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Approach to a Fifth-Order Boundary Value Problem, via Sperner's Lemma

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Abstract

We consider the five-point boundary value problem for a fifth-order differential equation, where the nonlinearity is superlinear at both the origin and +infinity. Our method of proof combines the Kneser's theorem with the well-known from combinatorial topology Sperner's lemma. We also notice that our geometric approach is strongly based on the associated vector field.

Keywords: Fifth-Order Differential Equation, Vector Field, Kneser's Theorem, Sperner's Lemma

1. Introduction

In this paper we study the boundary value problem

$$\left.\begin{array}{c} x^{(5)}(t)=c(t)F\left(t,x''(t),x'''(t),x^{(4)}(t)\right) \\ 0\le t\le 1 \\ ax(\xi_1)-\beta x'(\xi_1)=0 \\ \gamma x(\xi_2)+\delta x'(\xi_2)=0 \\ \text{and } \; x''(0)=x'''(\eta)=x^{(4)}(1)=0 \end{array}\right\} \quad (1)$$

under the following assumptions:

(A1) F is continuous and positive; *i.e.* $F\in C\left([0,1]\times[0,+\infty)\times\mathbb{R}\times\mathbb{R},[0,+\infty)\right)$;

(A2) c is continuous and positive; *i.e.* $c\in C\left((0,1),[0,+\infty)\right)$;

(A3) $\eta\in\left(\frac{1}{2},1\right), a,\beta,\gamma,\delta,\xi_1,\xi_2\ge 0$, with $0\le\xi_1<\xi_2\le 1$, and $p:=a\delta+\beta\gamma+a\gamma(\xi_2-\xi_1)\ne 0$.

In recent years, boundary-value problems for second and higher order differential equations have been extensively studied. Erbe and Wang [1] used a Green's function and the Krasnoselskii's fixed point theorem in a cone to prove the existence of a positive solution of the boundary value problem

$$x''(t)=f\left(t,x(t)\right),0\le t\le 1$$

$$ax(0)-bx'(0)=0,cx(1)+dx'(1)=0$$

Their technique assumed that the nonlinearity grew either superlinearly or sublinearly. The growth assumptions and calculations involving the Green's function followed by an application of Krasnoselskii's Theorem yielded the result.

Recently an increasing interest in studying the existence of solution and positive solutions to boundary-value problems for higher order differential equation is observed, see for example [2-7]. Especially, Graef and Yang [4] Hao *et al.* [8] Ge and Bai [9] and Kelevedjiev, Palamides and Popivanov [7] proved the existence of results on nonlinear boundary-value problem for fourth order equations. We are aware of limited number of works that study the boundary value problem for fifth order differential equations. We mention the work of Doronin and Larkin [10], which deals with the one-dimensional Kawahara equation that is a nonlinear fifth-order ODE with a convective nonlinearity, while Odda [11] obtains solution of 5th order differential equations under some conditions using a fixed-point theorem. Also, we refer to the works El-Shahed, Al-Mezel [12] and Noor, Mohyud-Din [13,14].

Our analysis of problem (1) will combine the well-known Kneser's theorem with the Sperner's lemma principle. The aim of this paper is to use Sperner's lemma as an alternative to the classical methodologies based on fixed point theory or degree theory under simple assumptions.

Let us recall some basic notions and results from the theory of simplex, which we will subsequently need. Let $p_0,p_1,\ldots,p_m$ be $m+1$ affinely independent points of the m-dimensional Euclidean space $\mathbb{R}^m$. Then the simple $S=[p_0,p_1,\cdots,p_m]$ is defined by

$$S=\left\{p\in\mathbb{R}^m:\exists\lambda_i>0 \text{ with } \sum_{i=1}^{m}\lambda_i=1 \text{ and } p=\sum_{i=1}^{m}\lambda_i p_i\right\}$$

The points $p_0,p_1,\ldots,p_m$ are called vertices of it and the simplex $[p_{i_0},p_{i_1},\ldots,p_{i_k}]$, $0\le k\le m-1$ is a phase of S. If $p_0=A,\ p_1=B,\ p_2=C$ then 2-dimensional simplex $S=[p_0,p_1,p_2]$ is the triangle $[A,B,C]$.

We make use of the following Sperner's (see [15]).

Lemma 1: If T^m be a closed m-simplex with vertices $\left\{e^0,e^1,\ldots,e^m\right\}$ and $\left\{E_0,E_1,\ldots,E_m\right\}$ be a closed covering of T^m such that each closed phase $\left[e^{i_0},e^{i_1},\ldots,e^{i_k}\right]$ of T^m is containing in the corresponding union $\left\{E_{i_0}\cup E_{i_1}\cup\cdots\cup E_{i_k}\right\}$ then the intersection $\bigcap_{i-0}^{m}E_i$ is nonempty.

For completeness, we recall the well-known Kneser's Theorem.

Theorem 1 ([16]): Consider a system

$$x'=f(t,x),\quad (t,x)\in\Omega:=[a,b]\times\mathbb{R} \tag{2}$$

with f continuous. Let $\tilde{E}_0$ be a continuum (compact and connected) in $\Omega_0:=\left\{(t,x)\in\Omega:t=a\right\}$ and let $\mathcal{X}\left(\tilde{E}_0\right)$ be the family of solutions of (2) emanating from $\tilde{E}_0$. If any solution $x\in\mathcal{X}\left(\tilde{E}_0\right)$ is defined on the interval $[a,\tau]$, then the set (cross-section)

$$\mathcal{X}\left(\tau,\tilde{E}_0\right):=\left\{x(\tau):x\in\mathcal{X}\left(\tilde{E}_0\right)\right\}$$

is a continuum in $\mathbb{R}^n$.

2. Main Results

The change of variable $u(x)=x''(t)$ reduces the boundary value problem (1) to:

$$\left.\begin{aligned} u'''(t)=c(t)F\left(t,u(t),u'(t),u''(t)\right),\quad t\in[0,1]\\ \text{and } u(0)=u'(\eta)=u''(1)=0\end{aligned}\right\} \tag{3}$$

where

$$x(t)=\int_{\xi_1}^{t}(t-s)u(s)\mathrm{d}s$$
$$+\frac{1}{p}\int_{\xi_1}^{\xi_2}\left(a(\xi_1-t)-\beta\right)\left(\gamma(\xi_2-s)+\delta\right)u(s)\mathrm{d}s$$

We may extend the nonlinearity as

$$f\left(t,u,u',u''\right)=F\left(t,0,u',u''\right),\quad u<0$$

From the sing property of F, we have

$$f\left(t,u,u',u''\right)\ge 0,\ \left(t,u,u',u''\right)\in[0,1]\times\mathbb{R}^3$$

We will initially study the following boundary value problem

$$u'''(t)=c(t)F\left(t,u(t),u'(t),u''(t)\right),\quad t\in[0,1] \tag{4}$$

$$u(0)=u'(\eta)=u''(1)=0 \tag{5}$$

Remark 1: The boundary value problem (4)-(5) defines a vector field, the properties of which will be crucial for our study. More specifically, let us look at the (u',u'') face semi-plane $(u'>0)$ By the sign condition on f and $c(t)$, we obtain that $u'''>0$ Thus any trajectory $\left(u'(t),u''(t)\right),t\ge 0$, emanating from any point in the fourth quarter:

$$\left\{\left(u',u''\right):u'>0,u''<0\right\}$$

"evolves'' naturally, initially (when $u'(t)>0$) toward the negative u''-semi-axis and then (when $u'(t)<0$) toward the negative u'-semi-axis. Setting a certain growth rate on f (say superlinearity), we can control the vector field, so that some trajectories will satisfy the given boundary conditions. These properties will be referred to as the nature of the vector field throughout the rest of the paper.

The hypotheses on the nonlinearity

$$f\in C\left([0,1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R},[0,\infty)\right)$$

are the following:

(H1) It is superlinear at origin; that is

$$\lim_{x\to 0^+}\max_{0\le t\le 1}\frac{f(t,x,y,z)}{x}=0$$

uniformly for every (y,z) in any compact subset of $\mathbb{R}^2$.

(H2) It is superlinear at infinitive; that is

$$\lim_{x\to+\infty}\min_{0\le t\le 1}\frac{f(t,x,y,z)}{x}=+\infty$$

uniformly for every (y,z) in any compact subset of $\mathbb{R}^2$.

The following result will be useful in our study of the problem (4)-(5).

Lemma 2: If $u=u(t)$ is a solution of the boundary value problem (4)-(5) which satisfies that:

$$u''(t)<0,\quad 0\le t<1 \tag{6}$$

then $u(t)\ge 0, 0\le t\le 1$.

Remark 2: From the above Lemma we have that every solution of the boundary value problem (4)-(5) is positive, provided that (6) holds.

Theorem 2: If the hypotheses (H1)-(H2) hold then the boundary value problem (4) has a positive solution.

Remark 3: The above positive function $u_0=u_0(t)$ solves the boundary value problem (3).

Our existence theorem reads as follows.

Theorem 3: If the hypotheses (A1)-(A2) and (H1)-

(H2) hold, then the boundary value problem (1) has a positive solution.

3. Proof of Main Results

Proof of Lemma 2: Arguing by contradiction, suppose that there exists $T \in (\eta,1)$ such that:

$$u(T)=0 \text{ and } \begin{cases} u(t)>0, t\in(0,T) \\ u(t)<0, t\in(T,1] \end{cases}$$

We have $2\eta - T > 0$. We consider $t \in [2\eta - T, \eta]$ and $t' \in [\eta, T]$ where t' is the symmetric point of t with respect to η (i.e. $t' = 2\eta - t$). Because of the concavity of $u = u(t)$ and the map $u = u''(t), 0 \le t \le 1$ is increasing and negative we obtain,

$$u'(t) > -u'(t')$$

and we have

$$\int_0^{\eta} u'(t)\,\mathrm{d}t \ge \int_{2\eta-T}^{\eta} u'(t)\,\mathrm{d}t > -\int_{\eta}^{T} u'(t')\,\mathrm{d}t'$$
$$\Rightarrow u(T) = \int_0^{\eta} u'(t)\,\mathrm{d}t + \int_{\eta}^{T} u'(t)\,\mathrm{d}t > 0$$

a contradiction to the fact that $u(T)=0$. *QED*

Proof Theorem 2: In view of the assumptions (H1) and (H2) there exist $r_0 > 0$ and $R_0 > 0$ such that:

1) For $\frac{1}{M} > 0$ where $M = \int_0^1 c(t)\,\mathrm{d}t$ and for every $(t,x,y,z) \in ([0,1]\times[0,r_0]\times[-R_0,r_0]\times[-R_0,0])$ we have

$$\frac{f(t,x,y,z)}{x} < \frac{1}{M} \Rightarrow f(t,x,y,z) < \frac{x}{M} \le \frac{r_0}{M} \tag{7}$$

2) For $\frac{1}{\theta N} > 0$, where $0 < \theta < \frac{1}{2}$, $N = \int_{\theta}^{1-\theta} c(t)\,\mathrm{d}t$ and for every

$$(t,x,y,z) \in \left([0,1]\times[\theta R_0, +\infty] \times \left[\frac{R_0}{\eta}, \frac{(2+\eta^2)R_0}{\eta}\right] \times \left[-R_0, \frac{(2+\eta^2)R_0}{\eta}\right] \right)$$

we have

$$\frac{f(t,x,y,z)}{x} > \frac{1}{\theta N}$$
$$\Rightarrow f(t,x,y,z) > \frac{x}{\theta N} \ge \frac{\theta R_0}{\theta N} = \frac{R_0}{N} \tag{8}$$

Claim 1: There exists a region V, which depends on r_0 and η such that any solution $u = u(t)$ of the problem (4), which emanates from every initial point of V, satisfies

$$u'(\eta) < 0 \text{ and } u''(t) < 0, t \in [0,1]$$

If we take the region V where every initial point $(u_0', u_0'') \in V$ satisfies

$$u_0' = r_0 \text{ and } -R_0 \le u_0'' \le -r_0 \frac{1+\eta^2}{\eta}$$

then any solution for the boundary value problem (4) which emanates from (u_0', u_0''), satisfies $u'(\eta) < 0$ and $u''(t) < 0, t \in [0,1]$.

We proceed by contradiction, suppose that $u''(1) > 0$. By the sign property of f and c we have, $u'''(t) > 0$, $0 \le t \le 1$ which implies that the function $u''(t)$, $0 \le t \le 1$ is increasing, so there exists a $\tilde{t} \in (0,1)$ such that

$$u''(\tilde{t}) = 0 \textit{ and } -R_0 \le u''(t) < 0, \forall t \in [0,\tilde{t})$$

Which implies that

$$u'(t) \ge -R_0 t \ge -R_0, \forall t \in [0,\tilde{t})$$

Moreover, because the derivative $u'(t)$, $t \in [0,\tilde{t})$. is decreasing we obtain

$$u'(t) \le u_0' = r_0, t \in [0,\tilde{t}) \Rightarrow u(t) \le \tilde{t}u_0' \le u_0' = r_0$$

From (7) and the Taylor's formula, we take the contradiction, hence

$$\begin{aligned} 0 &= u''(\tilde{t}) \\ &= u_0'' + \tilde{t}\int_0^1 c(s\tilde{t}) f\left(s\tilde{t}, u(s\tilde{t}), u'(s\tilde{t}), u''(s\tilde{t})\right)\mathrm{d}s \\ &\le u_0'' + \tilde{t}\frac{r_0}{M}\int_0^1 c(s\tilde{t})\,\mathrm{d}s \le u_0'' + \tilde{t}r_0 < u_0'' + r_0 < 0 \end{aligned}$$

In addition, again from (7) and the Taylor's formula, we obtain

$$\begin{aligned} u'(\eta) &= u_0' + \eta u_0'' \\ &\quad + \eta^2 \int_0^1 (1-s) c(s\eta) f\left(s\eta, u(s\eta), u'(s\eta), u''(s\eta)\right)\mathrm{d}s \\ &< u_0' + \eta u_0'' + \eta^2 r_0 \le 0 \end{aligned}$$

This proves Claim 1.

Let us fix a point $A(u_0', u'') \in V$ and let $B(u_0', 0)$. By the definition of B, every $u \in \mathcal{X}(B)$ ($\mathcal{X}(B)$ denotes the set of solutions of (4) emanating from the initial point B), has the property that $u'(\eta) > 0$.

Claim 2: There exists a region U which depends on R_0, r_0 and η such that any solution $u = u(t)$ of the problem (4), which emanates from every initial point of U, satisfies

$$\|u\| \ge \theta R_0, u'(t) > 0, t \in [0,1] \text{ and } u''(1) \ge 0$$

If we take the region U where every initial point $(u_*',u_0'') \in U$ satisfies

$$\frac{(2+\eta^2)R_0}{\eta} \ge u_*' \ge \frac{R_0}{\eta} - u_0''$$

then any solution of problem (4) emanating from (u_*',u_0'') satisfies

$$\|u\| \ge \theta R_0, u'(t) > 0, t \in [0,1] \text{ and } u''(1) \ge 0$$

Arguing by contradiction, assume $u''(1) < 0$. Then, since the function $u''(t)$, $0 \le t \le 1$ is increasing we obtain $u''(t) < 0$, $0 \le t \le 1$. That means the function $u'(t)$, $0 \le t \le 1$ is decreasing. It follows that

$$u'(t) \le u_*',\ 0 \le t \le 1 \Rightarrow u'(t) \le \frac{(2+\eta^2)R_0}{\eta}, 0 \le t \le 1$$

and, from the Taylor's formula, we have

$$\begin{aligned} u'(t) &= u_*' + tu_0'' \\ &+ t^2 \int_0^1 (1-s)c(st) f(st,u(st),u'(st),u''(st))\mathrm{d}s \\ &\ge u_*' + u_0''t \ge u_*' + u_0'' \ge \frac{R_0}{\eta} \end{aligned}$$

So, we have

$$u'(t) \ge \frac{R_0}{\eta} \ge 0 \text{ for every } t \in [0,1]$$

So, we obtain

$$\frac{R_0}{\eta} \le u'(t) \le \frac{(2+\eta^2)R_0}{\eta}, 0 \le t \le 1$$

hence we get

$$u(t) = \int_0^t u'(s)\mathrm{d}s \ge \eta^{-1} t R_0$$

Moreover, because of the fact that $u = u(t)$, $t \in [0,1]$ is increasing, we obtain

$$\min_{\theta \le x \le 1-\theta} u(t) = u(\theta) \ge \eta^{-1}\theta R_0 \ge \theta R_0 \tag{9}$$

Using (8) and (9) we take the contradiction

$$\begin{aligned} 0 > u''(1) &= u_0'' + \int_0^1 c(s) f(s,u(s),u'(s),u''(s))\mathrm{d}s \\ &> u_0'' + \int_0^{1-\theta} c(s) f(s,u(s),u'(s),u''(s))\mathrm{d}s \\ &\ge u_0'' + R_0 \ge 0, \text{ a contradiction} \end{aligned}$$

The Claim 2 is true.

Let us fix another point $\Gamma(u_\Gamma',u_\Gamma'') \in U$. We consider the Simplex $S = [A,B,\Gamma]$.

Claim 3: Every solution of the boundary value problem (4) emanating from any initial point $\Delta(u_1',u_1'') \in [\Gamma,\mathrm{B}]$ satisfies $u'(\eta) > 0$.

For $\Delta(u_1',u_1'') \in [\Gamma,\mathrm{B}]$ we have

$$\frac{u_0''}{u_1''} = \frac{u_\Gamma' - u_0'}{u_1' - u_0'} \Rightarrow u_1'' = \frac{u_0''(u_1' - u_0')}{u_\Gamma' - u_0'} \tag{10}$$

From (10) we have

$$\begin{aligned} u_1' + \eta u_1'' &= u_1' + \eta\left(\frac{u_0''(u_1' - u_0')}{u_\Gamma' - u_0'}\right) \\ &> u_1'\left(1 + \frac{\eta u_0''}{u_\Gamma' - u'}\right) - \frac{\eta u_0'' u_0'}{u_\Gamma' - u_0'} > u_0' > 0 \end{aligned} \tag{11}$$

$$\Rightarrow u_1' + \eta u_1'' > 0$$

From (11) and the Taylor's formula it follows that

$$\begin{aligned} u'(\eta) &= u_1' + \eta u_1'' \\ &+ \eta^2 \int_0^1 (1-s)c(s\eta) f(s\eta,u(s\eta),u'(s\eta),u''(s\eta))\mathrm{d}s \\ &\ge u_1' + \eta u_1'' > u_1' + u_1'' > 0 \end{aligned}$$

This proves Claim 3.

By the Kneser's Theorem 1 and the Claims 1 and 2 there exist points $\Delta_1, \Delta_2 \in [\mathrm{A},\Gamma]$ such that

$$u'(\eta) = 0, \text{for some solution } u \in \mathcal{X}(\Delta_1) \tag{12}$$

$$u''(1) = 0, \text{for some solution } u \in \mathcal{X}(\Delta_2) \tag{13}$$

By the Kneser's Theorem 1 and the Claim 1 and since $u'(\eta) > 0$ for $u \in \mathcal{X}(B)$ there exists point $\Delta_3 \in [A,B]$ such that

$$u'(\eta) = 0, \text{ for some solution } u \in \mathcal{X}(\Delta_1)$$

Claim 4: If $\Delta(u_0',u_\Delta'') \in [A,B]$ such that $u'(\eta) = 0$, for some solution $u \in \mathcal{X}(\Delta)$ then $u''(1) \le 0$.

Arguing by contradiction, assume $u''(1) > 0$. As in proof of Claim 1, by the sign property of f and c we have $u'''(t) > 0$, $0 \le t \le 1$ which implies that the function $u''(t)$, $0 \le t \le 1$ is increasing, so there exists a $\tilde{t} \in (0,1)$ such that

$$u''(\tilde{t}) = 0 \text{ and } -R_0 \le u''(t) < 0 \ \ \forall t \in [0,\tilde{t})$$

Which implies that

$$r_0 > u'(t) \ge -R_0 t \ge -R_0, \ \forall t \in [0,\tilde{t})$$

so, we have

$$\begin{aligned} 0 = u''(\tilde{t}) &= u_\Delta'' \\ &+ \tilde{t}\int_0^1 (1-s)c(s\tilde{t}) f(s\tilde{t},u(s\tilde{t}),u'(s\tilde{t}),u''(s\tilde{t}))\mathrm{d}s \\ &\le u_\Delta'' + \tilde{t}\frac{r_0}{A}\int_0^1 c(s\tilde{t})\mathrm{d}s \le u_\Delta'' + \tilde{t}r_0 < u_\Delta'' + r_0 \end{aligned}$$

we obtain

$$0 \le u''_{\Delta} + r_0 \tag{14}$$

On the other hand, we have

$$0 = u'(\eta) = u'_0 + \eta u''_{\Delta} + \int_0^{\eta} (\eta - s) c(s) f(s, u(s), u'(s), u''(s)) \mathrm{d}s \ge u'_0 + \eta u''_{\Delta} > u'_0 + u''_{\Delta}$$

we obtain

$$0 > u'_0 + u''_{\Delta} \tag{15}$$

From (14) and (15) we take the contradiction.

This proves Claim 4.

We consider now the sets

$$C_1 = \{\Delta(u'_1, u''_1) \in S : u'(\eta) = 0, u''(1) \le 0\}$$

and

$$C_2 = \{\Delta(u'_1, u''_1) \in S : u'(\eta) \ge 0, u''(1) = 0\}$$

From the Claim 4 we have $C_1 \ne \varnothing$ and from the Claims 2 and 4 we have $C_2 \ne \varnothing$.

We suppose that $C_1 \cap C_2 = \varnothing$, otherwise we don't have anything to prove.

Recalling that S is the simplex with vertices $A(u'_0, u''_0)$, $B(u'_0, 0)$, and $\Gamma(u'_{\Gamma}, u''_0)$.

We define the closed sets

$$E_A =: \{(\tilde{u}'_0, u''_0) \in S : u'(\eta) \le 0, u''(1) \le 0\}$$
$$E_B = \{(\tilde{u}'_0, u''_0) \in S : u''(1) \ge 0\}$$
$$E_{\Gamma} =: \{(\tilde{u}'_0, u''_0) \in S : u'(\eta) \ge 0\}$$

where $u(t)$ denotes a solution for the problem (4) emanating from the corresponding initial point in S.

We have $\Gamma \in E_A \ne \varnothing$ from the Claim 1, $B \in E_B \ne \varnothing$ from the nature of the vector field and $\Gamma \in E_{\Gamma} \ne \varnothing$ from the Claim 2.

Take a point Δ of the phase $[A, B]$ then

1) either $u''(1) \le 0$ and $u'(\eta) \le 0$ then $\Delta \in E_A \subset E_A \cup E_B$.

2) or $u''(1) \ge 0$ then $\Delta \in E_B \subset E_A \cup E_B$

3) or $u'(\eta) \ge 0$ and $u''(1) \le 0$ then we have a contradiction from the Claim 4.

Consequently, we have

$$[A, B] \subset E_A \cup E_B \tag{16}$$

On the other hand, let point Δ of the phase $[A, \Gamma]$ then

1) either $u''(1) \le 0$ and $u'(\eta) \le 0$ then $\Delta \in E_A \subset E_A \cup E_{\Gamma}$.

2) or $u'(\eta) \ge 0$ then $\Delta \in E_{\Gamma} \subset E_A \cup E_{\Gamma}$

3) or $u'(\eta) \le 0$ and $u''(1) \ge 0$ then $C_1 \cap C_2 \ne \varnothing$ that is a contradiction.

Consequently, we have

$$[A, \Gamma] \subset E_A \cup E_{\Gamma} \tag{17}$$

Finally, if $\Delta \in [\Gamma, B]$ then from the Claim 2 we have $u'(\eta) \ge 0$ which implies that $\Delta \in E_{\Gamma} \subset E_B \cup E_{\Gamma}$. Therefore $E_{\Gamma} \subset E_B \cup E_{\Gamma}$ is a suitable closed covering of S that satisfies the hypotheses of Sperner's lemma. Thus, there exists an initial point $\tilde{\Delta}(\tilde{u}'_0, \tilde{u}''_0)$ such that $\tilde{\Delta} \in E_{\Gamma} \subset E_B \cup E_{\Gamma}$.

The case that we have two solutions $u_1, u_2 \in \mathcal{X}(\tilde{\Delta})$ of the problem (4) with $u''_1(\eta) = 0$, $u''_1(1) \ne 0$ and $u'_2(\eta) \ne 0$, $u''_2(1) = 0$ has been addressed by Palamides, Infante and Pietramala [17]. They approached the continuous nonlinearity by a sequence of locally Lipschitz functions and then each such a Lipschitz boundary value problem ensure the existence of a solution. Finally the well-known Kamke theorem may be applied, to get a solution of the boundary value problem (3), as a limit solution.

This means that the corresponding solution $u_0 = u_0 \in \mathcal{X}(\tilde{\Delta})$ is a solution of the boundary value problem (4)-(5). <u>*QED*</u>

Proof Theorem 3: From the Remark 3 we have a positive solution u_0 for the boundary value problem (3).

We consider the boundary value problem

$$\left.\begin{aligned} & x''(t) = u_0(t), 0 \le t \le 1 \\ & ax(\xi_1) - \beta x'(\xi_1) = 0 \\ & \gamma x(\xi_2) + \delta x'(\xi_2) = 0 \end{aligned}\right\} \tag{18}$$

Then it is known (see for example [9]) that (18) has the solution

$$x(t) = \int_{\xi_1}^{t} (t - s) u_0(s) \mathrm{d}s + \frac{1}{p} \int_{\xi_1}^{\xi_2} (a(\xi_1 - t) - \beta)(\gamma(\xi_2 - s) + \delta) u_0(s) \mathrm{d}s,$$
$$0 \le t \le 1$$

Consequently in view of the transformation $u(t) = x''(t)$, a solution for the initial boundary value problem (1) is given by the last formula.

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5. References

[1] L. H. Erbe and H. Wang, "On Existence of Positive Solu-

tions of Ordinary Differential Equations," *Proceedings of the American Mathematical Society*, Vol. 120, 1994, pp. 743-748. doi:10.1090/S0002-9939-1994-1204373-9

[2] D. R. Anderson and J. M. Davis, "Multiple Solutions and Eigenvalues for Third-Order Right Focal Boundary-Value Problem," *Journal of Mathematical Analysis and Applications*, Vol. 267, No. 1, 2002, pp. 135-157. doi:10.1006/jmaa.2001.7756

[3] Z. Bai and H. Wang, "On Positive Solutions of Some Nonlinear Fourth-Order Beam Equations," *Journal of Mathematical Analysis and Applications*, Vol. 270, No. 2, 2002, pp. 357-368. doi:10.1016/S0022-247X(02)00071-9

[4] J. R. Graef and B. Yang, "On a Nonlinear Boundary-Value Problem for Fourth Order Equations," *Applied Analysis*, Vol. 72, 1999, pp. 439-448. doi:10.1080/00036819908840751

[5] J. R. Graef and B. Yang, "Positive Solutions to a Multi-Point Higher Order Boundary-Value Problem," *Journal of Mathematical Analysis and Applications*, Vol. 316, No. 2, 2006, pp. 409-421. doi:10.1016/j.jmaa.2005.04.049

[6] J. R. Graef, J. Henderson and B. Yang, "Positive Solutions of a Nonlinear Higher Order Boundary-Value Problem," *Electronic Journal of Differential Equations*, Vol. 2007, No. 45, 2007, pp. 1-10.

[7] P. S. Kelevedjievand P. K. Palamides and N. I. Polivanov "Another Understanding of Fourth-Order Four-Point Boundary-Value Problems," *Electronic Journal of Differential Equations*, Vol. 2008, No. 47, 2008, pp. 1-15.

[8] Z. Hao, L. Liu and L. Debnath, "A Necessary and Sufficiently Condition for the Existence of Positive Solution of Fourth-Order Singular Boundary-Value Problems," *Applied Mathematics Letters*, Vol. 16, No. 3, 2003, pp. 279-285. doi:10.1016/S0893-9659(03)80044-7

[9] J. Ge and C. Bai, "Solvability of a Four-Point Boundary-Value Problem for Fourth-Order Ordinary," *Electronic Journal of Differential Equations*, Vol. 2007, No. 123, 2007, pp. 1-9.

[10] G. G. Doronin and N. A. Larkin, "Boundary Value Problems for the Stationary Kawahara Equation," *Nonlinear Analysis*, Vol. 69, No. 5-6, 2008, pp. 1655-1665. doi:10.1016/j.na.2007.07.005

[11] S. N. Odda, "Existence Solution for 5th Order Differential Equations under Some Conditions," *Applied Mathematics*, Vol. 1, No. 4, 2010, pp. 279-282. doi:10.4236/am.2010.14035

[12] M. El-shahed and S. Al-Mezel, "Positive Solutions for Boundary Value Problems of Fifth-Order Differential Equations," *International Mathematical Forum*, Vol. 4, No. 33, 2009, pp. 1635-1640.

[13] M. A. Noor and S. T. Mohyud-Din, "A New Approach to Fifth-Order Boundary Value Problems," *International Journal of Nonlinear Science*, Vol. 7, No. 2, 2009, pp. 143-148.

[14] M. A. Noor and S. T. Mohyud-Din, "Variational Iteration Method for Fifth-Order Boundary Value Problems Using He's Polynomials," *Hindawi Publishing Corporation Mathematical Problems in Engineering*, Vol. 2008, 2008, Article ID 954794.

[15] A. Granas and J. Dugundji, "Fixed Point Theory," Springer-Verlag, New York, 2003.

[16] W. A. Copel, "Stability and Asymptotic Behavior of Differential Equations," Heath & Co. Boston, Boston, 1965.

[17] P. K. Palamides, G. Infante and P. Pietramala, "Nontrivial Solutions of a Nonlinear Heat Flow Problem via Sperner's Lemma," *Applied Mathematics Letters*, Vol. 22, No. 9, 2009, pp. 1444-1450. doi:10.1016/j.aml.2009.03.014

