Advances in Pure Mathematics, 2016, 6, 676-694
http://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

Freidlin-Wentzell's Large Deviations for Stochastic Evolution Equations with Poisson Jumps

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How to cite this paper: Zhao, H.Y. and Xu, S.Y. (2016) Freidlin-Wentzell's Large Deviations for Stochastic Evolution Equations with Poisson Jumps. *Advances in Pure Mathematics*, **6**, 676-694.
http://dx.doi.org/10.4236/apm.2016.610056

Received: August 4, 2016
Accepted: September 16, 2016
Published: September 19, 2016

Abstract

We establish a Freidlin-Wentzell's large deviation principle for general stochastic evolution equations with Poisson jumps and small multiplicative noises by using weak convergence method.

Keywords

Stochastic Evolution Equation, Poisson Jumps, Freidlin-Wentzell's Large Deviation, Weak Convergence Method

1. Introduction

The weak convergence method of proving a large deviation principle has been developed by Dupuis and Ellis in [1]. The main idea is to get sevral variational representation formulas for the Laplace transform of certain functionals, and then to prove an equivalence between Laplace principle and large deviation principle (LDP). For Brownian functionals, Boué and Dupuis [2] have proved an elegant variational representation formula (also can be found in Zhang [3]). For Poisson functionals, we can see Zhang [4]. Recently, a variational representation formula on Wiener-Poisson space has been established by Budhiraja, Dupuis, and Maroulas in [5]. These type variational representations have been proved to be very effective for both finite-dimensional and infinite-dimensional stochastic dynamical systems (cf. [6]-[10]). The main advantages of this method are that we only have to make some necessary moment estimates.

However, there are still few results on the large deviation for stochastic evolution equations with jumps. In [11], Röckner and Zhang considered the following type semi-linear stochastic evolutions driven by Lévy processes

$$\begin{cases} \mathrm{d}Z_t^\epsilon = AZ_t^\epsilon \mathrm{d}t + b\mathrm{d}t + \sqrt{\epsilon}\mathrm{d}W_t + \int_{\mathbb{X}} f(x)\left(\epsilon N^{\epsilon^{-1}}(\mathrm{d}x,\mathrm{d}t) - \nu(\mathrm{d}x)\mathrm{d}t\right), \epsilon \in (0,1], \\ Z_0^\epsilon = z \in H, \end{cases}$$

they established the LDP by proving some exponential integrability on different spaces. Later, Budhiraja, Chen and Dupuis developed a large deviation for small Poisson perturbations of a more general class of deterministic equations in infinite dimensional ([12]), but they did not consider the small Brownian perturbations simultaneously.

Motivated by the above work, we would like to prove a Freidlin-Wentzell's large deviation for nonlinear stochastic evolution equations with Poisson jumps and Brownian motions. At the same time, nonlinear stochastic evolution equations have been studied in various literatures (cf. [13]-[17]). So we consider the following stochastic evolution equation:

$$\begin{cases} \mathrm{d}Z_t^\epsilon = A(t,Z_t^\epsilon)\mathrm{d}t + \sqrt{\epsilon}B(t,Z_t^\epsilon)\mathrm{d}W_t + \int_{\mathbb{X}} f(t,Z_{t-}^\epsilon,x)\left(\epsilon N^{\epsilon^{-1}}(\mathrm{d}x,\mathrm{d}t) - \nu(\mathrm{d}x)\mathrm{d}t\right), \epsilon \in (0,1], \\ Z_0^\epsilon = z \in H, \end{cases}$$

in the framework of a Gelfand's triple:

$$V \subset H \equiv H^* \subset V^*$$

where V, H (see Section 2) are separable Banach and separable Hilbert space respectively. We will establish LDP for solutions of above evolution equation on $D([0,T];H) \cap L^2([0,T];V)$, where $D([0,T];H)$ is H-valued cádlág function space with the Skorokhod topology. For stochastic evolution equations without jumps, Ren and Zhang [9] and Liu [8] achieved the LDP on $C([0,T];H) \cap L^q([0,T];\mathrm{d}t)$ $(q \geq 2)$ and $C([0,T];H) \cap L^q([0,T];\mathrm{d}t)$ $(q > 1)$ respectively. In our case, there are two new difficulties. The first one is to find a sufficient condition to characterize a compact set in $D([0,T];V^*)$ (see Proposition 4) instead of Ascoli-Arzelà's theorem for continuous case, the second one is to control the jump parts. This form of equation contains a large class of (nonliear) stochastic partial differential equation of evolutional type, for applications and examples we refer the reader to [8], [9]. The equations we consider here are more general than the equations considered in [11], and we use a different method. We note that, the large deviations for semilinear SPDEs in the sense of mild solutions were considered in paper [18] recently. For other recent research on this topic, see also [12], [19].

In Section 2, we firstly give some notations and recall some results from [5], which are the basis of our paper, and then introduce our framework. In Section 3, we prove the large deviation principle. In the last section, we give an application. Note that notations c, C_M and $C_{T,M}$ below will only denote positive constants whose values may vary from line to line.

2. Preliminaries and Framework

We first recall some notations from [5].

Let $\mathbb{X}$ be a locally compact Polish space and denote by $\mathcal{M}(\mathbb{X})$ the space of all measures ν on $(\mathbb{X},\mathcal{B}(\mathbb{X}))$, satisfying $\nu(\Gamma)<\infty$ for every compact $\Gamma\subset\mathbb{X}$. Let $C_c(\mathbb{X})$ be the space of continuous functions with compact support. $\mathcal{M}(\mathbb{X})$ is a Polish space endowed with the weakest topology such that for every $f\in C_c(\mathbb{X})$, $\mathcal{M}(\mathbb{X})\ni\nu\to\langle f,\nu\rangle:=\int_{\mathbb{X}} f(u)\nu(\mathrm{d}u)$ is a continuous function.

Set $\mathbb{Y}:=\mathbb{X}\times[0,\infty)$. Fix $T\in(0,\infty)$ and let $\mathbb{Y}_T:=[0,T]\times\mathbb{Y}$. Let $\mathbb{M}:=\mathcal{M}(\mathbb{Y}_T)$ and denote by P_1 the unique probability measure on $(\mathbb{M},\mathcal{B}(\mathbb{M}))$ such that the canonical map, $\tilde{N}:\mathbb{N}\to\mathbb{M}$, $\tilde{N}(m):=m$, is a Poisson random measure with intensity $\nu_T:=\lambda_T\otimes\nu\otimes\lambda_\infty$, where $\nu\in\mathcal{M}(\mathbb{X})$, λ_T and λ_∞ are Lebesgue measures on $[0,T]$ and $[0,\infty)$ respectively.

Let G be a real separable Hilbert space and let Q be a positive definite and symmetric trace operator defined on G. Set $\mathbb{W}:=C([0,T];G)$ and $\Omega:=\mathbb{W}\times\mathbb{M}$. Let $N:\Omega\to\mathbb{M}$ be defined by $N(w,m)=m$, for $(w,m)\in\Omega$. Let W be the coordinate map on Ω defined as $W(w,m):=w$. Define $\mathcal{G}_t:=\sigma\{N((0,s]\times A),W_s:0\le s\le t,A\in\mathcal{B}(\mathbb{Y})\}$. We denote by P the unique probability measure on $(\Omega,\mathcal{B}(\Omega))$ such that under P:

1) W is a Q-Wiener process;

2) N is a Poisson random measure with intensity measure ν_T;

3) $\{W_t,t\in[0,T]\}$, $\{N([0,t],A),t\in[0,T]\}$ are $\mathcal{G}_t$-martingales for every $A\in\mathcal{B}(\mathbb{X})$.

We denote by $\mathcal{F}_t$ be P-completion of the filtration $\mathcal{G}_t$. From now on, we will work on the probability space $(\Omega,\mathcal{B}(\Omega),P)$ with filtration $\{\mathcal{F}_t,0\le t\le T\}$.

Denote by $\mathcal{P}$ the predictable σ-field on $[0,T]\times\Omega$ with the filtration $\{\mathcal{F}_t:0\le t\le T\}$ on $(\Omega,\mathcal{B}(\Omega))$. Let $\mathcal{A}:=\{f:[0,T]\times\mathbb{X}\times\Omega\to[0,\infty),\ f \text{ is } \mathcal{P}\otimes\mathcal{B}(\mathbb{X})\backslash\mathcal{B}([0,\infty)) \text{ measurable}\}$. $\forall\varphi\in\mathcal{A}$, define

$$L_T(\varphi):=\int_{[0,T]\times\mathbb{X}} l(\varphi(t,x,\omega))\mathrm{d}t\nu(\mathrm{d}x),$$

where

$$l(r):=r\log(r)-r+1,\quad r\in[0,\infty)$$

and define a counting process N^φ as

$$N^\varphi((0,t]\times U):=\int_{(0,t]\times U}\int_{(0,\infty)}1_{[0,\varphi(s,x)]}(r)N(\mathrm{d}s,\mathrm{d}x,\mathrm{d}r),\quad t\in[0,T],U\in\mathcal{B}(\mathbb{X}).$$

For fixed $M\in\mathbb{N}$, let

$$\tilde{S}_M:=\{g:[0,T]\times\mathbb{X}\to[0,\infty):L_T(g)\le M\}. \tag{1}$$

By [5], we can define $\nu_T^g(A):=\int_A g(s,x)\nu(\mathrm{d}x)\mathrm{d}s$, $A\in\mathcal{B}([0,T]\times\mathbb{X})$ for a function $g\in\tilde{S}_M$, and identify g with measure ν_T^g. Besides, $\{\nu_T^g:g\in\tilde{S}_M\}$ is a compact subset of $\mathcal{M}([0,T]\times\mathbb{X})$ through the superlinear groth of l. We can also consider the topology on $\tilde{S}_M$ which makes $\tilde{S}_M$ a compact space.

Remark 1. *We note that, for* g_n, $g\in\tilde{S}_M$, $g_n\to g$ *in this topology means* $\nu_T^{g_n}\to\nu_T^g$, *that is, for any* $f\in C_c([0,T]\times\mathbb{X};\mathbb{R})$, $\left|\int_0^T\int_{\mathbb{X}} f(s,x)(g_n(s,x)-g(s,x))\nu(\mathrm{d}x)\mathrm{d}s\right|\to 0$ *holds as* $n\to\infty$.

Set $G_Q := Q^{1/2}(G)$ and define $\|v\|_{G_Q} := \left\|\left(Q^{-1}\right)^{1/2} v\right\|_G$. Let

$$\mathcal{P}_2 := \left\{\psi : \mathcal{P}\setminus\mathcal{B}(G_Q) \text{ measurable and } \int_0^T \|\psi(s)\|_{G_Q}^2 \mathrm{d}s < \infty \ \ a.s.\ P\right\}$$

$$\bar{S}_M := \left\{h \in L^2\left([0,T];G_Q\right) : \frac{1}{2}\int_0^T \|h(s)\|_{G_Q}^2 \mathrm{d}s \le M\right\}. \tag{2}$$

We endow $\bar{S}_M$ with the weak topology on the Hilbert space such that $\bar{S}_M$ is a compact subset of $L^2\left([0,T];G_Q\right)$.

Let $S_M := \bar{S}_M \times \tilde{S}_M$ with the usual product topology. Set $\mathcal{U} := \mathcal{P}_2 \times \mathcal{A}$ and let $\mathcal{U}_M$ be the space of S_M-valued controls:

$$\mathcal{U}_M := \left\{u = (\psi,\varphi) \in \mathcal{U} : u(\omega) \in S_M, P\ \ a.e.\ \omega\right\}.$$

Let $\mathbb{D}$ be a Polish space and let $\left\{X^\epsilon\right\}_{\epsilon>0}$ be a set of $\mathbb{D}$-valued random variables defined on $\left(\Omega,\mathcal{B}(\Omega),P\right)$ by

$$X^\epsilon := \mathcal{G}^\epsilon\left(\sqrt{\epsilon}W, \epsilon N^{\epsilon^{-1}}\right),$$

where $\left\{\mathcal{G}^\epsilon\right\}_{\epsilon>0}$ is a family of measurable maps from Ω to $\mathbb{D}$.

Hypothesis. There exists a measurable map $\mathcal{G}^0 : \Omega \to \mathbb{D}$ such that the following hold.

1) For $M \in \mathbb{N}$, if a family $\left\{u_\epsilon = (\psi_\epsilon,\varphi_\epsilon), \epsilon \in (0,1)\right\} \subset \mathcal{U}_M$ converges in distribution to $u = (\psi,\varphi) \in \mathcal{U}_M$, then

$$\mathcal{G}^\epsilon\left(\sqrt{\epsilon}W + \int_0^\cdot \psi_\epsilon(s)\mathrm{d}s, \epsilon N^{\epsilon^{-1}\varphi_\epsilon}\right) \Rightarrow \mathcal{G}^0\left(\int_0^\cdot \psi(s)\mathrm{d}s, \nu_T^\varphi\right),$$

where $\Rightarrow$ denotes the weak convergence.

2) For $M \in \mathbb{N}$, let $(h_m,g_m),(h,g) \in S_M$ be such that $(h_m,g_m) \to (h,g)$. Then

$$\mathcal{G}^0\left(\int_0^\cdot h_m(s)\mathrm{d}s, \nu_T^{g_m}\right) \to \mathcal{G}^0\left(\int_0^\cdot h(s)\mathrm{d}s, \nu_T^g\right).$$

For $\phi \in \mathbb{D}$, define $S_\phi := \left\{(h,g) \in S : \phi = \mathcal{G}^0\left(\int_0^\cdot h(s)\mathrm{d}s, \nu_T^g\right)\right\}$. Let $I : \mathbb{D} \to [0,\infty]$ be

$$I(\phi) = \inf_{(h,g)\in S_\phi}\left\{\frac{1}{2}\int_0^T \|h(s)\|_{G_Q}^2 \mathrm{d}s + L_T(g)\right\}, \tag{3}$$

where $\inf \varnothing := \infty$.

We have the following important result due to [5].

Theorem 2. *Under the above* ***Hypothesis***, $\left\{X^\epsilon\right\}_{\epsilon>0}$ *satisfies a large deviation principle with rate function I.*

Now we introduce our framework and assumptions.

Let $\left(H,\langle\cdot,\cdot\rangle_H\right)$ be a real separable Hilbert space. Let V be a reflexive Banach space and V^* be the dual space of V and ${}_{V^*}\langle\cdot,\cdot\rangle_V$ denotes the corresponding dualization. Identify H with its dual H^* and the following assumptions are satisfied:

1) $V \subset H \equiv H^* \subset V^*$;

2) V is dense in H;

3) there exists a constant c such that for all $v \in V$, $\|v\|_H \le c\|v\|_V$;

4) ${}_{V^*}\langle\cdot,\cdot\rangle_V\big|_{H\times V} = \langle\cdot,\cdot\rangle_H$.

Let $L_2(G;H)$ be the space of Hilbert-Schmidt linear operators from G to H, which is a real separable Hilbert space with the inner product

$$\langle B_1, B_2\rangle_{L_2(G;H)} := \sum_{i\ge 1}\langle B_1 g_i, B_2 g_i\rangle_H ,$$

where $\{g_i\}$ is an orthonormal basis of G. We denote by $L_Q(G;H)$ the set of all linear operators C mapping $Q^{1/2}G$ into H such that $CQ^{1/2}\in L_2(G;H)$, and the norm $\|C\|_{L_Q} := \|CQ^{1/2}\|_{L_2}$.

Let

$$A:[0,T]\times V\times\Omega\to V^*,$$

$$B:[0,T]\times V\times\Omega\to L_Q(G;H),$$

$$f:[0,T]\times V\times\mathbb{X}\times\Omega\to V$$

be progressively measurable. For example, for every $t\in[0,T]$, A restricted to $[0,T]\times V\times\Omega$ is $\mathcal{B}([0,T])\otimes\mathcal{B}(V)\otimes\mathcal{F}_t$-measurable.

We assume throughout this paper that:

(H1) Hermicontinuity: For any $u,v,x\in V$, $\omega\in\Omega$ and any $t\in[0,T]$, the mapping

$$[0,1]\ni\varepsilon\mapsto {}_{V^*}\langle A(t,u+\varepsilon v,\omega),x\rangle_V$$

is continuous.

(H2) Weak monotonicity: There exist $\lambda_0,\lambda_1\ge 0$ such that for all $u,v\in V$

$$2\,{}_{V^*}\langle A(\cdot,u)-A(\cdot,v),u-v_V\rangle \le -\lambda_1\|u-v\|_V^2+\lambda_0\|u-v\|_H^2$$

holds on $[0,T]\times\Omega$.

(H3) Coercivity: For all $v\in V$ and $t\in[0,T]$, there exist $c_1,c_2>0$ such that

$$2\,{}_{V^*}\langle A(t,v),v\rangle_V \le c_1\|v\|_H^2-c_2\|v\|_V^2$$

holds on Ω.

(H4) For all $t\in[0,T]$ and $u\in V$, there exists $c_3\ge 0$ such that

$$\|A(t,u)\|_{V^*}\le c_3\left(1+\|u\|_V\right)$$

holds on Ω.

(H5) There exists $c_4>0$ such that for all $u,v\in V$, $x\in\mathbb{X}$ and $t\in[0,T]$

$$\|B(t,u)-B(t,v)\|_{L_Q}\le c_4\|u-v\|_H ,$$

$$\|f(t,u,x)-f(t,v,x)\|_H\le c_4\|u-v\|_H ,$$

$$\|B(t,u)\|_{L_Q}\le c_4\left(1+\|u\|_H\right),$$

and

$$\|f(t,u,x)\|_H\le c_4\left(1+\|u\|_{V^*}\right). \tag{4}$$

(H6) There exist some compact $\Gamma\subset\mathbb{X}$, $f(t,u,x)=0$, for all $(t,u,x)\in[0,T]\times V\times\Gamma^c$. For any $u\in V$, $f(\cdot,u,\cdot)$ is continuous on $[0,T]\times\Gamma$.

(H7) $V \hookrightarrow H$ compactly.

3. Large Deviation Principle

Consider small noise stochastic evolution equation as following:

$$\begin{cases} \mathrm{d}Z_t^\epsilon = A\left(t, Z_t^\epsilon\right)\mathrm{d}t + \sqrt{\varepsilon}B\left(t, Z_t^\epsilon\right)\mathrm{d}W_t + \int_{\mathbb{X}} f\left(t, Z_{t-}^\epsilon, x\right)\left(\epsilon N^{\epsilon^{-1}}\left(\mathrm{d}x, \mathrm{d}t\right) - \nu\left(\mathrm{d}x\right)\mathrm{d}t\right), \epsilon \in (0,1], \\ Z_0^\epsilon = z \in H. \end{cases} \tag{5}$$

Under the assumptions **(H1)-(H5)**, by [15], [17], there exists a unique solution in $D([0,T];H) \cap L^2([0,T];V)$ to Equation (5). By Yamada-Watanabe theorem, there exists a measurable mapping $\mathcal{G}^\epsilon : \Omega \to D([0,T];H) \cap L^2([0,T];V)$ such that

$$Z^\epsilon = \mathcal{G}^\epsilon\left(\sqrt{\epsilon}W, \epsilon N^{\epsilon^{-1}}\right).$$

We now fix a family of processes $(\psi_\epsilon, \varphi_\epsilon) \in \mathcal{U}_M$, and put

$$\tilde{Z}^\epsilon = \mathcal{G}^\epsilon\left(\sqrt{\epsilon}W + \int_0^\cdot \psi_\epsilon(s)\mathrm{d}s, \epsilon N^{\epsilon^{-1}\varphi_\epsilon}\right).$$

By Girsanov's theorem, $\tilde{Z}^\epsilon$ is the unique solution of the following controlled stochastic evolution equation:

$$\begin{cases} \mathrm{d}\tilde{Z}_t^\epsilon = \left(A\left(t, \tilde{Z}_t^\epsilon\right) + B\left(t, \tilde{Z}_t^\epsilon\right)\psi_\epsilon(t)\right)\mathrm{d}t + \sqrt{\epsilon}B\left(t, \tilde{Z}_t^\epsilon\right)\mathrm{d}W_t \\ \qquad + \int_{\mathbb{X}} f\left(t, \tilde{Z}_{t-}^\epsilon, x\right)\left(\epsilon N^{\epsilon^{-1}\varphi_\epsilon}\left(\mathrm{d}x, \mathrm{d}t\right) - \nu\left(\mathrm{d}x\right)\mathrm{d}t\right), \epsilon \in (0,1], \\ \tilde{Z}_0^\epsilon = z \in H. \end{cases} \tag{6}$$

Remark 3. *For* $(\psi_\epsilon, \varphi_\epsilon) \in \mathcal{U}_M$, *by* (1) *and* (2), *there exists a constant* $C_M > 0$ *such that for all* $\epsilon \in (0,1]$,

$$\int_0^T \left\|\psi_\epsilon(s)\right\|_{G_Q}^2 \mathrm{d}s + \int_0^T \int_{\mathbb{X}} \phi_\epsilon(s, x)\nu(\mathrm{d}x)\mathrm{d}s < C_M \quad a.s.. \tag{7}$$

We will verify that $\mathcal{G}^\epsilon$ satisfies the **Hypothesis** with $\mathbb{D}$ replaced by $D([0,T];H) \cap L^2([0,T];V)$. By using the similar method as in [9], we have the following uniform estimates about $\tilde{Z}^\epsilon$.

Lemma 1. *There exists a constant* $c_{T,M} > 0$ *such that, for all* $\epsilon \in (0,1]$,

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left\|\tilde{Z}_s^\epsilon\right\|_H^2\right) + \mathbb{E}\int_0^T \left\|\tilde{Z}_s^\epsilon\right\|_V^2 \mathrm{d}s \le c_{T,M}\left(\left\|z\right\|_H^2 + 1\right), \tag{8}$$

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left\|\tilde{Z}_s^\epsilon\right\|_H^4\right) \le c_{T,M}\left(\left\|z\right\|_H^4 + 1\right). \tag{9}$$

In order to characterize a compact set in $D([0,T];V^*)$, we need the following lemma.

Lemma 2. *For any* $\delta > 0$ *and* $\eta > 0$, *there exist* $\epsilon_0 > 0$ *and* $\theta > 0$ *such that for any* $\epsilon \le \epsilon_0$, *we have*

$$P\left(\sup_{t,s\in[0,T],|t-s|<\theta}\left\|\tilde{Z}_t^\epsilon - \tilde{Z}_s^\epsilon\right\|_{V^*} > \eta\right) \le \delta \tag{10}$$

Proof. For fixed $\theta > 0$ and any t such that $0 \le t \le t+\theta \le T$, we have

$$\begin{aligned}\tilde{Z}^{\epsilon}_{t+\theta} - \tilde{Z}^{\epsilon}_{t} &= \int_t^{t+\theta} \left(A\left(s, \tilde{Z}^{\epsilon}_s\right) + B\left(s, \tilde{Z}^{\epsilon}_s\right)\psi_{\epsilon}(s)\right)\mathrm{d}s + \int_t^{t+\theta} \sqrt{\epsilon} B\left(s, \tilde{Z}^{\epsilon}_s\right)\mathrm{d}W_s \\ &\quad + \int_t^{t+\theta}\int_{\mathbb{X}} f\left(s, \tilde{Z}^{\epsilon}_{s-}, x\right)\left(\epsilon N^{\epsilon^{-1}\varphi_{\epsilon}}(\mathrm{d}s,\mathrm{d}x) - \nu(\mathrm{d}x)\mathrm{d}s\right). \\ &= \int_t^{t+\theta} \left(A\left(s, \tilde{Z}^{\epsilon}_s\right) + B\left(s, \tilde{Z}^{\epsilon}_s\right)\psi_{\epsilon}(s) + \int_{\mathbb{X}} f\left(s, \tilde{Z}^{\epsilon}_{s-}, x\right)\left(\varphi_{\epsilon}(s,x) - 1\right)\nu(\mathrm{d}x)\right)\mathrm{d}s \\ &\quad + \int_t^{t+\theta} \sqrt{\epsilon} B\left(s, \tilde{Z}^{\epsilon}_s\right)\mathrm{d}W_s + \int_t^{t+\theta}\int_{\mathbb{X}} f\left(s, \tilde{Z}^{\epsilon}_{s-}, x\right)\left(\epsilon N^{\epsilon^{-1}\varphi_{\epsilon}}(\mathrm{d}s,\mathrm{d}x) - \varphi_{\epsilon}(s,x)\mathrm{d}s\nu(\mathrm{d}x)\right).\end{aligned}$$

Therefore

$$\mathbb{E}\left(\sup_{t:0\le t\le t+\theta\le T} \left\|\tilde{Z}^{\epsilon}_{t+\theta} - \tilde{Z}^{\epsilon}_{t}\right\|^2_{V^*}\right) \le 3\left(I_1 + I_2 + I_3\right),$$

where

$$I_1 := \mathbb{E}\left(\sup_{t:0\le t\le t+\theta\le T} \left\| \int_t^{t+\theta} \left(A\left(s, \tilde{Z}^{\epsilon}_s\right) + B\left(s, \tilde{Z}^{\epsilon}_s\right)\psi_{\epsilon}(s) + \int_{\mathbb{X}} f\left(s, \tilde{Z}^{\epsilon}_{s-}, x\right)\left(\varphi_{\epsilon}(s,x) - 1\right)\nu(\mathrm{d}x)\right)\mathrm{d}s\right\|^2_{V^*}\right),$$

$$I_2 := \mathbb{E}\left(\sup_{t:0\le t\le t+\theta\le T} \left\| \int_t^{t+\theta} \sqrt{\epsilon} B\left(s, \tilde{Z}^{\epsilon}_s\right)\mathrm{d}W_s\right\|^2_{V^*}\right),$$

$$I_3 := \mathbb{E}\left(\sup_{t:0\le t\le t+\theta\le T} \left\| \int_t^{t+\theta}\int_{\mathbb{X}} f\left(s, \tilde{Z}^{\epsilon}_{s-}, x\right)\left(\epsilon N^{\epsilon^{-1}\varphi_{\epsilon}}(\mathrm{d}s,\mathrm{d}x) - \varphi_{\epsilon}(s,x)\mathrm{d}s\nu(\mathrm{d}x)\right)\right\|^2_{V^*}\right).$$

For I_1, by **(H4)**, Hölder's inequality and Lemma 1, we have

$$\begin{aligned} I_1 &\le c\mathbb{E}\left(\sup_{t:0\le t\le t+\theta\le T} \int_t^{t+\theta}\left\|A\left(s, \tilde{Z}^{\epsilon}_s\right)\right\|_{V^*}\mathrm{d}s\right)^2 + c\mathbb{E}\left(\sup_{t:0\le t\le t+\theta\le T}\left\|\int_t^{t+\theta} B\left(s, \tilde{Z}^{\epsilon}_s\right)\psi_{\epsilon}(s)\mathrm{d}s\right\|^2_H\right) \\ &\quad + c\mathbb{E}\left(\sup_{t:0\le t\le t+\theta\le T}\left\|\int_t^{t+\theta}\int_{\mathbb{X}} f\left(s, \tilde{Z}^{\epsilon}_s, x\right)\left(\varphi_{\epsilon}(s,x) - 1\right)\nu(\mathrm{d}x)\mathrm{d}s\right\|^2_H\right) \\ &\le c\mathbb{E}\left(\sup_{t:0\le t\le t+\theta\le T}\int_t^{t+\theta}\left(1 + \left\|\tilde{Z}^{\epsilon}_s\right\|_V\right)\mathrm{d}s\right)^2 + c\mathbb{E}\left(\sup_{t:0\le t\le t+\theta\le T}\int_t^{t+\theta}\left\|B\left(s, \tilde{Z}^{\epsilon}_s\right)\right\|^2_{L_Q}\mathrm{d}s\int_0^T\left\|\psi_{\epsilon}(s)\right\|^2_{U_Q}\mathrm{d}s\right) \\ &\quad + c\nu(\Gamma)\mathbb{E}\left(\sup_{t:0\le t\le t+\theta\le T}\int_t^{t+\theta}\int_{\Gamma}\left\|f\left(s, \tilde{Z}^{\epsilon}_s, x\right)\right\|^2_H\nu(\mathrm{d}x)\mathrm{d}s\right) \\ &\quad + c\mathbb{E}\left(\sup_{t:0\le t\le t+\theta\le T}\int_t^{t+\theta}\int_{\mathbb{X}}\left\|f\left(s, \tilde{Z}^{\epsilon}_s, x\right)\right\|^2_H\varphi_{\epsilon}(s,x)\nu(\mathrm{d}x)\mathrm{d}s\int_t^{t+\theta}\int_{\mathbb{X}}\varphi_{\epsilon}(s,x)\nu(\mathrm{d}x)\mathrm{d}s\right) \\ &\le c\mathbb{E}\left(2\theta^2 + 2\theta\int_0^T\left\|\tilde{Z}^{\epsilon}_s\right\|^2_V\mathrm{d}s\right) + c\theta\mathbb{E}\left(1 + \sup_{s\in[0,T]}\left\|\tilde{Z}^{\epsilon}_s\right\|^2_H\right) + c\nu(\Gamma)^2\theta^2\mathbb{E}\left(1 + \sup_{s\in[0,T]}\left\|\tilde{Z}^{\epsilon}_s\right\|^2_H\right) \\ &\quad + c\mathbb{E}\left(\left(1 + \sup_{s\in[0,T]}\left\|\tilde{Z}^{\epsilon}_s\right\|^2_H\right)\sup_{t:0\le t\le t+\theta\le T}\int_t^{t+\theta}\int_{\mathbb{X}}\varphi_{\epsilon}(s,x)\nu(\mathrm{d}x)\mathrm{d}s\right) \\ &\le c\theta^2 + c\theta + cJ_{\theta}, \end{aligned}$$

where

$$J_{\theta} := \mathbb{E}\left(\left(1 + \sup_{s\in[0,T]}\left\|\tilde{Z}^{\epsilon}_s\right\|^2_H\right)\sup_{t:0\le t\le t+\theta\le T}\int_t^{t+\theta}\int_{\mathbb{X}}\varphi_{\epsilon}(s,x)\nu(\mathrm{d}x)\mathrm{d}s\right).$$

By (7), we have

$$\sup_{\epsilon\in(0,1]}\sup_{t:0\le t\le t+\theta\le T}\int_t^{t+\theta}\int_{\mathbb{X}}\varphi_{\epsilon}(s,x)\nu(\mathrm{d}x)\mathrm{d}s \to 0,\ a.s.,\quad \text{as } \theta\downarrow 0.$$

So by (9) and dominated convergence theorem, for all $\epsilon \in (0,1]$, we obtain

$$J_\theta \to 0, \quad \text{as } \theta \downarrow 0.$$

For I_2, I_3, by BDG's inequality, **(H5)** and Lemma 1, we obtain

$$I_2 \le 2\epsilon \mathbb{E}\int_0^T \left\| B\left(s, \tilde{Z}_s^\epsilon\right)\right\|_{L_Q}^2 \mathrm{d}s \le c \cdot \epsilon \mathbb{E}\int_0^T \left(1+\left\|\tilde{Z}_s^\epsilon\right\|_H^2\right)\mathrm{d}s \le c\epsilon$$

and

$$\begin{aligned} I_3 &\le 2\mathbb{E}\left(\left\|\int_0^T\int_{\mathbb{X}} f\left(s,\tilde{Z}_{s-}^\epsilon,x\right)\left(\epsilon N^{\epsilon^{-1}\varphi_\epsilon}(\mathrm{d}s,\mathrm{d}x)-\varphi_\epsilon(s,x)\mathrm{d}s\nu(\mathrm{d}x)\right)\right\|_{V^*}^2\right) \\ &\le \epsilon\mathbb{E}\int_0^T\int_{\mathbb{X}}\left\| f\left(s,\tilde{Z}_{s-}^\epsilon,x\right)\right\|^2 \varphi_\epsilon(s,x)\mathrm{d}s\nu(\mathrm{d}x) \\ &\le \epsilon\mathbb{E}\int_0^T\int_{\mathbb{X}}\left(1+\left\|\tilde{Z}_s^\epsilon\right\|_H^2\right)\varphi_\epsilon(s,x)\mathrm{d}s\nu(\mathrm{d}x) \\ &\le \epsilon\mathbb{E}\int_0^T\int_{\mathbb{X}}\left(1+\left\|\tilde{Z}_s^\epsilon\right\|_H^2\right)\varphi_\epsilon(s,x)\mathrm{d}s\nu(\mathrm{d}x)+\epsilon\mathbb{E}\left(\sup_{s\in[0,T]}\left\|\tilde{Z}_s^\epsilon\right\|_H^2\int_0^T\int_{\mathbb{X}}\varphi_\epsilon(s,x)\mathrm{d}s\nu(\mathrm{d}x)\right) \\ &\le c\epsilon. \end{aligned}$$

Hence, for any $\eta,\delta>0$

$$\begin{aligned} P\left(\sup_{t:0\le t\le t+\theta\le T}\left\|\tilde{Z}_{t+\theta}^\epsilon-\tilde{Z}_t^\epsilon\right\|_{V^*}>\eta\right) &\le P\left(\sup_{t:0\le t\le t+\theta\le T}\left\|\tilde{Z}_{t+\theta}^\epsilon-\tilde{Z}_t^\epsilon\right\|_{V^*}^2>\eta^2\right) \\ &\le \frac{3}{\eta^2}\left(I_1+I_2+I_3\right) \\ &\le \frac{3}{\eta^2}\left(c\theta^2+c\theta+cJ_\theta+2c\epsilon\right), \end{aligned}$$

By choosing θ and ϵ small enough, then (10) holds immediately.

Proposition 4. *For a sequence of* $D([0,T];V^*)$*-valued random variable* $\{X^n\}$, *if* $\{X^n\}$ *satisfies the following two conditons*:

1) For any $\delta>0$, there are $n_0\in\mathbb{N}$, $R\in\mathbb{R}_+$, with

$$n\ge n_0 \Rightarrow P\left(\sup_{t\in[0,T]}\left\|X_t^n\right\|_H>R\right)\le\delta;$$

2) For any $\delta>0$ and $\eta>0$, there are $n_0\in\mathbb{N}$, $\theta>0$, with

$$n\ge n_0 \Rightarrow P\left(\sup_{t,s\in[0,T],|t-s|<\theta}\left\|X_t^n-X_s^n\right\|_{V^*}>\eta\right)\le\delta;$$

Then $\{X^n\}$ is C-tight, that is, $\{X^n\}$ is tightness in $D([0,T];V^*)$ and if X is a limit point then $X\in C([0,T];V^*)$ a.s..

Proof. It's obvious that (2) implies the following condition (cf. [20], p. 290). For any $\delta>0$ and $\eta>0$, there are $n_0\in\mathbb{N}$, $\theta_0>0$, with

$$n\ge n_0 \Rightarrow P\left(w\left(X^n;\theta_0\right)>\eta\right)\le\delta, \tag{11}$$

where

$$w\left(X^n;\theta_0\right):=\inf\left\{\max_{i\le r}\sup_{t,s\in[t_{i-1},t_i)}\left\|X_t^n-X_s^n\right\|_{V^*}:0=t_0<\cdots<t_r=T,\inf_{i\le r}\left(t_i-t_{i-1}\right)\ge\theta_0\right\}.$$

For the finite family $\left(X^n\right)_{1\le n\le n_0}$, we can find $R'<\infty$ and $\theta'>0$ such that

$$\sup_{1\le n\le n_0} P\left(\sup_{t\in[0,T]}\left\|X_t^n\right\|_H > R'\right)\le \delta,\quad \sup_{1\le n\le n_0} P\left(w\left(X^n;\theta'\right)>\eta\right)\le\delta.$$

Hence, replacing R by $R\vee R'$ in (1) and θ_0 by $\theta_0\wedge\theta'$ in (11), we obtain that they still hold with $n_0=1$.

Fix $\delta>0$. Let $R_\delta<\infty$ and $\theta_{k,\delta}>0$ satisfy

$$\sup_n P\left(\sup_{t\in[0,T]}\left\|X_t^n\right\|_H > R_\delta\right)\le\frac{\delta}{2},\quad \sup_n P\left(w\left(X^n;\theta_{k,\delta}\right)>\frac{1}{k}\right)\le\frac{\delta}{2^{k+1}}.$$

Then

$$K_\delta := \bigcap_{k=1}^{\infty}\left\{X\in D\left([0,T];V^*\right): \sup_{t\in[0,T]}\left\|X_t\right\|_H\le R_\delta, w\left(X;\theta_{k,\delta}\right)\le\frac{1}{k}\right\}$$

satisfies

$$\sup_n P\left(X^n\notin K_\delta\right)\le \sup_n P\left(\sup_{t\in[0,T]}\left\|X_t^n\right\|_H>R_\delta\right)+\sum_{k=1}^{\infty}\sup_n P\left(w\left(X^n;\theta_{k,\delta}\right)>\frac{1}{k}\right)\le\delta.$$

By **(H7)**, we have $H\equiv H^*\hookrightarrow V^*$ compactly. So, K_δ satisfies the conditions of Theorem A2.2 ([21], p. 563), then it's relatively compact in $D\left([0,T];V^*\right)$. This implies tightness of $\left\{X^n\right\}$.

It remains to prove that if a subsequence, still denoted by $\left(X^n\right)$, converges in law to some *X*, then *X* is a.s. continuous. By taking the same scheme as in Proposition 3.26 (cf. [20], p. 315) and replacing R^d by V^* in the proof, we complete the proof.

According to Lemma 1 and Lemma 2, we have the following result:

Corollary 1. *The sequence* $\left\{\tilde{Z}^\epsilon\right\}_{\epsilon\in(0,1]}$ *is C-tight in* $D\left([0,T];V^*\right)$.

Lemma 3. *Assume that for almost all* ω, $\left\{u_\epsilon=\left(\psi_\epsilon,\varphi_\epsilon\right)\right\}_{\epsilon\in(0,1]}$ *weakly converges to* $\left\{u=\left(\psi,\varphi\right)\right\}$ *in* $\mathcal{U}_M$ *for fixed* $M\in\mathbb{N}$ *and there is a* $C\left([0,T];V^*\right)$*-valued process* $\tilde{Z}$ *such that*

$$\sup_{0\le t\le T}\left\|\tilde{Z}_t^\epsilon-\tilde{Z}_t\right\|_{V^*}\to 0\quad a.s.. \tag{12}$$

Then, $\tilde{Z}$ solves the following equation:

$$\tilde{Z}_t = z+\int_0^t A\left(s,\tilde{Z}_s\right)\mathrm{d}s+\int_0^t B\left(s,\tilde{Z}_s\right)\psi(s)\mathrm{d}s+\int_0^t\int_{\mathbb{X}} f\left(s,\tilde{Z}_s,x\right)\left(\varphi(s,x)-1\right)\nu(\mathrm{d}x)\mathrm{d}s.$$

Moreover, we have

$$\mathbb{E}\left(\sup_{0\le t\le T}\left\|\tilde{Z}_t^\epsilon-\tilde{Z}_t\right\|_H^2\right)\to 0 \tag{13}$$

and if $\lambda_1>0$ in **(H2)**, then

$$\mathbb{E}\int_0^T\left\|\tilde{Z}_t^\epsilon-\tilde{Z}_t\right\|_V^2\mathrm{d}t\to 0. \tag{14}$$

Proof. We divide our proof into several steps.

Step 1. By Lemma 1, we have

$$\sup_{\epsilon\in(0,1]}\mathbb{E}\left(\left\|\tilde{Z}_t^{\epsilon}\right\|_H^2\right)+\sup_{\epsilon\in(0,1]}\mathbb{E}\int_0^T\left\|\tilde{Z}_t^{\epsilon}\right\|_V^2\mathrm{d}s<\infty \tag{15}$$

and

$$\sup_{\epsilon\in(0,1]}\mathbb{E}\left(\sup_{t\in[0,T]}\left\|\tilde{Z}_t^{\epsilon}\right\|_H^4\right)<\infty. \tag{16}$$

Therefore, by the strong convergence of $\tilde{Z}^{\epsilon}(\cdot,\omega)$ to $\tilde{Z}(\cdot,\omega)$ as in (12). We get, for almost all ω, $\tilde{Z}^{\epsilon}(T,\omega)$ converges weakly to $\tilde{Z}(T,\omega)$ in H and $\tilde{Z}^{\epsilon}(\cdot,\omega)$ converges to $\tilde{Z}(\cdot,\omega)$ weakly in $L^2([0,T];V)$; and so we have

$$\mathbb{E}\left(\left\|\tilde{Z}(T)\right\|_H^2\right)\le\liminf_{\epsilon\downarrow 0}\mathbb{E}\left(\left\|\tilde{Z}_T^{\epsilon}\right\|_H^2\right)<\infty, \tag{17}$$

$$\mathbb{E}\int_0^T\left\|\tilde{Z}_s\right\|_V^2\mathrm{d}s\le\liminf_{\epsilon\downarrow 0}\mathbb{E}\int_0^T\left\|\tilde{Z}_s^{\epsilon}\right\|_V^2\mathrm{d}s<\infty. \tag{18}$$

By (12), (16) and dominated convergence theorem, we have

$$\lim_{\epsilon\downarrow 0}\mathbb{E}\left(\sup_{t\in[0,T]}\left\|\tilde{Z}_t^{\epsilon}-\tilde{Z}_t\right\|_{V^*}^2\right)=0.$$

Thus

$$\begin{aligned}\mathbb{E}\left(\int_0^T\left\|\tilde{Z}_t^{\epsilon}-\tilde{Z}_t\right\|_H^2\mathrm{d}t\right)&=\mathbb{E}\left(\int_0^T{}_{V^*}\left\langle\tilde{Z}_t^{\epsilon}-\tilde{Z}_t,\tilde{Z}_t^{\epsilon}-\tilde{Z}_t\right\rangle_V\mathrm{d}t\right)\\&\le\mathbb{E}\left(\int_0^T\left\|\tilde{Z}_t^{\epsilon}-\tilde{Z}_t\right\|_{V^*}\left\|\tilde{Z}_t^{\epsilon}-\tilde{Z}_t\right\|_V\mathrm{d}t\right)\\&\le\left(\int_0^T\mathbb{E}\left\|\tilde{Z}_t^{\epsilon}-\tilde{Z}_t\right\|_{V^*}^2\mathrm{d}t\int_0^T\mathbb{E}\left\|\tilde{Z}_t^{\epsilon}-\tilde{Z}_t\right\|_V^2\mathrm{d}t\right)^{1/2}\\&\le T^{1/2}\left(\mathbb{E}\left(\sup_{t\in[0,T]}\left\|\tilde{Z}_t^{\epsilon}-\tilde{Z}_t\right\|_{V^*}^2\right)\right)^{1/2}\mathbb{E}\left(\int_0^T\left\|\tilde{Z}_t^{\epsilon}-\tilde{Z}_t\right\|_V^2\mathrm{d}t\right)^{1/2}\to 0,\ \text{ as }\epsilon\downarrow 0.\end{aligned} \tag{19}$$

Step 2. In this step, we prove $\tilde{Z}$ solves Equation (13). By **(H4)** and (15), we have

$$\sup_{\epsilon\in(0,1]}\mathbb{E}\left(\int_0^T\left\|A\left(t,\tilde{Z}^{\epsilon}(t)\right)\right\|_{V^*}^2\mathrm{d}t\right)<\infty. \tag{20}$$

Hence, by (15) and (20), there exist subsequences of $\tilde{Z}^{\epsilon}$, $\tilde{Z}^{\epsilon}(T)$ and $A\left(\cdot,\tilde{Z}^{\epsilon}(\cdot)\right)$ (still denoted by themselves for simplicity) and $\bar{Z}\in L^2(\Omega\times[0,T];V)$, $\tilde{Z}_T'\in L^2(\Omega;H)$ and $Y\in L^2(\Omega\times[0,T];V^*)$ such that

$$\tilde{Z}^{\epsilon}\to\bar{Z}\quad\text{weakly in } L^2(\Omega\times[0,T];V), \tag{21}$$

$$\tilde{Z}^{\epsilon}(T)\to\tilde{Z}_T'\quad\text{weakly in } L^2(\Omega;H), \tag{22}$$

and

$$Y^{\epsilon}:=A\left(\cdot,\tilde{Z}^{\epsilon}(\cdot)\right)\to Y\quad\text{weakly in } L^2\left(\Omega\times[0,T];V^*\right). \tag{23}$$

Define

$$\hat{Z}_t:=z+\int_0^t Y_s\mathrm{d}s+\int_0^t B\left(s,\tilde{Z}_s\right)\psi(s)\mathrm{d}s+\int_0^t\int_{\mathbb{X}}f\left(s,\tilde{Z}_s,x\right)\left(\varphi(s,x)-1\right)\nu(\mathrm{d}x)\mathrm{d}s.$$

Note that

$$
\begin{aligned}
\tilde{Z}_t^\epsilon = z &+ \int_0^t A\left(s,\tilde{Z}_s^\epsilon\right)\mathrm{d}s + \int_0^t B\left(s,\tilde{Z}_s^\epsilon\right)\psi_\epsilon(s)\mathrm{d}s + \sqrt{\epsilon}\int_0^t B\left(s,\tilde{Z}_s^\epsilon\right)\mathrm{d}W_s \\
&+ \int_0^t\int_{\mathbb{X}} f\left(s,\tilde{Z}_{s-}^\epsilon,x\right)\left(\varphi_\epsilon(s,x)-1\right)\nu(\mathrm{d}x)\mathrm{d}s \\
&+ \epsilon\int_0^t\int_{\mathbb{X}} f\left(s,\tilde{Z}_{s-}^\epsilon,x\right)\left(N^{\epsilon^{-1}\varphi_\epsilon}(\mathrm{d}s,\mathrm{d}x)-\epsilon^{-1}\varphi_\epsilon(s,x)\nu(\mathrm{d}x)\mathrm{d}s\right).
\end{aligned}
$$

By taking weak limits and by (19), we can get

$$
\hat{Z}(t,\omega)=\bar{Z}(t,\omega)=\tilde{Z}(t,\omega) \quad \text{for } \mathrm{d}t\times\mathrm{d}P\text{-almost all } (t,\omega).
$$

Indeed, for any V-valued bounded and measurable process ξ,

$$
\begin{aligned}
\mathbb{E}\left(\int_0^T\left\langle \xi_t,\tilde{Z}_t^\epsilon - z\right\rangle_H \mathrm{d}t\right) &= \mathbb{E}\left(\int_0^T\int_0^t {}_{V^*}\left\langle A\left(s,\tilde{Z}_s^\epsilon\right),\xi_t\right\rangle_V \mathrm{d}s\mathrm{d}t\right) \\
&+ \mathbb{E}\left(\int_0^T\left\langle \xi_t,\int_0^t B\left(s,\tilde{Z}_s^\epsilon\right)\psi_\epsilon(s)\mathrm{d}s\right\rangle_H \mathrm{d}t\right) + \sqrt{\epsilon}\,\mathbb{E}\left(\int_0^T\left\langle \xi_t,\int_0^t B\left(s,\tilde{Z}_s^\epsilon\right)\mathrm{d}W_s\right\rangle_H \mathrm{d}t\right) \\
&+ \mathbb{E}\left(\int_0^T\left\langle \xi_t,\int_0^t\int_{\mathbb{X}} f\left(s,\tilde{Z}_{s-}^\epsilon,x\right)\left(\varphi_\epsilon(s,x)-1\right)\nu(\mathrm{d}x)\mathrm{d}s\right\rangle_H \mathrm{d}t\right) \\
&+ \epsilon\mathbb{E}\left(\int_0^T\left\langle \xi_t,\int_0^t\int_{\mathbb{X}} f\left(s,\tilde{Z}_{s-}^\epsilon,x\right)\left(N^{\epsilon^{-1}\varphi_\epsilon}(\mathrm{d}s,\mathrm{d}x)-\epsilon^{-1}\varphi_\epsilon(s,x)\nu(\mathrm{d}x)\mathrm{d}s\right)\right\rangle_H \mathrm{d}t\right).
\end{aligned}
$$

By (21), (23) and taking limits for $\epsilon\downarrow 0$, then we get (see also the proof of (27) and (29) below)

$$
\mathbb{E}\left(\int_0^T\left\langle \xi_t,\bar{Z}_t-\hat{Z}_t\right\rangle_H \mathrm{d}t\right)=0,
$$

which implies $\bar{Z}(t,\omega)=\hat{Z}(t,\omega)$ for almost all (t,ω). Similarly, we have $\hat{Z}_T(\omega)=\tilde{Z}_T'(\omega)=\tilde{Z}_T(\omega)$ for almost all ω.

We only have to prove

$$
Y(s,\omega)=A\left(s,\tilde{Z}(s,\omega)\right) \quad \text{for } \mathrm{d}t\times\mathrm{d}P\text{-almos tall } (t,\omega). \tag{24}
$$

Let $\Phi\in L^2\left([0,T];V\right)$. By Itô's formula

$$
\begin{aligned}
\mathbb{E}\left(\mathrm{e}^{-2\lambda_0 T}\left\|\tilde{Z}_T^\epsilon\right\|_H^2\right) &= \|z\|_H^2 - 2\lambda_0\mathbb{E}\left(\int_0^T \mathrm{e}^{-2\lambda_0 s}\left\|\tilde{Z}_s^\epsilon\right\|_H^2\mathrm{d}s\right) + 2\mathbb{E}\left(\int_0^T \mathrm{e}^{-2\lambda_0 s}\,{}_{V^*}\left\langle A\left(s,\tilde{Z}_s^\epsilon\right),\tilde{Z}_s^\epsilon\right\rangle_V \mathrm{d}s\right) \\
&+ 2\mathbb{E}\left(\int_0^T \mathrm{e}^{-2\lambda_0 s}\left\langle B\left(s,\tilde{Z}_s^\epsilon\right)\psi_\epsilon(s),\tilde{Z}_s^\epsilon\right\rangle_H \mathrm{d}s\right) + \epsilon\mathbb{E}\left(\int_0^T \mathrm{e}^{-2\lambda_0 s}\left\|B\left(s,\tilde{Z}_s^\epsilon\right)\right\|_{L_Q}^2\mathrm{d}s\right) \\
&+ 2\mathbb{E}\left(\int_0^T\int_{\mathbb{X}} \mathrm{e}^{-2\lambda_0 s}\left\langle f\left(s,\tilde{Z}_{s-}^\epsilon,x\right),\tilde{Z}_s^\epsilon\right\rangle_H \left(\varphi_\epsilon(s,x)-1\right)\nu(\mathrm{d}x)\mathrm{d}s\right) \\
&+ \epsilon\mathbb{E}\left(\int_0^T\int_{\mathbb{X}} \mathrm{e}^{-2\lambda_0 s}\left\|f\left(s,\tilde{Z}_s^\epsilon,x\right)\right\|_H^2 \varphi_\epsilon(s,x)\nu(\mathrm{d}x)\mathrm{d}s\right).
\end{aligned}
\tag{25}
$$

By **(H2)**

$$
\begin{aligned}
&2\mathbb{E}\int_0^T \mathrm{e}^{-2\lambda_0 s}\left({}_{V^*}\left\langle A\left(s,\tilde{Z}_s^\epsilon\right),\tilde{Z}_s^\epsilon\right\rangle_V - \lambda_0\left\|\tilde{Z}_s^\epsilon\right\|_H^2\right)\mathrm{d}s \\
&\le 2\mathbb{E}\left(\int_0^T \mathrm{e}^{-2\lambda_0 s}\,{}_{V^*}\left\langle A\left(s,\tilde{Z}_s^\epsilon\right)-A\left(s,\Phi_s\right),\Phi_s\right\rangle_V \mathrm{d}s\right) + 2\mathbb{E}\left(\int_0^T \mathrm{e}^{-2\lambda_0 s}\,{}_{V^*}\left\langle A\left(s,\Phi_s\right),\tilde{Z}_s^\epsilon\right\rangle_V \mathrm{d}s\right) \\
&\quad + 2\lambda_0\mathbb{E}\left(\int_0^T \mathrm{e}^{-2\lambda_0 s}\left(\left\|\Phi_s\right\|_H^2 - 2\left\langle \tilde{Z}_s^\epsilon,\Phi_s\right\rangle_H\right)\mathrm{d}s\right) \\
&\to 2\mathbb{E}\left(\int_0^T \mathrm{e}^{-2\lambda_0 s}\,{}_{V^*}\left\langle Y_s - A\left(s,\Phi_s\right),\Phi_s\right\rangle_V \mathrm{d}s\right) + 2\mathbb{E}\left(\int_0^T \mathrm{e}^{-2\lambda_0 s}\,{}_{V^*}\left\langle A\left(s,\Phi_s\right),\tilde{Z}_s\right\rangle_V \mathrm{d}s\right) \\
&\quad + 2\lambda_0\mathbb{E}\left(\int_0^T \mathrm{e}^{-2\lambda_0 s}\left(\left\|\Phi_s\right\|_H^2 - 2\left\langle \tilde{Z}_s,\Phi_s\right\rangle_H\right)\mathrm{d}s\right)
\end{aligned}
\tag{26}
$$

as $\epsilon \downarrow 0$.

We now prove

$$\mathbb{E}\left(\int_0^T e^{-2\lambda_0 s}\left\langle B\left(s,\tilde{Z}_s^\epsilon\right)\psi_\epsilon(s),\tilde{Z}_s^\epsilon\right\rangle_H \mathrm{d}s\right)\to\mathbb{E}\left(\int_0^T e^{-2\lambda_0 s}\left\langle B\left(s,\tilde{Z}_s\right)\psi(s),\tilde{Z}_s\right\rangle_H \mathrm{d}s\right). \quad (27)$$

Since $\{\psi_\epsilon(\cdot,\omega)\}$ weakly converges to $\psi(\cdot,\omega)$ in $\bar{S}_M$ (see (2)), then

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0}\mathbb{E}\left|\int_0^T e^{-2\lambda_0 s}\left\langle B\left(s,\tilde{Z}_s\right)\left(\psi_\epsilon(s)-\psi(s)\right),\tilde{Z}_s\right\rangle_H \mathrm{d}s\right| \\
&=\lim_{\epsilon\downarrow 0}\mathbb{E}\left|\int_0^T \left\langle\left(\psi_\epsilon(s)-\psi(s)\right),e^{-2\lambda_0 s}B^*\left(s,\tilde{Z}_s\right)\tilde{Z}_s\right\rangle_H \mathrm{d}s\right| \\
&=0,
\end{aligned}$$

the last limit follows by using dominated convergence theorem. By (2), **(H5)**, Lemma 1 and (19), we also have

$$\begin{aligned}
&\mathbb{E}\left|\int_0^T e^{-2\lambda_0 s}\left\langle B\left(s,\tilde{Z}_s^\epsilon\right)\psi_\epsilon(s),\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\rangle_H \mathrm{d}s\right| \\
&\le c\mathbb{E}\left(\int_0^T\left(\left\|\tilde{Z}_s^\epsilon\right\|_H+1\right)\left\|\psi_\epsilon(s)\right\|_{G_Q}\left\|\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\|_H \mathrm{d}s\right) \\
&\le c\mathbb{E}\left(\int_0^T\left(\left\|\tilde{Z}_s^\epsilon\right\|_H+1\right)^2\left\|\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\|_H^2 \mathrm{d}s\right)^{1/2}\left(\int_0^T\left\|\psi_\epsilon(s)\right\|_{G_Q}^2 \mathrm{d}s\right)^{1/2} \\
&\le c\left(\mathbb{E}\left(\sup_{s\in[0,T]}\left\|\tilde{Z}_s^\epsilon\right\|_H^2+1\right)\right)^{1/2}\left(\mathbb{E}\int_0^T\left\|\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\|_H^2 \mathrm{d}s\right)^{1/2} \\
&\le c\left(\mathbb{E}\int_0^T\left\|\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\|_H^2\right)^{1/2}\to 0,
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left|\int_0^T e^{-2\lambda_0 s}\left\langle\left(B\left(s,\tilde{Z}_s^\epsilon\right)-B\left(s,\tilde{Z}_s\right)\right)\psi_\epsilon(s),\tilde{Z}_s^\epsilon\right\rangle_H \mathrm{d}s\right| \\
&\le c\mathbb{E}\left(\int_0^T\left\|\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\|_H\left\|\psi_\epsilon(s)\right\|_{G_Q}\left\|\tilde{Z}_s\right\|_H \mathrm{d}s\right) \\
&\le c\mathbb{E}\left(\int_0^T\left\|\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\|_H^2\left\|\tilde{Z}_s\right\|_H^2 \mathrm{d}s\right)^{1/2} \\
&\le c\left(\mathbb{E}\left(\sup_{s\in[0,T]}\left\|\tilde{Z}_s\right\|_H^2\right)\right)^{1/2}\left(\mathbb{E}\int_0^T\left\|\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\|_H^2 \mathrm{d}s\right)^{1/2}\to 0.
\end{aligned}$$

Then limit (27) follows.

Moreover, it is easy to get that

$$\lim_{\epsilon\downarrow 0}\epsilon\mathbb{E}\left(\int_0^T e^{-2\lambda_0 s}\left\|B\left(s,\tilde{Z}_s^\epsilon\right)\right\|_{L_Q}^2 \mathrm{d}s\right)=0 \quad (28)$$

Now we prove the following limit:

$$\begin{aligned}
&2\mathbb{E}\left(\int_0^T\int_{\mathbb{X}} e^{-2\lambda_0 s}\left\langle\left(s,\tilde{Z}_s^\epsilon,x\right),\tilde{Z}_s^\epsilon\right\rangle_H\left(\varphi_\epsilon(s,x)-1\right)\nu(\mathrm{d}x)\mathrm{d}s\right) \\
&\to 2\mathbb{E}\left(\int_0^T\int_{\mathbb{X}} e^{-2\lambda_0 s}\left\langle\left(s,\tilde{Z}_s,x\right),\tilde{Z}_s\right\rangle_H\left(\varphi(s,x)-1\right)\nu(\mathrm{d}x)\mathrm{d}s\right)
\end{aligned} \quad (29)$$

By **(H5)**, Lemma 1 and (19), we have

$$\begin{aligned}
&\mathbb{E}\left|\int_0^T\int_{\mathbb{X}}\mathrm{e}^{-2\lambda_0 s}\left(\left\langle f\left(s,\tilde{Z}_s^\epsilon,x\right),\tilde{Z}_s^\epsilon\right\rangle_H \varphi_\epsilon(s,x)-\left\langle f\left(s,\tilde{Z}_s,x\right),\tilde{Z}_s\right\rangle_H \varphi(s,x)\right)\nu(\mathrm{d}x)\mathrm{d}s\right|\\
&\le \mathbb{E}\int_0^T\int_{\mathbb{X}}\left|\left\langle f\left(s,\tilde{Z}_s^\epsilon,x\right),\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\rangle_H \varphi_\epsilon(s,x)\right|\nu(\mathrm{d}x)\mathrm{d}s\\
&\quad+\mathbb{E}\int_0^T\int_{\mathbb{X}}\left|\left\langle f\left(s,\tilde{Z}_s^\epsilon,x\right)-f\left(s,\tilde{Z}_s,x\right),\tilde{Z}_s\right\rangle_H \varphi_\epsilon(s,x)\right| \mid \nu(\mathrm{d}x)\mathrm{d}s\\
&\quad+\mathbb{E}\left|\int_0^T\int_{\mathbb{X}}\mathrm{e}^{-2\lambda_0 s}\left\langle f\left(s,\tilde{Z}_s,x\right),\tilde{Z}_s\right\rangle_H\left(\varphi_\epsilon(s,x)-\varphi(s,x)\right)\nu(\mathrm{d}x)\mathrm{d}s\right|\\
&\le c\mathbb{E}\left(\int_0^T\int_{\mathbb{X}}\left(1+\left\|\tilde{Z}_s^\epsilon\right\|_H\right)\left\|\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\|_H \varphi_\epsilon(s,x)\nu(\mathrm{d}x)\mathrm{d}s\right)\\
&\quad+\mathbb{E}\left|\int_0^T\int_{\mathbb{X}}\mathrm{e}^{-2\lambda_0 s}\left\langle f\left(s,\tilde{Z}_s,x\right),\tilde{Z}_s\right\rangle_H\left(\varphi_\epsilon(s,x)-\varphi(s,x)\right)\nu(\mathrm{d}x)\mathrm{d}s\right|=cJ_1+J_2,
\end{aligned}\tag{30}$$

where

$$J_1:=\mathbb{E}\left(\int_0^T\int_{\mathbb{X}}\left(1+\left\|\tilde{Z}_s^\epsilon\right\|_H\right)\left\|\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\|_H \varphi_\epsilon(s,x)\nu(\mathrm{d}x)\mathrm{d}s\right),$$

and

$$J_2:=\mathbb{E}\left|\int_0^T\int_{\mathbb{X}}\mathrm{e}^{-2\lambda_0 s}\left\langle f\left(s,\tilde{Z}_s,x\right),\tilde{Z}_s\right\rangle_H\left(\varphi_\epsilon(s,x)-\varphi(s,x)\right)\nu(\mathrm{d}x)\mathrm{d}s\right|.$$

For J_1, by Young inequality, we have

$$\begin{aligned}
J_1&\le\left(\mathbb{E}\int_0^T\int_{\mathbb{X}}\varphi_\epsilon(s,x)\nu(\mathrm{d}x)\mathrm{d}s\right)^{1/2}\left\{\left(\mathbb{E}\int_0^T\int_{\mathbb{X}}\left\|\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\|_H^2\varphi_\epsilon(s,x)\nu(\mathrm{d}x)\mathrm{d}s\right)^{1/2}\right.\\
&\quad\left.+\mathbb{E}\left(\int_0^T\int_{\mathbb{X}}\left\|\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\|_H^2\left\|\tilde{Z}_s^\epsilon\right\|_H^2\varphi_\epsilon(s,x)\nu(\mathrm{d}x)\mathrm{d}s\right)^{1/2}\right\}\\
&\le c\left(\mathbb{E}\int_0^T\int_{\mathbb{X}}\left\|\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\|_H^2\varphi_\epsilon(s,x)\nu(\mathrm{d}x)\mathrm{d}s\right)^{1/2}\to 0,\quad \text{as } \epsilon\downarrow 0
\end{aligned}$$

by noting (16) and (19). For J_2, by (4), **(H6)** and $\tilde{Z}\in C\left([0,T];V^*\right)$, it's easy to verify $\mathrm{e}^{-2\lambda_0 s}\left\langle f\left(s,\tilde{Z}_s,x\right),\tilde{Z}_s\right\rangle_H$ is a continuous function on $[0,T]\times\mathbb{X}$ with the compact support $[0,T]\times\Gamma$, and by the weak convergence of $\nu_T^{\varphi_\epsilon}$ to ν_T^{φ} (see Remark 1) and dominated convergence theorem, $J_2\to 0$ as $\epsilon\downarrow 0$. Then (30) goes to 0 as $\epsilon\downarrow 0$. Similarly, we have

$$\mathbb{E}\left|\int_0^T\int_{\mathbb{X}}\mathrm{e}^{-2\lambda_0 s}\left(\left\langle f\left(s,\tilde{Z}_s^\epsilon,x\right),\tilde{Z}_s^\epsilon\right\rangle_H-\left\langle f\left(s,\tilde{Z}_s,x\right),\tilde{Z}_s\right\rangle_H\right)\nu(\mathrm{d}x)\mathrm{d}s\right|,\quad \text{as } \epsilon\downarrow 0.$$

Then, we get (29).

It is obvious that

$$\lim_{\epsilon\downarrow 0}\epsilon\mathbb{E}\int_0^T\int_{\mathbb{X}}\mathrm{e}^{-2\lambda_0 s}\left\|f\left(s,\tilde{Z}_s^\epsilon,x\right)\right\|_H^2\varphi_\epsilon(s,x)\mathrm{d}s\nu(\mathrm{d}x)\to 0.\tag{31}$$

Combining (26) to (31) yields that

$$\begin{aligned}
&\mathbb{E}\left(\mathrm{e}^{-2\lambda_0 T}\left\|\tilde{Z}_T\right\|_H^2\right)\le\liminf_{\epsilon\downarrow 0}\mathbb{E}\left(\mathrm{e}^{-2\lambda_0 T}\left\|\tilde{Z}_T^\epsilon\right\|_H^2\right)\\
&\le\|z\|_H^2+2\mathbb{E}\left(\int_0^T\mathrm{e}^{-2\lambda_0 s}{}_{V^*}\left\langle Y_s-A\left(s,\Phi_s\right),\Phi_s\right\rangle_V\mathrm{d}s\right)+2\mathbb{E}\left(\int_0^T\mathrm{e}^{-2\lambda_0 s}{}_{V^*}\left\langle A\left(s,\Phi_s\right),\tilde{Z}_s\right\rangle_V\mathrm{d}s\right)\\
&\quad+\lambda_0\mathbb{E}\left(\int_0^T\mathrm{e}^{-2\lambda_0 s}\left(\left\|\Phi_s\right\|_H^2-2\left\langle\tilde{Z}_s,\Phi_s\right\rangle_H\right)\mathrm{d}s\right)+2\mathbb{E}\left(\int_0^T\mathrm{e}^{-2\lambda_0 s}\left\langle B\left(s,\tilde{Z}_s\right)\psi(s),\tilde{Z}_s\right\rangle_H\mathrm{d}s\right)\\
&\quad+2\mathbb{E}\left(\int_0^T\int_{\mathbb{X}}\mathrm{e}^{-2\lambda_0 s}\left\langle f\left(s,\tilde{Z}_s,x\right),\tilde{Z}_s\right\rangle_H\left(\varphi(s,x)-1\right)\nu(\mathrm{d}x)\mathrm{d}s\right).
\end{aligned}$$

On the other hand, by Itô's formula we have

$$\begin{aligned}
e^{-2\lambda_0 T}\left\|\hat{Z}_T\right\|_H^2 &= \|z\|_H^2 - 2\lambda_0\int_0^T e^{-2\lambda_0 s}\left\|\tilde{Z}_s\right\|_H^2 ds + 2\int_0^T e^{-2\lambda_0 s}\,{}_{V^*}\left\langle Y_s,\tilde{Z}_s\right\rangle_V ds \\
&+ 2\int_0^T e^{-2\lambda_0 s}\left\langle B\left(s,\tilde{Z}_s\right)\psi(s),\tilde{Z}_s\right\rangle_H ds \\
&+ 2\int_0^T\int_{\mathbb{X}} e^{-2\lambda_0 s}\left\langle f\left(s,\tilde{Z}_s,x\right),\tilde{Z}_s\right\rangle_H \left(\varphi(s,x)-1\right)\nu(dx)ds
\end{aligned}$$

So, we have

$$\mathbb{E}\left(\int_0^T e^{-2\lambda_0 s}\,{}_{V^*}\left\langle Y_s - A(s,\Phi_s),\tilde{Z}_s-\Phi_s\right\rangle_V ds\right) \le \lambda_0\mathbb{E}\left(\int_0^T e^{-2\lambda_0 s}\left\|\tilde{Z}_s-\Phi_s\right\|_H^2 ds\right),$$

which implies (24) by **(H1)**.

Step 3. In this step we prove (13) and (14). Notice that

$$\begin{aligned}
\tilde{Z}_t^\epsilon - \tilde{Z}_t &= \int_0^t\left(A\left(s,\tilde{Z}_s^\epsilon\right)-A\left(s,\tilde{Z}_s\right)\right)ds + \int_0^t\left(B\left(s,\tilde{Z}_s^\epsilon\right)\psi_\epsilon(s)-B\left(s,\tilde{Z}_s\right)\psi(s)\right)ds \\
&+ \int_0^t\int_{\mathbb{X}}\left(f\left(s,\tilde{Z}_{s-}^\epsilon,x\right)\varphi_\epsilon(s,x) - f\left(s,\tilde{Z}_s,x\right)\varphi(s,x)\right)\nu(dx)ds \\
&- \int_0^t\int_{\mathbb{X}}\left(f\left(s,\tilde{Z}_{s-}^\epsilon,x\right) - f\left(s,\tilde{Z}_{s-},x\right)\right)\nu(dx)ds + \sqrt{\epsilon}\int_0^t B\left(s,\tilde{Z}_s^\epsilon\right)dW_s \\
&+ \epsilon\int_0^t\int_{\mathbb{X}} f\left(s,\tilde{Z}_{s-}^\epsilon,x\right)\left(N^{\epsilon^{-1}\varphi_\epsilon}(ds,dx) - \epsilon^{-1}\varphi_\epsilon(s,x)\nu(dx)ds\right).
\end{aligned}$$

By Itô's formula, we have

$$\left\|\tilde{Z}_t^\epsilon - \tilde{Z}_t\right\|_H^2 = \sum_{i=0}^9 I_i^\epsilon(t),$$

where

$$\begin{aligned}
I_0^\epsilon(t) &:= 2\int_0^t {}_{V^*}\left\langle A\left(s,\tilde{Z}_s^\epsilon\right)-A\left(s,\tilde{Z}_s\right),\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\rangle_V ds, \\
I_1^\epsilon(t) &:= 2\int_0^t\left\langle B\left(s,\tilde{Z}_s^\epsilon\right)\psi_\epsilon(s),\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\rangle_H ds, \\
I_2^\epsilon(t) &:= -2\int_0^t\left\langle B\left(s,\tilde{Z}_s\right)\psi(s),\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\rangle_H ds, \\
I_3^\epsilon(t) &:= 2\int_0^t\int_{\mathbb{X}}\left\langle f\left(s,\tilde{Z}_{s-}^\epsilon,x\right)\varphi_\epsilon(s,x),\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\rangle_H \nu(dx)ds, \\
I_4^\epsilon(t) &:= -2\int_0^t\int_{\mathbb{X}}\left\langle f\left(s,\tilde{Z}_s,x\right)\varphi(s,x),\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\rangle_H \nu(dx)ds, \\
I_5^\epsilon(t) &:= -2\int_0^t\int_{\mathbb{X}}\left\langle f\left(s,\tilde{Z}_{s-}^\epsilon,x\right)-f\left(s,\tilde{Z}_{s-},x\right),\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\rangle_H \nu(dx)ds, \\
I_6^\epsilon(t) &:= 2\sqrt{\epsilon}\int_0^t\left\langle B\left(s,\tilde{Z}_s^\epsilon\right)dW_s,\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\rangle_H, \\
I_7^\epsilon(t) &:= \epsilon\int_0^t\left\|B\left(s,\tilde{Z}_s^\epsilon\right)\right\|_{LQ} ds, \\
I_8^\epsilon(t) &:= \int_0^t\int_{\mathbb{X}}\left(\left\|f\left(s,\tilde{Z}_{s-}^\epsilon,x\right)\right\|_H^2 + 2\left\langle\tilde{Z}_{s-}^\epsilon, f\left(s,\tilde{Z}_{s-}^\epsilon,x\right)\right\rangle_H\right) \\
&\quad\times\left(\epsilon N^{\epsilon^{-1}\varphi_\epsilon}(dt,dx) - \varphi_\epsilon(s,x)\nu(dx)ds\right), \\
I_9^\epsilon(t) &:= \epsilon\int_0^t\int_{\mathbb{X}}\left\|f\left(s,\tilde{Z}_s^\epsilon,x\right)\right\|_H^2 \varphi_\epsilon(s,x)\nu(dx)ds.
\end{aligned}$$

By Lemma 1 and BDG's inequality, we get

$$\lim_{\epsilon\to 0}\mathbb{E}\left(\sup_{t\in[0,T]}\sum_{i=6}^{9}\left|I_i^\epsilon(t)\right|\right)\to 0.$$

For I_i^ϵ, we have

$$\begin{aligned}\mathbb{E}\left(\sup_{t\in[0,T]}\left|I_1^\epsilon(t)\right|\right) &\le c\mathbb{E}\left(\int_0^t\left(\left\|\tilde{Z}_s^\epsilon\right\|_H+1\right)\left\|\psi_\epsilon(s)\right\|_{G_Q}\left\|\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\|_H \mathrm{d}s\right)\\ &\le c\mathbb{E}\left(\int_0^T\left(\left\|\tilde{Z}_s^\epsilon\right\|_H+1\right)^2\left\|\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\|_H^2 \mathrm{d}s\right)^{1/2}\\ &\le c\mathbb{E}\left(\sup_{t\in[0,T]}\left\|\tilde{Z}_s^\epsilon\right\|_H^2\int_0^T\left\|\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\|_H^2 \mathrm{d}s\right)^{1/2}\\ &\le c\left(\mathbb{E}\sup_{t\in[0,T]}\left\|\tilde{Z}_s^\epsilon\right\|_H^2\right)^{1/2}\left(\mathbb{E}\int_0^T\left\|\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\|_H^2 \mathrm{d}s\right)^{1/2}\\ &\le c\left(\mathbb{E}\int_0^T\left\|\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\|_H^2 \mathrm{d}s\right)^{1/2}\to 0,\ \text{as } \epsilon\downarrow 0.\end{aligned}$$

Similarly

$$\lim_{\epsilon\downarrow 0}\mathbb{E}\left(\sup_{t\in[0,T]}\left|I_2^\epsilon(t)\right|\right)=0.$$

For I_3^ϵ, like J_1, we have

$$\begin{aligned}\mathbb{E}\left(\sup_{t\in[0,T]}\left|I_3^\epsilon(t)\right|\right) &\le 2\mathbb{E}\left(\int_0^t\int_{\mathbb{X}}\left\|f\left(s,\tilde{Z}_{s-}^\epsilon,x\right)\right\|_H\left\|\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\|_H\varphi_\epsilon(s,x)\nu(\mathrm{d}x)\mathrm{d}s\right)\\ &\le c\left(\mathbb{E}\int_0^T\int_{\mathbb{X}}\left\|\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\|_H^2\varphi_\epsilon(s,x)\nu(\mathrm{d}x)\mathrm{d}s\right)^{1/2}\\ &\le c_M\left(\mathbb{E}\int_0^T\left\|\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\|_H^2 \mathrm{d}s\right)^{1/2}\to 0,\ \text{as } \epsilon\downarrow 0.\end{aligned}$$

Similarly

$$\lim_{\epsilon\downarrow 0}\mathbb{E}\left(\sup_{t\in[0,T]}\left|I_4^\epsilon(t)\right|\right)=0.$$

For I_5^ϵ, by **(H5)** and **(H6)** we have

$$\begin{aligned}\mathbb{E}\left(\sup_{t\in[0,T]}\left|I_5^\epsilon(t)\right|\right) &\le 2\mathbb{E}\int_0^t\int_{\mathbb{X}}\left\|f\left(s,\tilde{Z}_{s-}^\epsilon,x\right)-f\left(s,\tilde{Z}_{s-},x\right)\right\|_H\left\|\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\|_H\nu(\mathrm{d}x)\mathrm{d}s\\ &\le 2c\nu(\Gamma)\mathbb{E}\left(\int_0^T\left\|\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\|_H^2 \mathrm{d}s\right)\to 0,\ \text{as } \epsilon\downarrow 0.\end{aligned}$$

Assume $\lambda_1>0$, then

$$I_0^\epsilon(t)\le -\lambda_1\int_0^t\left\|\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\|_V^2 \mathrm{d}s+\lambda_0\int_0^t\left\|\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\|_H^2 \mathrm{d}s \tag{32}$$

Set

$$F(t):=\limsup_{\epsilon\downarrow 0}\mathbb{E}\left(\sup_{s\in[0,t]}\left\|\tilde{Z}_s^\epsilon-\tilde{Z}_s\right\|_H^2\right),$$

then

$$F(t) \le \lambda_0 \int_0^t F(s)\,\mathrm{d}s = 0.$$

So

$$F(T) = 0.$$

Notice (32), we get (13) and (14) immediately.

We also have the following main lemma.

Lemma 4. *There exists a probability space* $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ *and a sequence* (*for convenience, still denote by* ϵ) $\left\{(\bar{\psi}_\epsilon, \bar{\varphi}_\epsilon), \bar{Z}^\epsilon, \left(\sqrt{\epsilon}\bar{W} + \int_0^\cdot \bar{\psi}_\epsilon(s)\mathrm{d}s, \epsilon \bar{N}^{\epsilon^{-1}\bar{\varphi}_\epsilon}\right)\right\}$ and $\left\{(\bar{\psi}, \bar{\varphi}), \bar{Z}, \left(\int_0^\cdot \bar{\psi}(s)\mathrm{d}s, \nu_T^{\bar{\varphi}}\right)\right\}$ defined on this space and taking value in $\mathcal{U}_M \times D([0,T];V^*) \times \Omega$ with $\bar{Z} \in C([0,T];V^*)$ such that:

1) For each ϵ, $\left\{(\bar{\psi}_\epsilon, \bar{\varphi}_\epsilon), \bar{Z}^\epsilon, \left(\sqrt{\epsilon}\bar{W} + \int_0^\cdot \bar{\psi}_\epsilon(s)\mathrm{d}s, \epsilon \bar{N}^{\epsilon^{-1}\bar{\varphi}_\epsilon}\right)\right\}$ has the same law as $\left\{(\psi_\epsilon, \varphi_\epsilon), \tilde{Z}^\epsilon, \left(\sqrt{\epsilon}\bar{W} + \int_0^\cdot \psi_\epsilon(s)\mathrm{d}s, \epsilon N^{\epsilon^{-1}\varphi_\epsilon}\right)\right\}$;

2) $\left\{(\bar{\psi}_\epsilon, \bar{\varphi}_\epsilon), \bar{Z}^\epsilon\right\} \to \left\{(\bar{\psi}, \bar{\varphi}), \bar{Z}\right\}$ in $\mathcal{U}_M \times D([0,T];V^*) \times \Omega$, $\bar{P}$-a.s., as $\epsilon \to 0$;

3) $\left\{(\bar{\psi}, \bar{\varphi}), \bar{Z}\right\}$ uniquely solves the following equation:

$$\begin{aligned} \bar{Z}_t &= z + \int_0^t A(s, \bar{Z}_s)\mathrm{d}s + \int_0^t B(s, \bar{Z}_s)\bar{\psi}(s)\mathrm{d}s \\ &\quad + \int_0^t \int_{\mathbb{X}} f(s, \bar{Z}_s, x)(\bar{\varphi}(s,x) - 1)\nu(\mathrm{d}x)\mathrm{d}s. \end{aligned} \tag{33}$$

Moreover, we have

$$\mathbb{E}^{\bar{P}}\left(\sup_{0\le t\le T}\left\|\bar{Z}_t^\epsilon - \bar{Z}_t\right\|_H^2\right) \to 0, \tag{34}$$

and if $\lambda_1 > 0$, then

$$\mathbb{E}^{\bar{P}}\int_0^T \left\|\bar{Z}_t^\epsilon - \bar{Z}_t\right\|_V^2 \mathrm{d}t \to 0. \tag{35}$$

Proof. From Corollary 1, we have $\{\tilde{Z}^\epsilon\}$ is *C*-tight in $D([0,T];V^*)$. By the compactness of S_M, the law of $\{(\psi_\epsilon, \varphi_\epsilon), \tilde{Z}^\epsilon\}$ in $S_M \times D([0,T];V^*)$ is tight. By Skorokhod's embedding theorem, (1) and (2) hold. Since $\bar{Z}_0^\epsilon = z$ $\bar{P}$-a.s. and

$$\begin{aligned} \bar{Z}_t^\epsilon &= z + \int_0^t A(s, \bar{Z}_s^\epsilon)\mathrm{d}s + \int_0^t B(s, \bar{Z}_s^\epsilon)\bar{\psi}_\epsilon(s)\mathrm{d}s + \sqrt{\epsilon}\int_0^t B(s, \bar{Z}_s^\epsilon)\mathrm{d}\bar{W}_s \\ &\quad + \int_0^t \int_{\mathbb{X}} f(s, \bar{Z}_{s-}^\epsilon, x)(\bar{\varphi}_\epsilon(s,x) - 1)\mathrm{d}s\nu(\mathrm{d}x) \\ &\quad + \epsilon\int_0^t \int_{\mathbb{X}} f(s, \bar{Z}_{s-}^\epsilon, x)\left(\bar{N}^{\epsilon^{-1}\bar{\varphi}_\epsilon}(\mathrm{d}s, \mathrm{d}x) - \epsilon^{-1}\bar{\varphi}_\epsilon(s,x)\mathrm{d}s\nu(\mathrm{d}x)\right). \end{aligned}$$

Then, the other conclusions follow from Lemma 3 and noting for $\bar{P}$ almost all ω, $\bar{Z}(\omega) \in C([0,T];V^*)$.

Remark 5. *Assume that* **(H1)-(H7)** *and* $\lambda_1 > 0$ *hold, we have verified* **Hypothesis** (1) *by the above lemma.*

For fixed $M \in \mathbb{N}$, let $(h, g) \in S_M$ and let $\mathcal{G}^0 : \Omega \to \mathbb{D}$ such that $\mathcal{G}^0\left(\int_0^\cdot h(s)\mathrm{d}s, \nu_T^g\right)$ is the unique solution of

$$Z_t = z + \int_0^t A(s, Z_s)\mathrm{d}s + \int_0^t B(s, Z_s)h(s)\mathrm{d}s + \int_0^t \int_{\mathbb{X}} f(s, Z_s, x)(g(s,x) - 1)\mathrm{d}s\nu(\mathrm{d}x).$$

We point out that the difference between (h,g) in the above equation and $(\psi_\epsilon,\varphi_\epsilon)$ in (13) is that (h,g) is not random. We have the following result.

Lemma 5. *Assume that* **(H1)-(H7)** *and* $\lambda_1>0$ *hold. Let* (h_m,g_m), $(h,g)\in S_M$ *be such that* $(h_m,g_m)\to(h,g)$ *in the weak topology of* S_M *(see Section* 2*), then*

$$\mathcal{G}^0\left(\int_0^\cdot h_m(s)\,\mathrm{d}s,\nu_T^{g_m}\right)\to\mathcal{G}^0\left(\int_0^\cdot h(s)\,\mathrm{d}s,\nu_T^{g}\right).$$

Proof. Similar to the proofs of Lemma 1 and 2, we can get $\left\{\mathcal{G}^0\left(\int_0^\cdot h_m(s)\,\mathrm{d}s,\nu_T^{g_m}\right)\right\}_{M\in\mathbb{N}}$ is *C*-tight. As in Lemma 4, there exist a subsequence m_k (still denoted by *m*) and $Z^0\in C([0,T];V^*)$ satisfying

$$\sup_{t\in[0,T]}\left\|\mathcal{G}^0\left(\int_0^\cdot h_m(s)\,\mathrm{d}s,\nu_T^{g_m}\right)-Z^0\right\|_{V^*}\to0,\quad\text{as } m\to\infty.$$

Combining with this convergence and the method used in the proof of Lemma 3, we have $\mathcal{G}^0\left(\int_0^\cdot h(s)\,\mathrm{d}s,\nu_T^{g}\right)=Z^0$, then the result holds.

Using Remark 5, Lemma 5 and Theorem 2, we obtain the following large deviation principle.

Theorem 6. *Under the same assumptions as in Lemma* 5, $\{Z^\epsilon\}_{\epsilon>0}$ *satisfies a large deviation principle with rate function I defined as in* (3), *i.e. for any* $A\in\mathcal{B}(\mathbb{D})$

$$-\inf_{\phi\in A^o}I(\phi)\le\liminf_{\epsilon\to0}\epsilon\log\mu_\epsilon(A)\le\limsup_{\epsilon\to0}\epsilon\log\mu_\epsilon(A)\le-\inf_{\phi\in\bar{A}}I(\phi),$$

where μ_ϵ is the law of Z^ϵ in $\mathbb{D}$ and $\mathbb{D}$ is $D([0,T];H)\cap L^2([0,T];V)$.

Remark 7. *If* $\lambda_1=0$, *then the conclusion still holds if* $\mathbb{D}$ *is replaced by* $D([0,T];H)$.

4. Application—Stochastic Porous Medium Equation

Similar to [9], consider a bounded domain $\mathcal{O}$ in $\mathbb{R}^d$ with smooth boundary. For $p\ge2$, let

$$V:=L^p(\mathcal{O}),\ H:=W^{-1,2}(\mathcal{O}),\ V^*:=L^{p/(p-1)}(\mathcal{O}).$$

The inner product in *H* is defined by

$$\langle x,y\rangle_H:=\int_{\mathcal{O}}(-\Delta)^{-1/2}x(s)\cdot(-\Delta)^{-1/2}y(s)\,\mathrm{d}s,\ \ x,y\in W^{-1,2}(\mathcal{O}).$$

$-\Delta$ establish an isomorphism between $W^{-1,2}(\mathcal{O})$ and $W_0^{1,2}(\mathcal{O})$. We identify $W_0^{1,2}(\mathcal{O})$ with the dual space H^* and *H*, then $H^*=W_0^{1,2}(\mathcal{O})\subset L^{p/(p-1)}(\mathcal{O})$. Therefore

$$V\subset H\equiv H^*\subset V^*,$$

and the inclusions are compact.

Let $\phi_p(r):=r|r|^{p-2}$. For $x\in V$, denote by

$$A(x):=\Delta\phi_p(x).$$

Then $A(x)\in V^*$ and **(H1)**-**(H4)** hold (cf. [9] [16]).

Let $B_1,\cdots,B_n\in L_2(Q^{1/2}G,H)$. Define

$$B(t,x) := \sum_{k=1}^{n} g_k\left(\left(e_{n_1},x\right)_H,\cdots,\left(e_{n_k},x\right)_H\right)B_k,\ e_{n_j}\in H,$$

where g_k are Lipschitz continuous on R^{n_k}. Let $\mathbb{X} := \mathbb{R}$, $h_1,\cdots,h_n \in H$, and define

$$f(t,x,y) := \sum_{k=1}^{n} f_k\left(\left(e_{n_1},x\right)_H,\cdots,\left(e_{n_k},x\right)_H\right)h_k 1_{[0,6]}(y),\ e_{n_j}\in H,$$

where f_k are Lipschitz continuous on R^{n_k}. Then B and f satisfy (**H5**)-(**H6**).

Consider the following stochastic porous medium equation

$$\begin{cases} \mathrm{d}Z_t^{\epsilon} = A\left(Z_t^{\epsilon}\right)\mathrm{d}t + \sqrt{\epsilon}B\left(t,Z_t^{\epsilon}\right)\mathrm{d}W_t + \int_{\mathbb{X}} f\left(t,Z_{t-}^{\epsilon},x\right)\left(\epsilon N^{\epsilon^{-1}}(\mathrm{d}x,\mathrm{d}t) - \nu(\mathrm{d}x)\mathrm{d}t\right), \epsilon\in(0,1], \\ Z^{\epsilon}(t,\xi) = 0, \forall \xi \in \partial\mathcal{O}, \\ Z^{\epsilon}(0,\xi) = z(\xi) \in H. \end{cases}$$

Let ν_{ϵ} be the law of Z^{ϵ} in $D([0,T];H)\cap L^p(0,T;V)$. Then the conclusion of Theorem 6 holds.

Acknowledgements

The authors thank the Editor and the referee for their valuable comments. This work is supported in part by Zhejiang Provincial Natural Science Foundation of China (Grant No. LQ13A010020) and the National Natural Science Foundation of China (Grant No. 11401029).

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