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H_∞ Finite-Time Control for Switched Linear Systems with Time-Varying Delay

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Abstract

Finite-time boundedness and H_∞ finite-time boundedness of switched linear systems with time-varying delay and exogenous disturbances are addressed. Based on average dwell time (ADT) and free-weight matrix technologies, sufficient conditions which can ensure finite-time boundedness and H_∞ finite-time boundedness are given. And then in virtue of the results on finite-time boundedness, the state memory feedback controller is designed to H_∞ finite-time stabilize a time-delay switched system. These conditions are given in terms of LMIs and are delay-dependent. An example is given to illustrate the efficiency of the proposed method.

Keywords: Switched System, Time-Delay, H_∞ Finite-Time Boundedness, ADT

1. Introduction

A switched system is a special kind of hybrid system, which is composed of a family of subsystems and a switching sequence orchestrating the switching between the subsystems. Recently, switched systems have received a great deal of attention, and commonly been found in automotive engine control systems, network control, process control, traffic control, etc. Many important progress and remarkable results have been made on basic problems concerning stability and design of switched systems [1-10]. For recent progress, readers can refer to survey papers [11-13] and the references therein. Many Lyapunov function techniques are effective tools dealing with switched systems [14-17]. Average dwell time and dwell time (DT) approaches were employed to study the stability and stabilization of time-dependent switched systems [18-20].

Time-delay, which is a common phenomenon encountered in many engineering process, is known to be great sources of poor performance and instability. For switched systems, because of the complicated behavior caused by the interaction between the continuous dynamics and discrete switching, the problem of time delays is more difficult to study [21]. The current methods of stabilization for time-delay systems can be classified into two categories: delay-independent and delay-dependent stabilization [22-24]. In [25], by using free weighting matrix scheme and average dwell time method incorporated with a piecewise Lyapunov functional, exponentially stability and L_2-gain were analyzed for a class of switched systems with time-varying delays and disturbance input. In [26], the robust stability, robust stabilization and H_∞ control problems for time-delay discrete switched singular systems with parameter uncertainties are discussed.

Up to now, most of existing literature related to stability of switched systems investigates Lyapunov asymptotic stability, which is defined over an infinite time interval. However, in practice, one is interested in not only system stability (usually in the sense of Lyapunov) but also a bound of system trajectories over a fixed short time [27]. The finite-time stability is a different stability concept which admits the state does not exceed a certain bound during a fixed finite-time interval. Some early results on finite-time stability can be found in [28-30]. Finite-time stability and stabilization for discrete linear system were investigated in [31]. In [32], finite-time stabilization of linear time-varying systems has been discussed. It should be pointed out that a finite-time stable system may not be Lyapunov asymptotical stable, and a Lyapunov asymptotical stable system may not be finite-time stable since the transient of a system response may exceed the bound [33]. So far, however, compared with numerous research results about Lyapunov stability, few results on finite-time stability have been given in literature about the finite-time boundedness switched systems with time-delay. This motivates us to study in this area.

In [27], finite-time boundedness and finite-time weighted L_2-gain for a class of switched delay systems with time-varying exogenous disturbances is investigated. In [33], the problems of finite-time stability analysis and stabilization for switched nonlinear discrete-time systems are addressed, and then the results are extended to H_∞ finite-time boundedness of switched nonlinear discrete-time systems. In [34], finite-time stability and stabilization problems for a class of switched linear systems were studied, and the state feedback controllers and a class of switching signals with average dwell-time have been designed to stabilize the switched linear control systems.

However, to the best of authors' knowledge, there is no result available yet on finite-time stability of switched systems with time-varying delay. Thus, it is necessary to investigate finite-time stability and finite-time boundedness for a class of switched linear systems with time-varying delay, which is an important property for switched system. Our contributions are given as follows: 1) Definitions of finite-time boundedness and H_∞ finite-time are extended to switched linear systems with time-varying delay. 2) Sufficient conditions for finite-time boundedness and H_∞ finite-time boundedness of switched linear systems with time-varying delay are given. 3) A set of memory state feedback controllers are designed to guarantee the closed-loop switched system with time-varying delay H_∞ finite-time bounded.

The paper is organized as follows. In Section 2, some definitions and problem formulations are presented. In Section 3, based on ADT technology and LMIs, sufficient conditions which ensure finite-time stability of switched linear systems with time-varying delay are given. In Section 4, sufficient conditions which guarantee the switched system has H_∞ finite-time are presented. In Section 5, a set of memory state feedback controllers are designed, which can guarantee the closed-loop switched system H_∞ finite-time bounded. Finally, an example is presented to illustrate the efficiency of the proposed method in Section 6. Conclusions are given in Section 7.

Notations: The notations used in this paper are standard. The notation $P > 0$ means that P is a real symmetric and positive definite; the symbol '*' within a matrix represents the symmetric term of the matrix; the superscript 'T' stands for matrix transposition; R^n denotes the n-dimensional Euclidean space; I and 0 represent the identity matrix and a zero matrix, respectively; $\mathrm{diag}\{\cdots\}$ stands for a block-diagonal matrix. $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximum and minimum eigen-values of matrix P, respectively; Notations 'sup' and 'inf' denote the supremum and infimum, respectively.

2. Preliminaries and Problem Formulation

In this paper, a switched linear system with time-varying delay is described as follows:

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + A_{d\sigma(t)}x(t-d(t)) + B_{\sigma(t)}u(t) + G_{\sigma(t)}\omega(t) \\ z(t) = C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t) + E_{\sigma(t)}\omega(t) \\ x(t) = \varphi(t) \qquad t \in [-\tau, 0] \end{cases} \tag{1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input, $z(t) \in R^m$ is the measurement output, $A_{\sigma(t)}$, $A_{d\sigma(t)}$, $B_{\sigma(t)}$, $G_{\sigma(t)}$, $C_{\sigma(t)}$, $D_{\sigma(t)}$ and $E_{\sigma(t)}$ are real known constant matrices with appropriate dimensions, $\varphi(t)$ is the continuous vector valued function specifying the initial state of the system, $\omega(t)$ is the time-varying exogenous noise signal and satisfies Assumption 1, $\sigma(t):[0 \quad \infty) \to I = \{1, 2, \cdots, N\}$ is the switching signal, corresponding to it, the switching sequence $\{x_0; (i_0, t_0), (i_1, t_1), \cdots, (i_k, t_k), \cdots, | i_k \in I, k = 0, 1, \cdots\}$ means that the i_k th subsystem is activated when $t \in [t_k, t_{k+1})$. $d(t)$ denotes the time-delay satisfying Assumption 2.

Assumption 1. The exogenous noise signal is time-varying and satisfies

$$\int_0^\infty \omega^T(t)\omega(t)\mathrm{d}t < d, \qquad d \ge 0. \tag{2}$$

Assumption 2. The time-varying delay satisfies

$$0 \le \mathrm{d}(t) < \tau, \qquad \dot{\mathrm{d}}(t) \le h < 1. \tag{3}$$

Remark 1. It should be pointed out that the Assumption 2 about time-varying delay d(t) in this paper is different from that of [27], where the time-delay is constant. In [33], the concept of finite-time boundedness and H_∞ finite-time boundedness for discrete switched system were proposed. In this paper, we extend the definitions to continuous switched linear system with time-varying delay. First, the following three lemmas are presented, which play important roles in our further derivation.

Lemma 1 [35]. The linear matrix inequality $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} < 0$, where $S_{11} = S_{11}^T$ and $S_{22} = S_{22}^T$ are equivalent to $S_{22} < 0$, $S_{11} - S_{12}S_{22}^{-1}S_{12}^T < 0$.

Lemma 2 [36]. For any $T \ge t \ge 0$, let $N_\sigma(t, T)$ denote the switching number of $\sigma(t)$ over (t, T). If

$$N_\sigma(t, T) \le N_0 + (T - t)/\tau_a \tag{4}$$

holds for and an integer , then is called an average dwell-time.

Lemma 3 [37]. For given symmetrical matrix X, $\begin{bmatrix} P_1+X & Q_1 \\ * & R_1 \end{bmatrix} > 0$ and $\begin{bmatrix} P_2-X & Q_2 \\ * & R_2 \end{bmatrix} > 0$ are satisfied simultaneously, if and only if the following inequality holds

$$\begin{bmatrix} P_1+P_2 & Q_1 & Q_2 \\ * & R_1 & 0 \\ * & * & R_2 \end{bmatrix} > 0 \tag{5}$$

Definition 1. (*Finite-time stability*) Switched system (1) with $u(t) \equiv 0$ and $\omega(t) \equiv 0$ is said to be finite-time stability with respect to $(\delta, \varepsilon, T_f, d, R, \sigma)$, where $0 \le \delta < \varepsilon$ and $d \ge 0$, R is positive definite matrix and $\sigma(t)$ is a switching signal. If $x^T(t)Rx(t) < \varepsilon$, $\forall t \in [0, T_f]$, whenever $\sup_{-\tau \le \theta \le 0}\{x^T(\theta)Rx(\theta)\} < \delta$. If the above condition holds for any switching signal $\sigma(t)$, system (1) is said to be uniformly finite-time stability with respect to $(\delta, \varepsilon, T_f, d, R)$.

Remark 2. As can be seen from Definition 1, the concept of finite-time stability and Lyapunov asymptotic stability are different. A Lyapunov asymptotically stable switched system may not be finite-time stable if its states exceed the prescribed bounds.

Remark 3. The meaning of "uniformity" in Definition 1 is with respect to the switching signal, rather than the time, which is identical to that of [11].

Next, the definitions of finite-time boundedness and H_∞ finite-time boundedness for switched system with time-varying delay are introduced.

Definition 2. (*Finite-time boundedness*) Switched system (1) with $u(t) \equiv 0$ is said to be finite-time boundedness with respect to $(\delta, \varepsilon, T_f, d, R, \sigma)$, where $0 \le \delta < \varepsilon$ and $d \ge 0$, R is positive definite matrix and $\sigma(t)$ is a switching signal. If $x^T(t)Rx(t) < \varepsilon$, $\forall t \in [0, T_f]$, $\forall \omega(t): \int_0^{T_f} \omega^T(t)\omega(t)\,\mathrm{d}t < d$, whenever $\sup_{-\tau \le \theta \le 0}\{x^T(\theta)Rx(\theta)\} < \delta$.

Definition 3. (*H_∞ finite-time boundedness*) Switched system (1) with $u(t) \equiv 0$ is said to be H_∞ finite-time boundedness with respect to $(\delta, \varepsilon, T_f, d, R, \sigma)$, where $0 \le \delta < \varepsilon$, $d \ge 0$, $\gamma > 0$, R is positive definite matrix and $\sigma(t)$ is a switching signal, following conditions should be satisfied:

1) Switched system (1) is finite-time bounded.

2) Under zero-initial condition $\varphi(t) = 0$, $\forall t \in [-\tau, 0]$, the output $z(t)$ satisfies

$$\int_0^{T_f} z^T(t)z(t)\,\mathrm{d}t < \gamma^2 \int_0^{T_f} \omega^T(t)\omega(t)\,\mathrm{d}t. \tag{6}$$

In this paper, the main purpose is to find sufficient conditions, which can ensure the finite-time boundedness and H_∞ finite-time boundedness, and apply these conditions to design H_∞ finite-time stabilizing controller.

Remark 4. Definition 3 means that once a switching signal is given, a switched system is H_∞ finite-time boundedness if, given a bound on initial state and a H_∞-gain γ, the state remains within the prescribed bound in the fixed finite-time interval.

3. Finite-Time Stability and Bounded Analysis

In this section, we focus on finite-time boundedness of switched time-delay system (1) with $u(t) \equiv 0$, that is

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + A_{d\sigma(t)}x(t-d(t)) + G_{\sigma(t)}\omega(t), & t > 0 \\ x(t) = \varphi(t) & t \in [-\tau, 0] \end{cases} \tag{7}$$

Now, let us discuss the finite-time boundedness of switched time-delay system (6). For a symmetric positive definite matrix $R \in R^{n \times n}$, it is easy to verify that R can be factorized according to $R = (R^{1/2})^T(R^{1/2})$, where $R^{1/2}$ is also a symmetric positive definite matrix.

Theorem 1. For any $i \in I$, let $P_i = R^{1/2}\tilde{P}_i R^{1/2}$, $Q_i = R^{1/2}\tilde{Q}_i R^{1/2}$, $S_i = R^{1/2}\tilde{S}_i R^{1/2}$. Suppose that there exist matrices $\tilde{P}_i > 0$, $\tilde{Q}_i > 0$, $\tilde{S}_i > 0$, $W_i > 0$, $N_{1,i}$, $N_{2,i}$, $X_i = \begin{bmatrix} X_{11,i} & X_{12,i} \\ * & X_{22,i} \end{bmatrix} \ge 0$ and constants $\alpha_i \ge 0$, $\beta \ge 0$ such that

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ * & \Omega_{22} & \Omega_{23} \\ * & * & \Omega_{33} \end{bmatrix} < 0 \tag{8}$$

$$\Psi = \begin{bmatrix} X_{11,i} & X_{12,i} & N_{1,i} \\ * & X_{22,i} & N_{2,i} \\ * & * & e^{-\alpha_i \tau}S_i \end{bmatrix} > 0 \tag{9}$$

$$(\lambda_2 + \tau e^{\tau\alpha_i}\lambda_3)\delta + \tau e^{\tau\alpha_i}\lambda_4\beta + d\sup_{i \in I}(\lambda_{\max}(W_i)) < \lambda_1 e^{-\alpha_i T_f}\varepsilon \tag{10}$$

where

$$\Omega_{11} = A_i^T P_i + P_i A_i - \alpha_i P_i + Q_i + \tau A_i^T S_i A_i + N_{1,i} + N_{1,i}^T + \tau X_{11,i}.$$
$$\Omega_{12} = P_i A_{di} + \tau A_i^T S_i A_{di} - N_{1,i} + N_{2,i}^T + \tau X_{12,i},$$
$$\Omega_{13} = P_i G_i + \tau A_i^T S_i G_i,$$
$$\Omega_{22} = -(1-h)e^{\alpha_i \tau}Q_i + \tau A_{di}^T S_i A_{di} - N_{2,i} - N_{2,i}^T + \tau X_{22,i},$$
$$\Omega_{23} = \tau A_{di}^T S_i G_i,$$
$$.\Omega_{33} = \tau G_i^T S_i G_i - W_i.$$

If the average dwell time of the switching signal satisfies

$$\tau_a > \tau_a^* = \frac{T_f \ln \mu}{\ln(\lambda_1 \varepsilon) - \ln\left(\lambda_2 + \tau e^{\tau\alpha_i}\lambda_3\right)\delta + \tau e^{\tau\alpha_i}\lambda_4 \beta + \upsilon} \quad (11)$$

then the switched systems is finite-time boundedness with respect to $(\delta, \varepsilon, T_f, d, R, \sigma)$, where $\mu \ge 1$, $\upsilon = d\sup_{i\in I}\left(\lambda_{\max}(W_i)\right) - \alpha T_f - N_0 \ln \mu$, $\tilde{P}_i \le \mu \tilde{P}_j$, $\tilde{Q}_i \le \mu \tilde{Q}_j$, $\tilde{S}_i \le \mu \tilde{S}_j$, $\forall i, j \in I$, $\alpha = \max_{i\in I}\{\alpha_i\}$, $\lambda_1 = \inf_{i\in I}\left\{\lambda_{\min}\left(\tilde{P}_i\right)\right\}$ $\lambda_2 = \sup_{i\in I}\left\{\lambda_{\max}\left(\tilde{P}_i\right)\right\}$, $\lambda_3 = \sup_{i\in I}\left\{\lambda_{\max}\left(\tilde{Q}_i\right)\right\}$, $\lambda_4 = \sup_{i\in I}\left\{\lambda_{\max}\left(\tilde{S}_i\right)\right\}$.

Proof. Choose a Lyapunov-like function as follows

$$V(t) = V_i(t) = V_{1,i}(t) + V_{2,i}(t) + V_{3,i}(t) \quad (12)$$

where

$$V_{1,i}(t) = x^T(t) P_i x(t),$$
$$V_{2,i}(t) = \int_{t-d(t)}^{t} e^{\alpha_i(t-s)} x^T(s) Q_i x(s)\,\mathrm{d}s,$$
$$V_{3,i}(t) = \int_{-\tau}^{0}\int_{t+\theta}^{t} e^{\alpha_i(t-s)} \dot{x}^T(s) S_i \dot{x}(s)\,\mathrm{d}s\mathrm{d}\theta.$$

When $t \in [t_k, t_{k+1})$, taking the derivative of *V*(t) with respect to *t* along the trajectory of switched system (7), we have

$$\begin{aligned}\dot{V}_{1,i}(t) &= \dot{x}^T(t) P_i x(t) + x^T(t) P_i \dot{x}(t) \\ &= x^T(t)\left(A_i^T P_i + P_i A_i\right)x(t) \\ &\quad + x^T(t-d(t)) A_{di}^T P_i x(t) \\ &\quad + x^T(t) P_i A_{di} x(t-d(t)) \\ &\quad + \omega^T(t) G_i^T P_i x(t) \\ &\quad + x^T(t) P_i G_i \omega(t)\end{aligned} \quad (13)$$

$$\begin{aligned}\dot{V}_{2,i}(t) &= \alpha_i V_{2,i}(t) + x^T(t) Q_i x(t) \\ &\quad - \left(1 - \dot{d}(t)\right) e^{\alpha_i d(t)} x^T(t-d(t)) Q_i x(t-d(t)) \\ &\le \alpha_i V_{2,i}(t) + x^T(t) Q_i x(t) \\ &\quad - (1-h) e^{\alpha_i \tau} x^T(t-d(t)) Q_i x(t-d(t))\end{aligned} \quad (14)$$

$$\begin{aligned}\dot{V}_{3,i}(t) &= \alpha_i V_{3,i}(t) + \tau \dot{x}^T(t) S_i \dot{x}(t) \\ &\quad - \int_{-\tau}^{0} e^{\alpha_i \theta} \dot{x}^T(t+\theta) S_i \dot{x}(t+\theta)\,\mathrm{d}\theta \\ &\le \alpha_i V_{3,i}(t) + \tau \dot{x}^T(t) S_i \dot{x}(t) \\ &\quad - \int_{t-\tau}^{t} e^{-\alpha_i(s-t)} \dot{x}^T(s) S_i \dot{x}(s)\,\mathrm{d}s\end{aligned} \quad (15)$$

From the Leibniz-Newton formula, the following equation is true for any matrices $N_{1,i}$, $N_{2,i}$, $(i \in I)$ with appropriate dimensions

$$\begin{aligned}&2\left[x^T(s) x^T(t-\mathrm{d}(s))\right]\begin{bmatrix} N_{1,i} \\ N_{2,i}\end{bmatrix} \\ &\times\left[x(t) - x(t-d(t)) - \int_{t-d(t)}^{t} \dot{x}(s)\,\mathrm{d}s\right] = 0\end{aligned} \quad (16)$$

For any matrices $X_i \ge 0$, $(i \in I)$ with appropriate dimensions, we have

$$\tau \eta_1^T(t) X_i \eta_1(t) - \int_{t-d(t)}^{t} \eta_1^T(s) X_i \eta_1(s)\,\mathrm{d}s \ge 0 \quad (17)$$

where $\eta_1(t) = \left[x^T(t) x^T(t-d(t))\right]^T$.

Then, it follows from (13)-(17) that

$$\begin{aligned}&\dot{V}(t) - \alpha_i V(t) = \dot{V}_{1,i}(t) + \dot{V}_{2,i}(t) + \dot{V}_{3,i}(t) - \alpha_i V(t) \\ &\le \begin{bmatrix} x(t) \\ x(t-d(t)) \\ \omega(t) \end{bmatrix}^T \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ * & \Omega_{22} & \Omega_{23} \\ * & * & \Omega_{33} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-d(t)) \\ \omega(t) \end{bmatrix} \\ &- \int_{t-\tau}^{t} e^{-\alpha_i(s-t)} \dot{x}^T(s) S_i \dot{x}(s)\,ds \\ &- 2\left[x^T(t) N_{1,i} + x^T(t-d(t)) N_{2,i}\right] \\ &\times \int_{t-d(t)}^{t} \dot{x}(s)\,\mathrm{d}s \\ &- \int_{t-d(t)}^{t} \eta_1^T(s) X_i \eta_1(s)\,\mathrm{d}s + \omega^T(t) W_i \omega(t) \\ &\le \eta_2^T(t) \Omega \eta_2(t) \\ &- \int_{t-d(t)}^{t} \eta_3^T(t,s) \Psi \eta_3(t,s)\,\mathrm{d}s + \omega^T(t) W_i \omega(t)\end{aligned} \quad (18)$$

Assuming conditions (8) and (9) are satisfied, we obtain

$$\dot{V}(t) - \alpha_i V(t) < \omega^T(t) W_i \omega(t) \quad (19)$$

By calculation, we have

$$\begin{aligned}V(t) &< e^{\alpha_{\sigma(t_k)}(t-t_k)} V_{\sigma(t_k)}(t_k) \\ &+ \int_{t_k}^{t} e^{\alpha_{\sigma(t_k)}(t-s)} \omega^T(s) W_{\sigma(t_k)} \omega(s)\,\mathrm{d}s\end{aligned} \quad (20)$$

Since $\mu \ge 1$, $\tilde{P}_i \le \mu \tilde{P}_j$, $\tilde{Q}_i \le \mu \tilde{Q}_j$, $\tilde{S}_i \le \mu \tilde{S}_j$ and $P_i = R^{1/2} \tilde{P}_i R^{1/2}$, $Q_i = R^{1/2} \tilde{Q}_i R^{1/2}$, $S_i = R^{1/2} \tilde{S}_i R^{1/2}$, then

$$\begin{aligned}&P_i \le \mu P_j, \quad Q_i \le \mu Q_j, \\ &S_i \le \mu S_j, \quad \forall i, j \in I\end{aligned} \quad (21)$$

Assume that $\sigma(t_k) = i$ and $\sigma(t_k^-) = j$ at switching instant t_k. According to (19), we obtain

$$V_{\sigma(t_k)}(t_k) \le \mu V_{\sigma(t_k^-)}(t_k^-) \quad (22)$$

For any $t \in (0, T_f)$, let *N* be the switching number of *σ*(*t*) over (0, T_f). Using the iterative method, we have

$$
\begin{aligned}
V(t) &< e^{\alpha t}\mu^{N} V_{\sigma(0)}(0) \\
&\quad +\mu^{N}\int_{0}^{t_1} e^{\alpha(t-s)}\omega^{T}(s)W_{\sigma(0)}\omega(s)\,\mathrm{d}s \\
&\quad +\mu^{N-1}\int_{t_1}^{t_2} e^{\alpha(t-s)}\omega^{T}(s)W_{\sigma(t_1)}\omega(s)\,\mathrm{d}s \\
&\quad +\cdots+\int_{t_k}^{t} e^{\alpha(t-s)}\omega^{T}(s)W_{\sigma(t_k)}\omega(s)\,\mathrm{d}s \\
&= e^{\alpha t}\mu^{N} V_{\sigma(0)}(0) \\
&\quad +\int_{0}^{t} e^{\alpha(t-s)}\mu^{N_{\sigma}(s,t)}\omega^{T}(s)W_{\sigma(s)}\omega(s)\,\mathrm{d}s \\
&\le e^{\alpha T_f}\mu^{N} V_{\sigma(0)}(0) \\
&\quad +\int_{0}^{t} e^{\alpha T_f}\mu^{N}\omega^{T}(s)W_{\sigma(s)}\omega(s)\,\mathrm{d}s \\
&\le e^{\alpha T_f}\mu^{N}\left(V_{\sigma(0)}(0)+d\sup_{i\in I}\left(\lambda_{\max}(W_i)\right)\right)
\end{aligned} \tag{23}
$$

where $\alpha=\max_{i\in I}\{\alpha_i\}$.

Noticing that $N\le N_0+T_f/\tau_a$, then

$$
V(t) < e^{\alpha T_f}\mu^{N_0+T_f/\tau_a}\left(V_{\sigma(0)}(0)+d\sup_{i\in I}\left(\lambda_{\max}(W_i)\right)\right) \tag{24}
$$

On the other hand,

$$
\begin{aligned}
V(t) &\ge x^{T}(t)P_i x(t) = x^{T}(t)R^{1/2}\tilde{P}_i R^{1/2}x(t) \\
&\ge \inf_{i\in I}\left\{\lambda_{\min}(\tilde{P}_i)\right\}x^{T}(t)Rx(t)=\lambda_1 x^{T}(t)Rx(t)
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
V_{\sigma((0)}(0) &\le x^{T}(0)P_{\sigma(0)}x(0)+\int_{-\tau}^{0} e^{-\alpha s}x^{T}(s)Q_{\sigma(0)}x(s)\,\mathrm{d}s \\
&\quad +\int_{-\tau}^{0}\int_{\theta}^{0} e^{-\alpha s}\dot{x}^{T}(s)S_{\sigma(0)}\dot{x}(s)\,\mathrm{d}s\mathrm{d}\theta \\
&\le \lambda_{\max}(\tilde{P}_{\sigma(0)})x^{T}(0)Rx(0) \\
&\quad +\tau e^{\tau\alpha}\lambda_{\max}(\tilde{Q}_{\sigma(0)})\sup_{-\tau\le\theta\le 0}\left\{x^{T}(\theta)Rx(\theta)\right\} \\
&\quad +\tau e^{\tau\alpha}\lambda_{\max}(\tilde{S}_{\sigma(0)})\sup_{-\tau\le\theta\le 0}\left\{\dot{x}^{T}(\theta)R\dot{x}(\theta)\right\} \\
&\le (\lambda_2+\tau e^{\tau\alpha}\lambda_3)\delta+\tau e^{\tau\alpha}\lambda_4\beta
\end{aligned} \tag{26}
$$

Taking (24)-(26) into account, we obtain

$$
x^{T}(t)Rx(t) < \frac{(\lambda_2+\tau e^{\tau\alpha}\lambda_3)\delta+\tau e^{\tau\alpha}\lambda_4\beta+d\sup_{i\in I}\left(\lambda_{\max}(W_i)\right)}{\lambda_1}e^{\alpha T_f}\mu^{N_0+T_f/\tau_a} \tag{27}
$$

1) When $\mu=1$, from (10),

$$
x^{T}(t)Rx(t) < e^{\alpha T_f}e^{-\alpha T_f}\varepsilon=\varepsilon \tag{28}
$$

2) When $\mu>1$, from (11),

$$
\frac{T_f}{\tau_a} < \frac{\ln\mu}{\ln(\lambda_1\varepsilon)-\ln(\lambda_2+\tau e^{\tau\alpha_i}\lambda_3)\delta+\tau e^{\tau\alpha_i}\lambda_4\beta+\upsilon} \tag{29}
$$

Substituting (29) into (27) yields

$$
x^{T}(t)Rx(t) < \varepsilon \tag{30}
$$

According to definition 2, we can conclude that the switched time-delay system (6) is finite-time bounded with respect to $(\delta,\varepsilon,T_f,d,R,\sigma)$. The proof is completed.

Remark 5. In the proof of Theorem 1, there is no requirement of negative definitiveness on $\dot{V}(t)$, which is different from the classical Lyapunov function for switched systems in the case of asymptotical stability. In order to reduce the conservatism of the theorem conditions, free-weighing matrix method is introduced. When $\mu=1$, one obtains τ_a, in other words, there is no restriction on the average dwell time for switching signal.

When the time-varying exogenous noise signal $\omega(t)\equiv 0$, the results about finite-time stability can be obtained and given in the following corollary.

Corollary 1. Assume that the switched time-delay system (6) satisfies $u(t)\equiv 0$ and $\omega(t)\equiv 0$. For any $i\in I$, let $P_i=R^{1/2}\tilde{P}_iR^{1/2}$, $Q_i=R^{1/2}\tilde{Q}_iR^{1/2}$, $S_i=R^{1/2}\tilde{S}_iR^{1/2}$. Suppose that there exist matrices $\tilde{P}_i>0$, $\tilde{Q}_i>0$, $\tilde{S}_i>0$, $X_i=\begin{bmatrix} X_{11,i} & X_{12,i} \\ * & X_{22,i}\end{bmatrix}\ge 0$, $N_{1,i}$, $N_{2,i}$ and constants $\alpha_i\ge 0$, $\beta\ge 0$ such that

$$
\Upsilon=\begin{bmatrix}\Upsilon_{11} & \Upsilon_{12} \\ * & \Upsilon_{22}\end{bmatrix}<0 \tag{31}
$$

$$
\Psi=\begin{bmatrix} X_{11,i} & X_{12,i} & N_{1,i} \\ * & X_{22,i} & N_{2,i} \\ * & * & e^{-\alpha_i\tau}S_i\end{bmatrix}>0 \tag{32}
$$

$$
(\lambda_2+\tau e^{\tau\alpha_i}\lambda_3)\delta+\tau e^{\tau\alpha_i}\lambda_4\beta<\lambda_1 e^{-\alpha_i T_f}\varepsilon \tag{33}
$$

where

$$
\begin{aligned}
\Upsilon_{11} &= A_i^{T}P_i+P_iA_i-\alpha_iP_i+Q_i+\tau A_i^{T}S_iA_i \\
&\quad +N_{1,i}+N_{1,i}^{T}+\tau X_{11,i}, \\
\Upsilon_{12} &= P_iA_{di}+\tau A_i^{T}S_iA_{di}-N_{1,i}+N_{2,i}^{T}+\tau X_{12,i}, \\
\Upsilon_{22} &= -(1-h)e^{\alpha_i\tau}Q_i+\tau A_{di}^{T}S_iA_{di}-N_{2,i}-N_{2,i}^{T}+\tau X_{22,i}.
\end{aligned}
$$

If the ADT of the switching signal σ satisfies

$$
\tau_a>\tau_a^{*} = \frac{T_f\ln\mu}{\ln(\lambda_1\varepsilon)-\ln\left((\lambda_2+\tau e^{\tau\alpha_i}\lambda_3)\delta+\tau e^{\tau\alpha_i}\lambda_4\beta\right)-\bar{\upsilon}} \tag{34}
$$

then the switched system is finite-time stability with respect to $(\delta,\varepsilon,T_f,R,\sigma)$, where $\bar{\upsilon}=\alpha T_f+N_0\ln\mu$, $\mu\ge 1$, $\tilde{P}_i\le\mu\tilde{P}_j$, $\tilde{Q}_i\le\mu\tilde{Q}_j$, $\tilde{S}_i\le\mu\tilde{S}_j$, $\forall i,j\in I$, $\alpha=\max_{i\in I}\{\alpha_i\}$, $\lambda_1=\inf_{i\in I}\left\{\lambda_{\min}(\tilde{P}_i)\right\}$,

$\lambda_2 = \sup_{i\in I}\left\{\lambda_{\max}\left(\tilde{P}_i\right)\right\}$, $\lambda_3 = \sup_{i\in I}\left\{\lambda_{\max}\left(\tilde{Q}_i\right)\right\}$,
$\lambda_4 = \sup_{i\in I}\left\{\lambda_{\max}\left(\tilde{S}_i\right)\right\}$.

Remark 6. It is easy to find that some differences between Lyapunov asymptotical stability and finite-time stability. Conditions (33) and (34) must be satisfied for finite-time stability, which is not necessary for asymptotical stability. Thus, the two concepts are independent. However, in previous research, there are few results on finite-time stability, which needs our full investigation.

4. H_∞ Finite-Time Boundedness Analysis

In this section, we discuss H_∞ finite-time boundedness of switched time-delay system (1) with $u(t)\equiv 0$. First, consider the following switched time-delay system

$$\begin{cases}\dot{x}(t) = A_{\sigma(t)}x(t) + A_{d\sigma(t)}x(t-d(t)) + G_{\sigma(t)}\omega(t)\\ z(t) = C_{\sigma(t)}x(t) + E_{\sigma(t)}\omega(t)\\ x(t) = \varphi(t) \quad t\in[-\tau,0]\end{cases} \tag{35}$$

Theorem 2. For any $i\in I$, let $P_i = R^{1/2}\tilde{P}_iR^{1/2}$, $Q_i = R^{1/2}\tilde{Q}_iR^{1/2}$, $S_i = R^{1/2}\tilde{S}_iR^{1/2}$. Suppose that there exist matrices $\tilde{P}_i > 0$, $\tilde{Q}_i > 0$, $\tilde{S}_i > 0$, $X_i = \begin{bmatrix} X_{11,i} & X_{12,i}\\ * & X_{22,i}\end{bmatrix} \ge 0$, $N_{1,i}$, $N_{2,i}$ and constants $\alpha_i \ge 0$ and $\gamma > 0$ such that

$$\begin{bmatrix}\Omega_{11} + C_i^TC_i & \Omega_{12} & \Omega_{13} + C_i^TE_i\\ * & \Omega_{22} & \Omega_{23}\\ * & * & -\gamma^2I + \tau G_i^TS_iG_i + E_i^TE_i\end{bmatrix} < 0 \tag{36}$$

$$\begin{bmatrix} X_{11,i} & X_{12,i} & N_{1,i}\\ * & X_{22,i} & N_{2,i}\\ * & * & e^{-\alpha_i\tau}S_i\end{bmatrix} > 0 \tag{37}$$

$$\tau e^{\tau\alpha_i}\lambda_4\beta + \gamma^2 d < \lambda_1 e^{-\alpha_i T_f}\varepsilon \tag{38}$$

If the ADT of the switching signal σ satisfies

$$\tau_a > \tau_a^* = \frac{T_f \ln\mu}{\ln(\lambda_1\varepsilon) - \ln(\gamma^2 d) - \alpha T_f - N_0\ln\mu} \tag{39}$$

then the switched systems is H_∞ finite-time boundedness with respect to $(0,\varepsilon,T_f,d,R,\sigma)$, where $\mu\ge 1$, $\tilde{P}_i \le \mu\tilde{P}_j$, $\tilde{Q}_i \le \mu\tilde{Q}_j$, $\tilde{S}_i \le \mu\tilde{S}_j$, $\forall i,j\in I$,
$\alpha = \max_{i\in I}\{\alpha_i\}$, $\lambda_1 = \inf_{i\in I}\left\{\lambda_{\min}\left(\tilde{P}_i\right)\right\}$,
$\lambda_2 = \sup_{i\in I}\left\{\lambda_{\max}\left(\tilde{P}_i\right)\right\}$, $\lambda_3 = \sup_{i\in I}\left\{\lambda_{\max}\left(\tilde{Q}_i\right)\right\}$,
$\lambda_4 = \sup_{i\in I}\left\{\lambda_{\max}\left(\tilde{S}_i\right)\right\}$.

Proof. Assuming condition (36) is satisfied, then we obtain

$$\begin{bmatrix}\Omega_{11} & \Omega_{12} & \Omega_{13}\\ * & \Omega_{22} & \Omega_{23}\\ * & * & -\gamma^2I + \tau G_i^TS_iG_i\end{bmatrix} + \begin{bmatrix} C_i^TC_i & 0 & C_i^TE_i\\ * & 0 & 0\\ * & * & E_i^TE_i\end{bmatrix} < 0 \tag{40}$$

Since

$$\begin{bmatrix} C_i^TC_i & 0 & C_i^TE_i\\ * & 0 & 0\\ * & * & E_i^TE_i\end{bmatrix} = \begin{bmatrix} C_i\\ 0\\ E_i\end{bmatrix}^T\begin{bmatrix} C_i & 0 & E_i\end{bmatrix} \ge 0 \tag{41}$$

which implies that

$$\begin{bmatrix}\Omega_{11} & \Omega_{12} & \Omega_{13}\\ * & \Omega_{22} & \Omega_{23}\\ * & * & -\gamma^2I + \tau G_i^TS_iG_i\end{bmatrix} < 0 \tag{42}$$

From Theorem 1, conditions (37)-(39) can ensure that the switched time-delay system (35) is finite-time bounded with respect to $(0,\varepsilon,T_f,d,R,\sigma)$.

Next, we will prove condition (6) is satisfied under zero initial condition. Choose the following Lyapunov function $V(t) = V_i(t) = V_{1,i}(t) + V_{2,i}(t) + V_{3,i}(t)$, where

$$V_{1,i}(t) = x^T(t)P_ix(t),$$
$$V_{2,i}(t) = \int_{t-d(t)}^{t} e^{\alpha_i(t-s)}x^T(s)Q_ix(s)\,\mathrm{d}s,$$
$$V_{3,i}(t) = \int_{-\tau}^{0}\int_{t+\theta}^{t} e^{\alpha_i(t-s)}\dot{x}^T(s)S_i\dot{x}(s)\,\mathrm{d}s\mathrm{d}\theta.$$

When $t\in[t_k, t_{k+1})$, by virtue of (36), we can obtain

$$\begin{aligned} V(t) &< e^{\alpha_{\sigma(t_k)}(t-t_k)}V_{\sigma(t_k)}(t_k)\\ &\quad + \int_{t_k}^{t} e^{\alpha_{\sigma(t_k)}(t-s)}(\gamma^2\omega^T(s)\omega(s) - z^T(s)z(s))\mathrm{d}s\\ &= e^{\alpha_{\sigma(t_k)}(t-t_k)}V_{\sigma(t_k)}(t_k) + \int_{t_k}^{t} e^{\alpha_{\sigma(t_k)}(t-s)}\Gamma(s)\mathrm{d}s \end{aligned} \tag{43}$$

Since $\mu\ge 1$, $\tilde{P}_i \le \mu\tilde{P}_j$, $\tilde{Q}_i \le \mu\tilde{Q}_j$, $\tilde{S}_i \le \mu\tilde{S}_j$ and $P_i = R^{1/2}\tilde{P}_iR^{1/2}$, $Q_i = R^{1/2}\tilde{Q}_iR^{1/2}$, $S_i = R^{1/2}\tilde{S}_iR^{1/2}$, then $P_i \le \mu P_j, Q_i \le \mu Q_j, S_i \le \mu S_j, \forall i,j\in I$. In what follows, assume that $\sigma(t_k) = i$ and $\sigma(t_k^-) = j$ at switching instant t_k. We have

$$V_{\sigma(t_k)}(t_k) \le \mu V_{\sigma(t_k^-)}(t_k^-) \tag{44}$$

Since $\alpha = \max_{i\in I}\{\alpha_i\}$, then it follows from (43) and (44) that

$$V(t) < e^{\alpha(t-t_k)} V_{\sigma(t_k)}(t_k) + \int_{t_k}^{t} e^{\alpha(t-s)} \Gamma(s) \mathrm{d}s \tag{45}$$

When $t \in (0, T_f)$, let N be the switching number of $\sigma(t)$ over $(0, T_f)$. Using the iterative method, we have

$$\begin{aligned} V(t) &< e^{\alpha t} \mu^N V_{\sigma(0)}(0) + \mu^N \int_0^{t_1} e^{\alpha(t-s)} \Gamma(s) \mathrm{d}s \\ &\quad + \mu^{N-1} \int_{t_1}^{t_2} e^{\alpha(t-s)} \Gamma(s) \mathrm{d}s + \cdots \\ &\quad + \int_{t_k}^{t} e^{\alpha(t-s)} \Gamma(s) \mathrm{d}s \\ &= e^{\alpha t} \mu^N V_{\sigma(0)}(0) \\ &\quad + \int_0^t e^{\alpha(t-s)} \mu^{N_\sigma(s,t)} \Gamma(s) \mathrm{d}s \\ &\le e^{\alpha T_f} \mu^N V_{\sigma(0)}(0) \\ &\quad + \int_0^t e^{\alpha T_f} \mu^N \Gamma(s) \mathrm{d}s \end{aligned} \tag{46}$$

Under zero initial condition, (46) implies

$$0 \le V(t) < \int_0^t e^{\alpha T_f} \mu^N \Gamma(s) \mathrm{d}s \tag{47}$$

that is

$$\begin{aligned} &\int_0^t e^{\alpha T_f} \mu^{N_\sigma(s,t)} z^T(s) z(s) \mathrm{d}s \\ &< \gamma^2 \int_0^t e^{\alpha T_f} \mu^{N_\sigma(s,t)} \omega^T(s) \omega(s) \mathrm{d}s \end{aligned} \tag{48}$$

Setting $t = T_f$, we obtain

$$\int_0^{T_f} z^T(s) z(s) \mathrm{d}s < \gamma^2 \int_0^{T_f} \omega^T(s) \omega(s) \mathrm{d}s \tag{49}$$

Therefore, according to Definition 3, the proof is completed.

5. Finite-Time Stabilization

In this section, the static state feedback controllers are designed. Based on the results in the previous section, the closed-loop system H_∞ finite-time bounded with respect to $(0, \varepsilon, T_f, d, R, \sigma)$ can be ensured by memory state feedback controllers $u(t) = K_{1,i} x(t) + K_{2,i} x(t - d(t))$. Applying the memory state feedback controllers into switched time-delay system (1), we can obtain the closed-loop switched system as follows

$$\begin{cases} \dot{x}(t) = \bar{A}_{\sigma(t)} x(t) + \bar{A}_{d\sigma(t)} x(t - d(t)) + G_{\sigma(t)} \omega(t) \\ z(t) = \bar{C}_{\sigma(t)} x(t) + \bar{D}_{\sigma(t)} x(t - d(t)) + E_{\sigma(t)} \omega(t) \\ x(t) = \varphi(t) \qquad t \in [-\tau, 0] \end{cases} \tag{50}$$

where $\bar{A}_{\sigma(t)} = A_{\sigma(t)} + B_{\sigma(t)} K_{1,\sigma(t)}$, $\bar{A}_{d\sigma(t)} = A_{d\sigma(t)} + B_{\sigma(t)} K_{2,\sigma(t)}$, $\bar{C}_{\sigma(t)} = C_{\sigma(t)} + D_{\sigma(t)} K_{1,\sigma(t)}$, $\bar{D}_{\sigma(t)} = D_{\sigma(t)} K_{2,\sigma(t)}$.

From condition (36), we have

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & P_i G_i & \bar{A}_i^T & \bar{C}_i^T \\ * & \Xi_{22} & 0 & \bar{A}_{di}^T & \bar{D}_i^T \\ * & * & -\gamma^2 I + E_i^T E_i & G_i^T & 0 \\ * & * & * & -\tau^{-1} S_i^{-1} & 0 \\ * & * & * & * & -I \end{bmatrix} < 0 \tag{51}$$

where

$$\begin{aligned} \Xi_{11} &= \bar{A}_i^T P_i + P_i \bar{A}_i - \alpha_i P_i + Q_i + N_{1,i} + N_{1,i}^T + \tau X_{11,i}, \\ \Xi_{12} &= P_i \bar{A}_{di} - N_{1,i} + N_{2,i}^T + \tau X_{12,i}, \\ \Xi_{22} &= -(1-h) e^{\alpha_i \tau} Q_i - N_{2,i} - N_{2,i}^T + \tau X_{22,i}. \end{aligned}$$

According to Lemma 3, (37) and (51) are equivalent to the following inequality

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} & P_i G_i & \bar{A}_i^T & -\tau N_{1,i} & \bar{C}_i^T \\ * & \Theta_{22} & 0 & \bar{A}_{di}^T & -\tau N_{2,i} & \bar{D}_i^T \\ * & * & -\gamma^2 I + E_i^T E_i & G_i^T & 0 & 0 \\ * & * & * & -\tau^{-1} S_i^{-1} & 0 & 0 \\ * & * & * & * & -\tau e^{-\alpha_i \tau} S_i & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \tag{52}$$

where

$$\begin{aligned} \Theta_{11} &= \bar{A}_i^T P_i + P_i \bar{A}_i - \alpha_i P_i + Q_i + N_{1,i} + N_{1,i}^T, \\ \Theta_{12} &= P_i \bar{A}_{di} - N_{1,i} + N_{2,i}^T, \\ \Theta_{22} &= -(1-h) e^{\alpha_i \tau} Q_i - N_{2,i} - N_{2,i}^T. \end{aligned}$$

For matrix Inequality (52), let $M_i = \begin{bmatrix} P_i & 0 \\ N_{1,i}^T & N_{2,i}^T \end{bmatrix}$, $\tilde{A} = \begin{bmatrix} \bar{A}_i & \bar{A}_{di} \\ I & -I \end{bmatrix}$, then

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} \\ * & \Theta_{22} \end{bmatrix} = \tilde{A}_i^T M_i + M_i^T \tilde{A}_i + \begin{bmatrix} Q_i - \alpha_i P_i & 0 \\ 0 & -(1-h) e^{\alpha_i \tau} Q_i \end{bmatrix} \tag{53}$$

Let $M_i^{-1} = \begin{bmatrix} P_i^{-1} & 0 \\ L_{1,i} & L_{2,i} \end{bmatrix}$ and $T = \mathrm{diag}\{M_i^{-1}, I, I, S_i^{-1}, I\}$. Pre-multiplying Equation (52) by T^T and post-multiplying Equation (52) by T, we have

$$\begin{bmatrix} \theta_{11} & \theta_{12} & G_i & \theta_{13} & 0 & \theta_{14} \\ * & \theta_{22} & 0 & \rho_i Q_i^{-1}\overline{A}_{di}^T & -\tau I & \rho_i Q_i^{-1}\overline{D}_i^T \\ * & * & -\gamma^2 I + E_i^T E_i & G_i^T & 0 & 0 \\ * & * & * & -\tau^{-1}S_i^{-1} & 0 & 0 \\ * & * & * & * & -\tau e^{-\alpha_i\tau}S_i^{-1} & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \tag{54}$$

where

$$\theta_{11} = P_i^{-1}\overline{A}_i^T + \xi_i Q_i^{-1}\overline{A}_{di}^T + \overline{A}_i P_i^{-1} + \xi_i \overline{A}_{di} Q_i^{-1} + P_i^{-1}Q_i P_i^{-1} - \alpha_i P_i^{-1} - \xi_i^2 (1-h) e^{\alpha_i\tau} Q_i^{-1}$$
$$\theta_{12} = P_i^{-1} - \xi_i Q_i^{-1} + \rho_i \overline{A}_{di} Q_i^{-1} - \rho_i \xi_i (1-h) e^{\alpha_i\tau} Q_i^{-1}$$
$$\theta_{22} = -2\rho_i Q_i^{-1} - \rho_i^2 (1-h) e^{\alpha_i\tau} Q_i^{-1}$$
$$\theta_{13} = P_i^{-1}\overline{A}_i^T + \xi_i Q_i^{-1}\overline{A}_{di}^T$$
$$\theta_{14} = P_i^{-1}\overline{C}_i^T + \xi_i Q_i^{-1}\overline{D}_i^T$$

where $L_{1,i} = \xi_i Q_i^{-1}$, $L_{2,i} = \rho_i Q_i^{-1}$, $(\xi_i,\ \rho_i \in R,\ \rho_i \neq 0)$. Denote $\overline{P}_i = P_i^{-1}$, $\overline{S}_i = S_i^{-1}$, $\overline{Q}_i = Q_i^{-1}$, $Y_{1,i} = K_{1,i}P_i^{-1}$, $Y_{2,i} = K_{2,i}Q_i^{-1}$. By Schur complement (Lemma 1), we can obtain the following Theorem.

Theorem 3. For given $\gamma > 0$, $\xi_i \in R$, $0 \neq \rho_i \in R$. Suppose that there exist matrices $\overline{P}_i > 0$, $\overline{Q}_i > 0$, $\overline{S}_i > 0$, $Y_{1,i}$, $Y_{2,i}$ and constants $\alpha_i \geq 0$, $\beta \geq 0$ and such that the following conditions are satisfied $\forall i \in I$

$$\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & G_i & \varepsilon_{13} & 0 & \varepsilon_{14} & \overline{P}_i \\ * & \varepsilon_{22} & 0 & \varepsilon_{23} & -\tau I & \rho_i Y_{2,i}^T D_i^T & 0 \\ * & * & -\gamma^2 I + E_i^T E_i & G_i^T & 0 & 0 & 0 \\ * & * & * & -\tau^{-1}\overline{S}_i & 0 & 0 & 0 \\ * & * & * & * & -\tau e^{-\alpha_i\tau}\overline{S}_i & 0 & 0 \\ * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & -\overline{Q}_i \end{bmatrix} < 0 \tag{55}$$

$$\tau e^{\tau\alpha_i}\lambda_4\beta + \gamma^2 d < \lambda_1 e^{-\alpha_i T_f}\varepsilon \tag{56}$$

where

$$\varepsilon_{11} = \overline{P}_i A_i^T + A_i\overline{P}_i + \xi_i\overline{Q}_i A_{di}^T + \xi_i A_{di}\overline{Q}_i + Y_{1,i}^T B_i^T + B_i Y_{1,i} + \xi_i Y_{2,i}^T B_i^T + \xi_i B_i Y_{2,i} - \alpha_i \overline{P}_i - \xi_i^2 (1-h) e^{\alpha_i\tau}\overline{Q}_i,$$
$$\varepsilon_{12} = \overline{P}_i - \xi_i\overline{Q}_i + \rho_i A_{di}\overline{Q}_i + \rho_i B_i Y_{2,i} - \rho_i\xi_i (1-h) e^{\alpha_i\tau}\overline{Q}_i,$$
$$\varepsilon_{13} = \overline{P}_i A_i^T + \xi_i\overline{Q}_i A_{di}^T + Y_{1,i}^T B_i^T + \xi_i Y_{2,i}^T B_i^T,$$
$$\varepsilon_{14} = \overline{P}_i C_i^T + Y_{1,i}^T D_i^T + \xi_i Y_{2,i}^T D_i^T,$$
$$\varepsilon_{22} = -2\rho_i\overline{Q}_i - \rho_i^2 (1-h) e^{\alpha_i\tau}\overline{Q}_i,$$
$$\varepsilon_{23} = \rho_i\overline{Q}_i A_{di}^T + \rho_i Y_{2,i}^T B_i^T.$$

If the ADT of the switching signal σ satisfies

$$\tau_a > \tau_a^* = \frac{T_f \ln\mu}{\ln(\lambda_1\varepsilon) - \ln(\gamma^2 d) - \alpha T_f - N_0 \ln\mu} \tag{57}$$

then the memory state feedback gains $K_{1,i} = Y_{1,i}\overline{P}_i^{-1}$ and $K_{2,i} = Y_{2,i}\overline{Q}_i^{-1}$ ensure closed-loop switched time-delay system (50) H_∞ finite-time bounded with respect to $(0, \varepsilon, T_f, d, R, \sigma)$.

Remark 7. In Theorem 3, ξ_i and ρ_i are adjustable parameters. By virtue of the method in [38], these parameters can be obtained.

Remark 8. It should be pointed out that the conditions in Theorems 1, 2, 3 and Corollary 1 are not standard LMIs conditions. However, once some values are fixed for α_i, these conditions, *i.e.*, (10) and (38) can be translated into LMIs conditions. As in [27], (10) and (38) can be rewritten in the following forms

1) The condition (10) can be guaranteed by the following LMI condition, that is, for any $i \in I$, there exists some positive numbers κ_1, κ_2, κ_3, κ_4 and κ_5 such that

$$\kappa_1 I < P_i \leq \kappa_2 I \tag{58}$$

$$0 < Q_i \leq \kappa_3 I \tag{59}$$

$$0 < S_i \le \kappa_4 I \quad (60)$$

$$0 < W_i \le \kappa_5 I \quad (61)$$

$$\left(\kappa_2 + \tau e^{\tau\alpha_i}\kappa_3\right)\delta + \tau e^{\tau\alpha_i}\kappa_4\beta + d\kappa_5 < \kappa_1 e^{-\alpha_i T_f}\varepsilon \quad (62)$$

2) The condition (38) can be guaranteed by the following LMI condition, that is, for any $i \in I$, there exists some positive numbers κ_1, κ_2, κ_3 and κ_4 satisfying (58)-(60) such that

$$\left(\kappa_2 + \tau e^{\tau\alpha_i}\kappa_3\right)\delta + \tau e^{\tau\alpha_i}\kappa_4\beta < \kappa_1 e^{-\alpha_i T_f}\varepsilon . \quad (63)$$

6. Numerical Simulation and Results

In this section, for given ε and μ, an example is employed to verify the method proposed above. Consider a switched linear system with time-varying delay as follows

$$\dot{x}(t) = A_{\sigma(t)}x(t) + A_{d\sigma(t)}x(t - d(t)) + G_{\sigma(t)}\omega(t) \quad (64)$$

with $A_1 = \begin{bmatrix} -1.7 & 1.7 & 0 \\ 1.3 & -1 & 0.7 \\ 0.7 & 1 & -0.6 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0.7 & 0 & -0.6 \\ 1.7 & 0 & -1.7 \end{bmatrix}$,

$A_{d1} = \begin{bmatrix} 1.5 & -1.7 & 0.1 \\ -1.3 & 1 & -0.3 \\ -0.7 & 1 & 0.6 \end{bmatrix}$, $G_1 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$,

$x(t) = \begin{bmatrix} 0.7 \\ 0 \\ 0 \end{bmatrix}$, $A_{d2} = \begin{bmatrix} -1 & 0 & 0.1 \\ 1.3 & -0.1 & 0.6 \\ 1.5 & 0.1 & 1.8 \end{bmatrix}$, $t \in [-h, 0]$,

$G_2 = G_1$, $\tau = 0.2$, $h = 0.02$.

The values of δ, T_f, d and R are selected as follows:

$\delta = 0.5$, $T_f = 10$, $d = 0.01$, $R = I$, $\alpha_i = 0.05$, $\beta = 0.01$.

When $\mu = 2$ and $\varepsilon = 30$, by virtue of Theorem 1, one obtains $\tau_a^* = 2.4659$. For any switching signal $\sigma(t)$ with average dwell time $\tau_a > \tau_a^*$, switched linear system with time-delay is finite-time bounded with respect to (0.5, 30, 10, 0.01, I, σ). The state trajectory over 0~10 s under a periodic switching signal with interval time $\Delta T = 2.5s$ is shown in **Figure 1**. It is obvious that switched linear system (64) is finite-time bounded. The state trajectory over 0 ~ 10 s under a periodic switching signal with interval time $\Delta T = 2s$ is shown in **Figure 2**. As can be seen from figure 2, switched linear system (64) is not finite-time bounded any more.

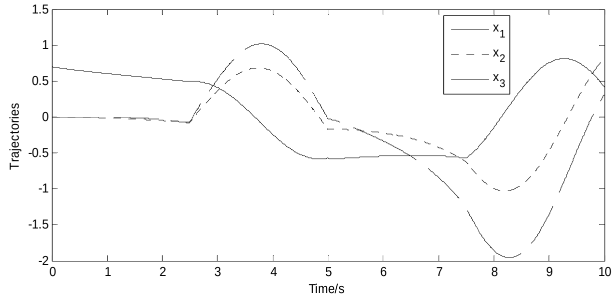

Figure 1. The histories of the state trajectory of switched system under a periodic switching signal with interval time ΔT = 2.5s.

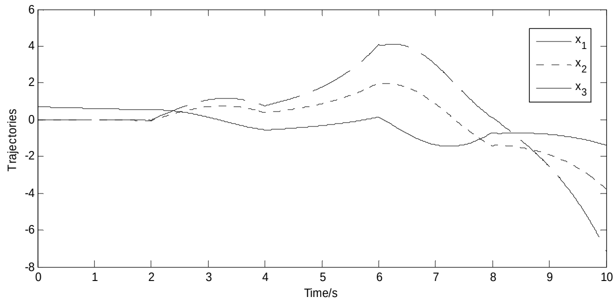

Figure 2. The histories of the state trajectory of switched system under a periodic switching signal with interval time ΔT = 2s.

7. Conclusions

In this paper, unlike most existing research results focusing on Lyapunov stability property of switched time-varying delay system, we mainly discuss finite-time boundedness and H_∞ finite-time boundedness of switched linear systems with time-varying delay. As the main contribution of this paper, sufficient conditions which can guarantee finite-time boundedness and H_∞ finite-time boundedness of switched linear systems with time-varying delay are proposed. And then based on the results on finite-time boundedness, the memory state feedback controller is designed to H_∞ finite-time stabilize a switched linear system with time-varying delay. An important and challenging further investigation is how to extend the results in this paper to uncertain switched systems and switched nonlinear systems.

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