in [5], the solutions $y_j(x,\lambda)(j=1,2)$ of the Equation

(1) satisfying the initial conditions

$$y_j(0,\lambda)=1,\ y_j'(0,\lambda)=(-1)^{j+1}i\lambda^n$$

and it is proved that in the sectors of complex plane

$$S_m=\left\{\lambda:\frac{m\pi}{n}\le\arg\lambda\le\frac{(m+1)\pi}{n}\right\},\ m=\overline{0,2n-1}$$

the solutions $y_j(x,\lambda)$ have the following integral representations:

$$y_j(x,\lambda)=\mathrm{e}^{(-1)^{j+1}i\lambda^n x}\left[1+\int_{\frac{(-1)^{j+m}-1}{2}x}^{+\infty}K_{v,m}(x,t)\mathrm{e}^{(-1)^m 2i\lambda^n t}\mathrm{d}t\right] \tag{2}$$

where $v=j+\frac{1}{2}\left[(-1)^{j+m}-(-1)^j\right]$, $K_{1,m}(x,.),D_xK_{1,m}(x,.)$ and $K_{2,m}(x,.)$, $D_xK_{2,m}(x,.)$ belong to $L_1(-x;+\infty)$ and $L_1(0;+\infty)$ respectively. Moreover, if $D_{a,t}^{\alpha}\varphi(x,t)$ denotes Riemann-Liouville fractional derivative of order $\alpha(0<\alpha<1)$ (see [6]) with respect to *t*, *i.e.*

$$D_{a,t}^{\alpha}\varphi(x,t)\overset{\det}{=}\frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t(t-s)^{\alpha-1}\varphi(x,s)\mathrm{d}s$$

then for all $x\in[0,\pi]$ the functions $\left(D_{-x,t}^{\frac{1}{n}}\right)^p K_{1,n}(x,t)$ and $\left(D_{0,t}^{\frac{1}{n}}\right)^p K_{2,m}(x,t)\left(p=\overline{1,n}\right)$ belong to $L_1(-x,+\infty)$ and $L_2(0,+\infty)$ respectively. Furthermore, the following equalities are valid:

$$\begin{aligned}\int_{-x}^{+\infty}K_{1,m}(x,t)\mathrm{e}^{(-1)^m 2i\lambda^n t}\mathrm{d}t&=-\sum_{k=0}^{n-1}\gamma_{k+1}\lambda^{-k-1}\alpha_k^{(1)}(x)\\&\quad+(-1)^m\left(2i\lambda^n\right)^{-1}\int_{-x}^{+\infty}\mathrm{e}^{(-1)^m 2i\lambda^n t}\left(D_{-x,t}^{\frac{1}{n}}\right)^n K_{1,m}(x,t)\mathrm{d}t,\end{aligned} \tag{3}$$

$$\begin{aligned}\int_{0}^{+\infty}K_{2,m}(x,t)\mathrm{e}^{(-1)^m 2i\lambda^n t}\mathrm{d}t&=\sum_{k=0}^{n-1}\gamma_{k+1}\lambda^{-k-1}\alpha_k^{(2)}(x)\\&\quad+(-1)^m\left(2i\lambda^n\right)^{-1}\int_{0}^{+\infty}\mathrm{e}^{(-1)^m 2i\lambda^n t}\left(D_{0,t}^{\frac{1}{n}}\right)^n K_{2,m}(x,t)\mathrm{d}t,\end{aligned} \tag{4}$$

where

$$\gamma_k=2^{-\frac{k}{n}}\mathrm{e}^{\frac{i\pi k}{2n}},k=\overline{1,n-1},$$

$$\alpha_0^{(1)}(x)=\alpha_0^{(2)}(x)=\gamma_{n-1}\int_0^x q_{n-1}(s)\mathrm{d}s,$$

$$\alpha_k^{(j)}(x)=\gamma_{n-k-1}\int_0^x q_{n-k-1}(s)\mathrm{d}s+(-1)^j\sum_{p=1}^{k}\gamma_{n-p}\int_0^x q_{n-p}(s)\alpha_{k-p}^{(j)}(s)\mathrm{d}s,\ j=1,2;k=\overline{1,n-1}. \tag{5}$$

In the present paper we use the above facts about special solutions of the Equation (1) to obtain some trace formulas for the boundary value problem generated by the Equation (1) in the segment $[0,\pi]$ with simple boundary conditions

$$y(0)=y(\pi)=0.$$

2. Asymptotic Formulas and the Trace Formulas

Using (2), (3) and (4) it is easy to prove the following theorem where we seek two solutions which have special representations.

Theorem 1. If $q_0(x)\in C[0,a]$, $q_k(x)\in C^1[0,a]$ $\left(k=\overline{1,n-1}\right)$, and $\lambda\neq 0$ then the Equation (1) has solutions

$$f_1(x,\lambda)=\mathrm{e}^{i\lambda^n x}\left(1-\sum_{k=1}^{n-1}\frac{\gamma_k}{\lambda^k}u_{k,1}(x)-\frac{\gamma_n}{\lambda^n}u_{n,1}(x,\lambda)\right) \tag{6}$$

and

$$f_2(x,\lambda)=\mathrm{e}^{-i\lambda^n x}\left(1+\sum_{k=1}^{n-1}\frac{\gamma_k}{\lambda^k}u_{k,2}(x)+\frac{\gamma_n}{\lambda^n}u_{n,2}(x,\lambda)\right), \tag{7}$$

where

$$u_{k,j}(x)=\gamma_{n-k}\int_0^x q_{n-k}(s)\mathrm{d}s+(-1)^j\sum_{p=1}^{k-1}\gamma_{n-p}\int_0^x q_{n-p}(s)u_{k-p,j}(s)\mathrm{d}s, k=\overline{2,n-1};\ j=1,2,$$

$$u_{1,j}(x)=\gamma_{n-1}\int_0^x q_{n-1}(s)\mathrm{d}s,\ \ \gamma_k=2^{-\frac{k}{n}}\mathrm{e}^{\frac{i\pi k}{2n}},\ \ k=\overline{1,n} \tag{8}$$

$$\begin{aligned}u_{n,j}(x,\lambda)&=u_{2n-1,j}(x)+(-1)^j\sum_{k=0}^{n-1}\frac{1}{\lambda^{n-k}}\int_0^x A_{n-k,j}(x,t)\mathrm{d}t-\int_0^x u'_{2n-1,j}\mathrm{e}^{(-1)^j 2i\lambda^n t}\mathrm{d}t\\&+\sum_{k=1}^{n-1}2i\lambda^k\int_0^x u_{n+k-1,j}(x-t)\mathrm{e}^{(-1)^j 2i\lambda^n t}\mathrm{d}t+\sum_{k=0}^{n-1}2i\lambda^k\int_0^x B_{k,j}(x,t)\mathrm{e}^{(-1)^j 2i\lambda^n t}\mathrm{d}t\\&+(-1)^j\sum_{k=1}^{n-1}\frac{1}{\lambda^{n-k}}\int_0^x C_{n-k,j}(x,t)\mathrm{e}^{(-1)^j 2i\lambda^n t}\mathrm{d}t+\mathrm{e}^{\left|Im\lambda^n x\right|}o\left(\lambda^{-n}\right),\lambda\to\infty\,(j=1,2)\end{aligned}$$

$$\begin{aligned}u_{n+k-1,j}(x)&=(-1)^{j+1}\gamma_{n-k}u'_{n-k,j}(x)+(-1)^j\int_0^x\sum_{p=0}^{k-1}\gamma_{n-k+p}q_p(t)u_{n-k+p,j}(t)\mathrm{d}t,\\&\left(u_{0,j}(x)=(-1)^j\right),k=\overline{1,n},\end{aligned} \tag{9}$$

$$A_{n-k,j}(x,t)=\sum_{\nu=0}^{n-k-1}\gamma_{\nu+1}u_{n+k+\nu,j}(x-t)\frac{\partial}{\partial t}a_{\nu,j}(x,t), \tag{10}$$

$$C_{n-k,j}(x,t)=\sum_{\nu=0}^{n-k-1}\gamma_{\nu+1}u_{n+k+\nu,j}(x-t)\frac{\partial}{\partial t}a_{\nu,1+\frac{1-(-1)^j}{2}}(x,t), \tag{11}$$

$$B_{k,j}(x,t)=\sum_{\nu=0}^{n-k-1}\gamma_{\nu+1}u_{n+k+\nu,j}(x-t)\frac{\partial}{\partial t}a_{\nu,1+\frac{1-(-1)^j}{2}}(x,t),\ j=1,2, \tag{12}$$

$$a_{\nu,j}(x,t)=\gamma_{n-\nu}\int_0^t q_{n-\nu}(x-s)\mathrm{d}s+(-1)^j\sum_{k=1}^{\nu-1}\gamma_{n-k}\int_0^t q_{n-k}(x-s)a_{\nu-k,j}(x,s)\mathrm{d}s,\nu=\overline{2,n-1}$$

$$a_{1,j}(x,t)=\gamma_{n-1}\int_0^t q_{n-1}(x-s)\,\mathrm{d}s,\ j=1,2. \tag{13}$$

Since the solutions $f_1(x,\lambda)$ and $f_2(x,\lambda)$ are linearly independent for $\lambda\neq 0$ we have

$$s(x,\lambda)=\frac{1}{2i\lambda^n}\big(f_1(x,\lambda)-f_2(x,\lambda)\big)$$

for the solution $s(x,\lambda)$ of the Equation (1) with initial conditions

$$s(0,\lambda)=0, s'(0,\lambda)=1.$$

Then the Theorem 1 gives

$$\begin{aligned}
s(x,\lambda)=\frac{\sin\lambda^n x}{\lambda^n}\Bigg[&1+\sum_{k=2}^{n-1}\frac{\gamma_n d_k^-(x)}{\lambda^k}+ictg\lambda^n x\sum_{k=1}^{n-1}\frac{\gamma_k d_k^+(x)}{\lambda^k}+\frac{1}{2i\lambda^n}d_{2n-1}^-(x)\\
&-ctg\lambda^n x\frac{1}{2\lambda^n}d_{2n-1}^+(x)+\frac{1}{2i\lambda^n}\sum_{k=1}^{n}\frac{1}{\lambda^k}\int_0^x A_k^-(x,t)\,\mathrm{d}t\\
&+\frac{ctg\lambda^n x}{2\lambda^n}\sum_{k=1}^{n}\frac{1}{\lambda^k}\int_0^x A_k^+(x,t)\,\mathrm{d}t-\left(4\lambda^n\sin\lambda^n x\right)^{-1}\Bigg\{-\mathrm{e}^{i\lambda^n x}\int_0^x u'_{2n-1,1}(x-t)\mathrm{e}^{-2i\lambda^n t}\mathrm{d}t\\
&+\mathrm{e}^{i\lambda^n x}\sum_{k=1}^{n-1}2i\lambda^k\int_0^x u_{n+k-1,1}(x-t)\mathrm{e}^{-2i\lambda^n t}\mathrm{d}t+\mathrm{e}^{i\lambda^n x}\sum_{k=0}^{n-1}2i\lambda^k\int_0^x B_{k,1}(x,t)\mathrm{e}^{-2i\lambda^n t}\mathrm{d}t\\
&+\mathrm{e}^{-i\lambda^n x}\int_0^x u'_{2n-1,2}(x-t)\mathrm{e}^{2i\lambda^n t}\mathrm{d}t-\mathrm{e}^{-i\lambda^n x}\sum_{k=1}^{n-1}2i\lambda^k\int_0^x u_{n+k-1,2}(x-t)\mathrm{e}^{2i\lambda^n t}\mathrm{d}t\\
&-\mathrm{e}^{-i\lambda^n x}\sum_{k=0}^{n-1}2i\lambda^k\int_0^x B_{k,2}(x,t)\mathrm{e}^{2i\lambda^n t}\mathrm{d}t\Bigg\}+\left(\lambda^{2n}\sin\lambda^n x\right)^{-1}\mathrm{e}^{|\mathrm{Im}\,\lambda^n x|}o(1),|\lambda|\to+\infty,
\end{aligned} \tag{14}$$

where

$$d_k^{\pm}(x)=\beta^{\pm}\gamma_{n-k}\int_0^x q_{n-k}(s)\,\mathrm{d}s+\sum_{p=1}^{k-1}\gamma_{n-p}\int_0^x q_{n-p}(s)d_{k-p}^{\mp}(s)\,\mathrm{d}s, k=\overline{2,n-1} \tag{15}$$

$$\beta^+=1,\beta^-=0,$$

$$d_1^+(x)=\gamma_{n-1}\int_0^x q_{n-1}(s)\,\mathrm{d}s,\ d_1^-(x)=0, d_0^+(x)=0,\ d_0^-(x)=1.$$

$$A_{n-k}^{\pm}(x,t)=\frac{1}{2}\sum_{\nu=0}^{n-k-1}\gamma_{\nu+1}\left[d_{n+k+\nu}^-(x-t)\frac{\partial}{\partial t}a_\nu^{\pm}(x,t)-d_{n+k+\nu}^+(x-t)\frac{\partial}{\partial t}a_\nu^{\mp}(x,t)\right], \tag{16}$$

$$\begin{cases}
a_1^+(x,t)=\gamma_{n-1}\int_0^t q_{n-1}(x-s)\,\mathrm{d}s, a_1^-(x,t)=0,\\
a_\nu^+(x,t)=\gamma_{n-\nu}\int_0^x q_{n-\nu}(x-s)\,\mathrm{d}s+\sum_{k=1}^{\nu-1}\gamma_{n-k}\int_0^t q_{n-k}(s)a_{\nu-k}^-(x,s)\,\mathrm{d}s,\\
a_\nu^-(x,t)=\sum_{k=1}^{\nu-1}\gamma_{n-k}\int_0^t q_{n-k}(x-s)a_{\nu-k}^+(x,s)\,\mathrm{d}s,\ \nu=\overline{2,n-1}.
\end{cases} \tag{17}$$

Now let us connect the Equation (1) to the boundary conditions

$$y(0)=y(\pi)=0. \tag{18}$$

In [2] it is obtained the asymptotic formulas for the eigenvalues $\{\lambda_{\nu,m}\}$ of the boundary value problem (1)-(2). Let $\Delta(\lambda)=s(\pi,\lambda)$ be a characteristic function of this boundary value problem. Then

$$\Delta(\lambda)=\frac{\sin\lambda^n\pi}{\lambda^n}\left[1+r(\lambda)\right], \tag{19}$$

where

$$\begin{aligned} r(\lambda)={}&\sum_{k=2}^{n-1}\frac{\gamma_k}{\lambda_k}d_k^-(\pi)+ictg\lambda^n\pi\sum_{k=1}^{n-1}\frac{\gamma_k}{\lambda_k}d_k^+(\pi)+\frac{1}{2i\lambda^n}d_{2n-1}^-(\pi)-\frac{ctg\lambda^n\pi}{2\lambda^n}d_{2n-1}^+(\pi)\\ &+\frac{1}{2i\lambda^n}\sum_{k=1}^{n}\frac{1}{\lambda^k}\int_0^\pi A_k^-(\pi,t)\mathrm{d}t+\frac{ctg\lambda^n\pi}{2\lambda^n}\sum_{k=1}^{n}\frac{1}{\lambda^k}A_k^+(\pi,t)\mathrm{d}t-\left(4\lambda^n\sin\lambda^n\pi\right)^{-1}\\ &\times\left\{-\mathrm{e}^{i\lambda^n\pi}\int_0^\pi u'_{2n-1,1}(\pi-t)\mathrm{e}^{-2i\lambda^n t}\mathrm{d}t+\mathrm{e}^{i\lambda^n\pi}\sum_{k=1}^{n-1}2i\lambda^k\int_0^\pi u_{n+k-1,1}(\pi-t)\mathrm{e}^{-2i\lambda^n t}\mathrm{d}t\right.\\ &+\mathrm{e}^{i\lambda^n\pi}\sum_{k=1}^{n-1}2i\lambda^k\int_0^\pi B_{k,1}(\pi,t)\mathrm{e}^{-2i\lambda^n t}\mathrm{d}t+\mathrm{e}^{-i\lambda^n\pi}\int_0^\pi u'_{2n-1,2}(\pi-t)\mathrm{e}^{2i\lambda^n t}\mathrm{d}t\\ &-\mathrm{e}^{-i\lambda^n\pi}\sum_{k=1}^{n-1}2i\lambda^k\int_0^\pi u_{n+k-1,2}(\pi-t)\mathrm{e}^{2i\lambda^n t}\mathrm{d}t-\mathrm{e}^{-i\lambda^n\pi}\sum_{k=1}^{n-1}2i\lambda^k\int_0^\pi B_{k,2}(\pi,t)\mathrm{e}^{2i\lambda^n t}\mathrm{d}t\\ &+\left(\lambda^n\sin\lambda^n\pi\right)^{-1}\mathrm{e}^{\left|\operatorname{Im}\lambda^n\pi\right|}o(1),\ |\lambda|\to\infty. \end{aligned} \tag{20}$$

Let us consider the circles $D_k=\left\{\lambda:|\lambda|=\sqrt[k]{k+\frac{1}{2}},0\le\arg\lambda\le2\pi\right\}$ where k is sufficiently large integer. On circles D_k the functions $\left|ctg\lambda^n\pi\right|$ and $\left|\sin\lambda^n\pi\right|^{-1}\mathrm{e}^{\left|\operatorname{Im}\lambda^n\pi\right|}$ are bounded by the constants independent of k. So we have that the module of the maximum of the function $r(\lambda)$ approaches to zero when $k\to\infty$. Hence, if $\lambda_{k,m}$ $\left(m=\overline{0,2n-1}\right)$ are the series of eigenvalues of the problem (1), (18) we have

$$\sum_{m=0}^{2n-1}\sum_{\nu=1}^{k}\lambda_{\nu,m}^j=\frac{1}{2\pi i}\oint_{D_k}\lambda^j\mathrm{d}\ln\Delta(\lambda),\ j=\overline{1,2n}. \tag{21}$$

Using (19) and (20) we compute the integrals on the right hand side of the Equation (21) and prove the following theorem.

Theorem 2. If $\lambda_{\nu,m}$, $m=\overline{0,2n-1}$, $\nu=1,2,\cdots$ are the series of eigenvalues of the boundary value problem (1), (18) then

$$\lim_{k\to\infty}\sum_{\nu=1}^{k}\sum_{m=0}^{2n-1}\lambda_{\nu,m}=0, \tag{22}$$

$$\lim_{k\to\infty}\sum_{\nu=1}^{k}\sum_{m=0}^{2n-1}\lambda_{\nu,m}^j=C_j(\pi),\ j=\overline{1,n}, \tag{23}$$

$$\lim_{k\to\infty}\left[\sum_{\nu=1}^{k}\sum_{m=0}^{2n-1}\left\{\lambda_{\nu,m}^{n+j}-M_j^{(1)}(\pi)\right\}-\frac{1}{2in\pi}\sum_{\nu=-k}^{k}f_{j,\nu}\right]=M_j^{(2)}(\pi),\ \ j=\overline{1,n-1}, \tag{24}$$

$$\lim_{k\to\infty}\left[\sum_{\nu=1}^{k}\sum_{m=0}^{2n-1}\left\{\lambda_{\nu,m}^2-\nu^2-M_n^{(1)}(\pi)\right\}-\frac{1}{2in\pi}\sum_{\nu=-k}^{k}f_{n,\nu}\right]=M_n^{(2)}(\pi), \tag{25}$$

where $C_j(\pi)$, $j=\overline{1,n-1}$, $M_j^{(k)}(\pi)$, $j=\overline{1,n}$; $k=1,2$ are constants defined by the help of the functions $q_k(x)$, $k=\overline{0,n-1}$. Here

$$f_{1,\nu}=\int_0^\pi\left[u_{2n-2,1}(t)-u_{2n-2,2}(t)\right]\mathrm{e}^{-2i\nu t}\mathrm{d}t,$$

$$\begin{aligned}f_{j,\nu}=\int_0^\pi&\Big\{u_{2n-j-1,1}(t)+B_{n-j,1}(\pi,t)-u_{2n-j-1,2}(t)-B_{n-j,2}(\pi,t)\\&-(n+j)\sum_{s=1}^{j-2}h_{j-s}(\pi)\big[u_{2n-s-1,1}(t)+B_{n-s,1}(\pi,t)-B_{n-s,2}(\pi,t)\\&-u_{2n-s-1,2}(t)\big]\Big\}\mathrm{e}^{-2i\nu t}\mathrm{d}t,\ j=\overline{2,n-1},\end{aligned}$$

$$\begin{aligned}f_{n,\nu}=\int_0^\pi&\Big\{i\left(u'_{2n-1,2}(t)+2iB_{0,1}(\pi,t)-u'_{2n-1,1}(t)-2iB_{0,2}(\pi,t)\right)\\&+\sum_{s=1}^{n-2}2nh_{n-s}(\pi)\left[u_{2n-s-1,1}(t)+B_{n-s,1}(\pi,t)-u_{2n-s-1,2}(t)-B_{n-s,2}(\pi,t)\right]\Big\}\mathrm{e}^{-2i\nu t}\mathrm{e}t\end{aligned}\tag{26}$$

in which the numbers $h_k(\pi)(k=2,\cdots,j)$ are defined from the asymptotic equality

$$\frac{1}{1+\sum_{k=2}^{n-1}\frac{\gamma_k}{\lambda^k}d_k^-(\pi)}=1-\sum_{k=2}^{j}\frac{h_k(\pi)}{\lambda^k}+o\left(\lambda^{-j}\right),|\lambda|\to\infty.$$

From Theorem 2 we have that if the Fourier series $\sum_{\nu=-\infty}^{+\infty}f_{j,\nu}$ $\left(j=\overline{1,n}\right)$ are convergent and denoting their sums by $F_j(+0)-F_j(\pi-0)$ we obtain the following regularized trace formulas for the eigenvalues of the boundary value problem (1), (18):

$$\sum_{k=1}^{\infty}\sum_{m=0}^{2n-1}\lambda_{k,m}=0,$$

$$\sum_{k=1}^{\infty}\sum_{m=0}^{2n-1}\lambda_{k,m}^{j}=C_j(\pi),\ j=\overline{1,n-1},$$

$$\sum_{k=1}^{\infty}\sum_{m=0}^{2n-1}\left\{\lambda_{k,m}^{n+j}-M_j^{(1)}(\pi)\right\}=M_j^{(2)}(\pi)+\frac{1}{2in\pi}\left[F_j(+0)-F_j(\pi-0)\right],\ j=\overline{1,n-1},$$

$$\sum_{k=1}^{\infty}\sum_{m=0}^{2n-1}\left\{\lambda_{k,m}^{2n}-k^2-M_n^{(1)}(\pi)\right\}=M_n^{(2)}((\pi))+\frac{1}{2in\pi}\left[F_n(+0)-F_n(\pi-0)\right].$$

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