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Semi-Markovian Model of Two-Line Queuing System with Losses

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Abstract

In the present paper, to build model of two-line queuing system with losses GI/G/2/0, the approach introduced by V.S. Korolyuk and A.F. Turbin, is used. It is based on application of the theory of semi-Markov processes with arbitrary phase space of states. This approach allows us to omit some restrictions. The stationary characteristics of the system have been defined, assuming that the incoming flow of requests and their service times have distributions of general form. The particular cases of the system were considered. The used approach can be useful for modeling systems of various purposes.

Keywords

Two-Line Queuing System with Losses, Semi-Markov Process, Stationary Distribution of Embedded Markov Chain, Stationary Characteristics of System

1. Introduction

A large number of works, in particular [1]-[5], have been dedicated to the queuing systems (QS) with losses. Building of QS models and determining their characteristics are simplified, if it is assumed that the incoming flow of requests or their service times are exponentially distributed. The rejection of this assumption leads to a considerable complication of the models. In this paper, the model of two-line QS with losses was built on the assumption that the incoming flow of requests and their service times have distributions of general form. For building QS model and determining its stationary characteristics, the theory of semi-Markov processes with arbitrary phase state space [5]-[10] was used.

2. System Description and Building of the Semi-Markov Model

Two-line QS with losses GI/G/2/0 is being considered. It is assumed that the system receives requests, and the time between their arrival is a random variable (RV) β with the distribution function (DF) $G(x)=P\{\beta\le x\}$. A received request, with equal probability, starts to be served by one of the available servers or gets lost, if no servers are available. The service time of request by the i^{th} server-RV α_i with DF $F_i(x)=P\{\alpha_i\le x\}$, $i=1,2$. It is assumed that RV α_i, β are independent, and have densities $f_i(x)$, $g(x)$, finite mathematical expectations and variances.

To describe the QS operation, the semi-Markov process [5]-[7] $\xi(t)$ with the following set of states is used:

$$E=\{10, 01, 100z, 200z, 111x, 211x, 101xz, 210xz, 311x_1x_2\}.$$

The meaning of state codes is the following:

$10\,(01)$: first (second) server started serving the received request, and second (first) server is available;

$100z\,(200z)$: first (second) server became available; second (first) server is available; $z>0$ is the time until the arrival of the next request;

$111x\,(211x)$: first (second) server started serving the received request; $x>0$ is the time until the end of the request service by second (first) server;

$101xz\,(210xz)$: first (second) server became available; $x>0$ is the time until the end of the request service by second (first) server; $z>0$ is the time until the arrival of the next request;

$311x_1x_2$: the received request was lost; the times until the end of the request service by first (second) servers are respectively equal $x_1>0,\ x_2>0$.

The time diagram of the system is shown in **Figure 1**.

Let us define the sojourn times in states of the system. For instance, the sojourn time θ_{211x}, in the state $211x$ is determined by three factors: the time x left until the end of request service by the first server, the time α_2 of request service by the second server, and the time β between the request arrivals.

Therefore, $\theta_{211x}=x\wedge\alpha_2\wedge\beta$, where $\wedge$ is the minimum sign. Similarly, the sojourn times in other states are determined as follows:

$$\begin{aligned}&\theta_{10}=\alpha_1\wedge\beta,\ \theta_{01}=\alpha_2\wedge\beta,\ \theta_{100z}=z,\ \theta_{200z}=z,\ \theta_{111x}=x\wedge\alpha_1\wedge\beta\\&\theta_{101xz}=x\wedge z,\ \theta_{210xz}=x\wedge z,\ \theta_{311x_1x_2}=x_1\wedge x_2\wedge\beta.\end{aligned}\tag{1}$$

We define the transition probabilities of the embedded Markov chain (EMC) $\{\xi_n;\ n>0\}$ for states $10, 100z, 111x, 101xz, 311x_1x_2$, in the context of other states, they are determined similarly.

$$\begin{aligned}&p_{10}^{211x}=\int_0^\infty f_1(x+t)g(t)\,\mathrm{d}t,\ \ x>0;\ \ p_{10}^{100z}=\int_0^\infty g(z+t)f_1(t)\,\mathrm{d}t,\ \ z>0,\\&P_{100z}^{10}=P_{100z}^{01}=\frac{1}{2};\ \ p_{111x}^{210yz}=f_1(x+y)g(x+z),\ \ y>0,\ z>0,\\&p_{111x}^{101yz}=f_1(x-y)g(x-y+z),\ \ 0<y<x,\ \ z>0,\\&p_{111x}^{311y_1y_2}=g(x-y_1)f_1(x-y_2+y_1),\ \ 0<y_2<x,\ \ y_1>0,\\&P_{101xz}^{200z-x}=1,\text{ if } z>x;\ \ P_{101xz}^{111x-z}=1,\text{ if } x>z,\\&p_{311x_1x_2}^{101x_2-x_1z}=g(x_1+z),\ \ x_1<x_2,\ \ z>0;\ p_{311x_1x_2}^{311x_1-tx_2-t}=g(t),\ \ x_1<x_2,\ \ 0<t<x_1,\\&p_{311x_1x_2}^{210x_1-x_2z}=g(x_2+z),\ \ x_2<x_1,\ \ z>0;\ p_{311x_1x_2}^{311x_1-tx_2-t}=g(t),\ \ x_2<x_1,\ \ 0<t<x_2.\end{aligned}\tag{2}$$

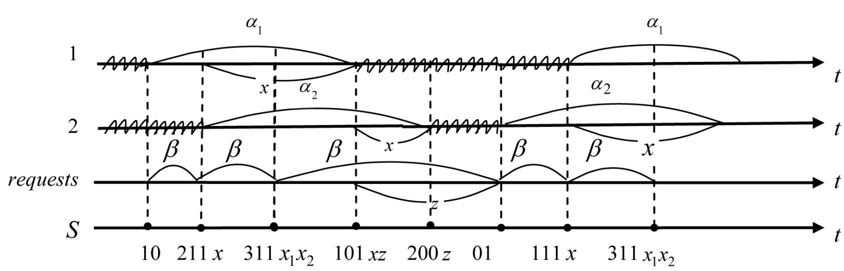

Figure 1. The time diagram of the system functioning.

3. Definition of the Stationary Distribution of the Embedded Markov Chain

We will find the stationary distribution of EMC $\{\xi_n;\ n>0\}$. Let us denote $\rho(01), \rho(10)$ the values of stationary distribution in states $01, 10$ and assume the existence of stationary densities $\rho(i00z)$, $\rho(i11x), i=1,2,$ $\rho(101xz)$, $\rho(210xz), \rho(311x_1x_2)$.

Introduce the notations:

$$\rho_0=\rho(01)=\rho(10),\ \varphi_1(x_1,x_2)=\rho(311x_1x_2),\ \varphi_2(x,z)=\rho(101xz),$$

$$\varphi_3(x,z)=\rho(210xz),\ \varphi_4(x)=\rho(111x),\ \varphi_5(x)=\rho(211x),$$

$$\varphi_6(z)=\rho(100z),\ \varphi_7(z)=\rho(200z).$$

Using (2), set up a system of integral equations to determine the stationary distribution:

$$\begin{cases}
\rho_0=\dfrac{1}{2}\displaystyle\int_0^\infty\varphi_6(z)\,\mathrm{d}z+\dfrac{1}{2}\int_0^\infty\varphi_7(z)\,\mathrm{d}z,\\
\varphi_1(x_1,x_2)=\displaystyle\int_0^\infty\varphi_1(x_1+t,x_2+t)g(t)\,\mathrm{d}t+\int_0^\infty\varphi_4(x_2+t)f_1(x_1+t)g(t)\,\mathrm{d}t\\
\qquad\qquad+\displaystyle\int_0^\infty\varphi_5(x_1+t)f_2(x_2+t)g(t)\,\mathrm{d}t,\\
\varphi_2(x,z)=\displaystyle\int_0^\infty\varphi_4(x+t)f_1(t)g(z+t)\,\mathrm{d}t+\int_0^\infty\varphi_5(t)f_2(x+t)g(z+t)\,\mathrm{d}t\\
\qquad\qquad+\displaystyle\int_0^\infty\varphi_1(t,x+t)g(z+t)\,\mathrm{d}t,\\
\varphi_3(x,z)=\displaystyle\int_0^\infty\varphi_5(x+t)f_2(t)g(z+t)\,\mathrm{d}t+\int_0^\infty\varphi_4(t)f_1(x+t)g(z+t)\,\mathrm{d}t\\
\qquad\qquad+\displaystyle\int_0^\infty\varphi_1(x+t,t)g(z+t)\,\mathrm{d}t,\\
\varphi_4(x)=\rho_0\displaystyle\int_0^\infty f_2(x+t)g(t)\,\mathrm{d}t+\int_0^\infty\varphi_2(x+t,t)\,\mathrm{d}t,\\
\varphi_5(x)=\rho_0\displaystyle\int_0^\infty f_1(x+t)g(t)\,\mathrm{d}t+\int_0^\infty\varphi_3(x+t,t)\,\mathrm{d}t,\\
\varphi_6(z)=\rho_0\displaystyle\int_0^\infty f_1(t)g(z+t)\,\mathrm{d}t+\int_0^\infty\varphi_3(t,z+t)\,\mathrm{d}t,\\
\varphi_7(z)=\rho_0\displaystyle\int_0^\infty f_2(t)g(z+t)\,\mathrm{d}t+\int_0^\infty\varphi_2(t,z+t)\,\mathrm{d}t,\\
\rho_0+\displaystyle\int_0^\infty\int_0^\infty\varphi_1(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2+\int_0^\infty\int_0^\infty\varphi_2(x,z)\,\mathrm{d}x\mathrm{d}z+\int_0^\infty\int_0^\infty\varphi_3(x,z)\,\mathrm{d}x\mathrm{d}z\\
\quad+\displaystyle\int_0^\infty\varphi_4(x)\,\mathrm{d}x+\int_0^\infty\varphi_5(x)\,\mathrm{d}x+\int_0^\infty\varphi_6(z)\,\mathrm{d}z+\int_0^\infty\varphi_7(z)\,\mathrm{d}z=1.
\end{cases}\tag{3}$$

The last equation in the system (3) is the normalization requirement.

Next, for the sake of simplicity, a homogenous case is considered, and a inhomogeneous case leads to lengthy transformations and results. Let $F_1(t)=F_2(t)=F(t)$. Then, due to the symmetry of states, we get that $\varphi_4(x)=\varphi_5(x)$, $\varphi_2(x,z)=\varphi_3(x,z)$, $\varphi_6(z)=\varphi_7(z)$.

The system (3) is reduced to the following system of equations:

$$\begin{cases} \rho_0 = \int_0^\infty \varphi_6(z)\,\mathrm{d}z, \\ \varphi_1(x_1,x_2) = \int_0^\infty \varphi_1(x_1+t,x_2+t)g(t)\,\mathrm{d}t + \int_0^\infty \varphi_4(x_2+t)f(x_1+t)g(t)\,\mathrm{d}t + \int_0^\infty \varphi_4(x_1+t)f(x_2+t)g(t)\,\mathrm{d}t, \\ \varphi_2(x,z) = \int_0^\infty \varphi_4(x+t)f(t)g(z+t)\,\mathrm{d}t + \int_0^\infty \varphi_4(t)f(x+t)g(z+t)\,\mathrm{d}t + \int_0^\infty \varphi_1(t,x+t)g(z+t)\,\mathrm{d}t, \\ \varphi_4(x) = \rho_0\int_0^\infty f(x+t)g(t)\,\mathrm{d}t + \int_0^\infty \varphi_2(x+t,t)\,\mathrm{d}t, \\ \varphi_6(z) = \rho_0\int_0^\infty f(t)g(z+t)\,\mathrm{d}t + \int_0^\infty \varphi_2(t,z+t)\,\mathrm{d}t, \\ \rho_0 + \int_0^\infty\int_0^\infty \varphi_1(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2 + 2\int_0^\infty\int_0^\infty \varphi_2(x,z)\,\mathrm{d}x\mathrm{d}z + 2\int_0^\infty \varphi_4(x)\,\mathrm{d}x + 2\int_0^\infty \varphi_6(z)\,\mathrm{d}z = 1. \end{cases} \tag{4}$$

Let us introduce the following functions, which are used to record the stationary distribution of EMC:

$h_g(t) = \sum_{n=1}^\infty g^{*(n)}(t)$—is the density of the renewal function [11] $H_g(t)$ of the renewal process generated by RV β;

$v_g(y,z) = g(y+z) + \int_0^y g(y+z-s)h_g(s)\,\mathrm{d}s$—is the density of the direct residual time distribution [11] for the renewal process generated by RV β;

$$\tilde{\gamma}(y) = \int_0^y f(t)v_g(t,y-t)\,\mathrm{d}t, \qquad \beta(x,t) = \int_0^\infty f(x+t+y)v_g(t,y)\,\mathrm{d}y,$$

$h(y) = \sum_{n=1}^\infty \tilde{\gamma}^{*(n)}(y)$—is the density of the renewal function, renewal process generated by RV with the distribution density $\tilde{\gamma}(y)$;

$$\begin{gathered} \tilde{h}(t) = g(t) + (h*g)(t), \qquad \gamma(x,t) = \beta(x,t) + \int_0^\infty \beta(x+y,t)h(y)\,\mathrm{d}y, \\ \pi(x,y) = \sum_{n=1}^\infty \gamma^{(n)}(x,y), \quad \gamma^{(n)}(x,y) = \int_0^\infty \gamma^{(1)}(x,t)\gamma^{(n-1)}(t,y)\,\mathrm{d}t, \quad n\ge 2, \quad \gamma^{(1)}(x,t) = \gamma(x,t), \\ \varphi(x) = \int_0^\infty f(x+t)\tilde{h}(t)\,\mathrm{d}t + \int_0^\infty \pi(x,y)\,\mathrm{d}y\int_0^\infty f(y+t)\tilde{h}(t)\,\mathrm{d}t. \end{gathered} \tag{5}$$

Using the method of successive approximations [12], we can show that the system (4) has the following solution:

$$\begin{aligned} &\varphi_4(x) = \varphi_5(x) = \rho_0\varphi(x), \\ &\varphi_1(x_1,x_2) = \rho_0\left(\int_0^\infty \varphi(x_2+t)f(x_1+t)h_g(t)\,\mathrm{d}t + \int_0^\infty \varphi(x_1+t)f(x_2+t)h_g(t)\,\mathrm{d}t\right), \\ &\varphi_2(x,z) = \varphi_3(x,z) = \rho_0\left(\int_0^\infty \varphi(x+y)f(y)v_g(y,z)\,\mathrm{d}y + \int_0^\infty \varphi(y)f(x+y)v_g(y,z)\,\mathrm{d}y\right), \\ &\varphi_6(z) = \varphi_7(z) = \rho_0\left(\int_0^\infty g(z+t)f(t)\,\mathrm{d}t + \int_0^\infty \mathrm{d}t\int_0^\infty \varphi(t+y)f(y)v_g(y,z+t)\,\mathrm{d}y \right. \\ &\qquad\qquad \left. + \int_0^\infty \mathrm{d}t\int_0^\infty \varphi(y)f(t+y)v_g(y,z+t)\,\mathrm{d}y\right). \end{aligned} \tag{6}$$

The constant ρ_0 is found by means of normalization requirement; its explicit form is not used when finding the QS stationary characteristics.

The system of equations, which is almost identical to the system (3), and its solution method are covered in [13].

4. Definition of Stationary Characteristics of System

Let us turn to the determination of the stationary characteristics of the QS. Using Formulas (1), we will define the average sojourn times in states of the system:

$$\begin{gathered} m(10)=m(01)=\int_0^\infty \bar{F}(t)\bar{G}(t)\mathrm{d}t,\quad m(100z)=m(200z)=z,\quad m(101xz)=m(210xz)=x\wedge z,\\ m(111x)=m(211x)=\int_0^x \bar{F}(t)\bar{G}(t)\mathrm{d}t,\ m(311x_1x_2)=\int_0^{x_1\wedge x_2}\bar{G}(t)\mathrm{d}t. \end{gathered} \tag{7}$$

We divide the set of states E into three following subsets:

$E_0=\{100z, 200z\}$—all servers are available;

$E_1=\{10, 01, 101xz, 210xz\}$—one server is in service;

$E_2=\{111x, 211x, 311x_2x_2\}$—two servers are in service;

$$E=\bigcup_{i=0}^{2}E_i,\quad E_i\cap E_j\neq\varnothing,\ i\neq j.$$

We will introduce the transition probabilities of the semi-Markov processes $\xi(t)$:

$$\Phi(t,e,E_i)=P\{\xi(t)\in E_i\mid\xi(0)=e\},\ e\in E,\ i=\overline{0,2},$$

and

$P_i=\lim_{t\to\infty}\Phi(t,e,E_i)$—stationary probabilities, $i=\overline{0,2}$.

We will show that the stationary probabilities of QS $GI/G/2/0$ are defined by the following formulas:

$$\begin{aligned} P_0&=\frac{1}{C}\Bigg(\int_0^\infty \bar{G}(t)F(t)\mathrm{d}t+\int_0^\infty M\beta_y\bar{\Phi}(y)f(y)\mathrm{d}y+\int_0^\infty M\beta_y\bar{F}(y)\varphi(y)\mathrm{d}y\\ &\quad-\int_0^\infty f(y)\mathrm{d}y\int_0^\infty \bar{V}_g(y,z)\bar{\Phi}(z+y)\mathrm{d}z-\int_0^\infty \varphi(y)\mathrm{d}y\int_0^\infty \bar{V}_g(y,z)\bar{F}(z+y)\mathrm{d}z\Bigg),\\ P_1&=\frac{1}{C}\Bigg(\int_0^\infty \bar{F}(t)\bar{G}(t)\mathrm{d}t+\int_0^\infty f(y)\mathrm{d}y\int_0^\infty \bar{\Phi}(x+y)\bar{V}_g(y,x)\mathrm{d}x+\int_0^\infty \varphi(y)\mathrm{d}y\int_0^\infty \bar{F}(x+y)\bar{V}_g(y,x)\mathrm{d}x\Bigg),\\ P_2&=\frac{1}{C}\int_0^\infty \bar{\Phi}(t)\bar{F}(t)\mathrm{d}t,\\ C&=M\beta\left(1+\int_0^\infty \bar{\Phi}(x)\bar{F}(x)\mathrm{d}\hat{H}_g(x)\right)=M\beta\left(1+\bar{\Phi}(0)+\int_0^\infty \bar{\Phi}(x)\bar{F}(x)h_g(x)\mathrm{d}x\right), \end{aligned} \tag{8}$$

where

$$\begin{aligned} \varphi(x)&=\int_0^\infty f(x+t)\tilde{h}(t)\mathrm{d}t+\int_0^\infty \pi(x,y)\mathrm{d}y\int_0^\infty f(y+t)\tilde{h}(t)\mathrm{d}t,\\ \bar{\Phi}(x)&=\int_x^\infty \varphi(t)\mathrm{d}t=\int_0^\infty \bar{F}(x+y)\tilde{h}(y)\mathrm{d}y+\int_0^\infty \tilde{h}(t)\mathrm{d}t\int_0^\infty \bar{\Pi}(x,y)f(y+t)\mathrm{d}y,\\ \bar{\Pi}(x,y)&=\int_x^\infty \pi(t,y)\mathrm{d}t, \end{aligned} \tag{9}$$

$\hat{H}_g(x)=\sum_{n=0}^{\infty}G^{*(n)}(x)$—is the renewal function [11];

$V_g(y,x)$—is DF of the direct residual time [11]; $\bar{V}_g(y,x)=1-V_g(y,x)$;
$M\beta_y = M\beta\hat{H}_g(y)-t$—is the mathematical expectation of the direct residual time [11].
The proof. As is known [5] [6], the following equalities are true:

$$\lim_{t\to\infty}\Phi(t,e,E_i)=\frac{\int_{E_i} m(e)\rho(\mathrm{d}e)}{\int_E m(e)\rho(\mathrm{d}e)},\ i=\overline{0,2}, \tag{10}$$

where $m(e)$—is the average sojourn time of SMP $\xi(t)$ in state $e\in E$;
$\rho(\mathrm{d}e)$—is the stationary distribution of EMC $\{\xi_n; n\ge 0\}$.
Let us calculate the integrals entering into the right side of equalities (10). Using (6), (7), we get:

$$\begin{aligned}\int_{E_2} m(e)\rho(\mathrm{d}e) &= \rho_0\Bigg(2\int_0^\infty \varphi(x)\mathrm{d}x\int_0^x \bar{F}(t)\bar{G}(t)\mathrm{d}t+\int_0^\infty \mathrm{d}x_1\int_0^{x_1}\mathrm{d}x_2\int_0^{x_2}\bar{G}(t)\mathrm{d}t\int_0^\infty \varphi(x_2+z)f(x_1+z)h_g(z)\mathrm{d}z\\
&+\int_0^\infty \mathrm{d}x_1\int_0^{x_1}\mathrm{d}x_2\int_0^{x_2}\bar{G}(t)\mathrm{d}t\int_0^\infty \varphi(x_1+z)f(x_2+z)h_g(z)\mathrm{d}z+\int_0^\infty \mathrm{d}x_1\int_{x_1}^\infty \mathrm{d}x_2\int_0^{x_1}\bar{G}(t)\mathrm{d}t\int_0^\infty \varphi(x_2+z)f(x_1+z)h_g(z)\mathrm{d}z\\
&+\int_0^\infty \mathrm{d}x_1\int_{x_1}^\infty \mathrm{d}x_2\int_0^{x_1}\bar{G}(t)\mathrm{d}t\int_0^\infty \varphi(x_1+z)f(x_2+z)h_g(z)\mathrm{d}z\Bigg)=2\rho_0\int_0^\infty \bar{\Phi}(t)\bar{F}(t)\mathrm{d}t.\end{aligned}$$

In the transformations, the following formula was used:

$$\int_0^x \bar{G}(t)h_g(x-t)\mathrm{d}t=G(x).$$

$$\begin{aligned}\int_{E_1} m(e)\rho(\mathrm{d}e) &= 2\rho_0\Bigg(\int_0^\infty \bar{F}(t)\bar{G}(t)\mathrm{d}t+\int_0^\infty \mathrm{d}x\int_0^x z\mathrm{d}z\int_0^\infty \varphi(x+y)f(y)v_g(y,z)\mathrm{d}y\\
&+\int_0^\infty \mathrm{d}x\int_0^x z\mathrm{d}z\int_0^\infty \varphi(y)f(x+y)v_g(y,z)\mathrm{d}y\\
&+\int_0^\infty x\mathrm{d}x\int_x^\infty \mathrm{d}z\int_0^\infty \varphi(x+y)f(y)v_g(y,z)\mathrm{d}y+\int_0^\infty x\mathrm{d}x\int_x^\infty \mathrm{d}z\int_0^\infty \varphi(y)f(x+y)v_g(y,z)\mathrm{d}y\Bigg)\\
&=2\rho_0\Bigg(\int_0^\infty \bar{F}(t)\bar{G}(t)\mathrm{d}t+\int_0^\infty f(y)\mathrm{d}y\int_0^\infty \bar{\Phi}(x+y)\bar{V}_g(y,x)\mathrm{d}x+\int_0^\infty \varphi(y)\mathrm{d}y\int_0^\infty \bar{F}(x+y)\bar{V}_g(y,x)\mathrm{d}x\Bigg),\end{aligned}$$

$$\begin{aligned}\int_{E_0} m(e)\rho(\mathrm{d}e) &= 2\rho_0\Bigg(\int_0^\infty z\mathrm{d}z\int_0^\infty g(z+t)f(t)\mathrm{d}t+\int_0^\infty z\mathrm{d}z\int_0^\infty \mathrm{d}t\int_0^\infty \varphi(t+y)f(y)v_g(y,z+t)\mathrm{d}y\\
&+\int_0^\infty z\mathrm{d}z\int_0^\infty \mathrm{d}t\int_0^\infty \varphi(y)f(t+y)v_g(y,z+t)\mathrm{d}y\Bigg)\\
&=2\rho_0\Bigg(\int_0^\infty \bar{G}(t)F(t)\mathrm{d}t+\int_0^\infty M\beta_y\bar{\Phi}(y)f(y)\mathrm{d}y+\int_0^\infty M\beta_y\bar{F}(y)\varphi(y)\mathrm{d}y\\
&-\int_0^\infty f(y)\mathrm{d}y\int_0^\infty \bar{V}_g(y,z)\bar{\Phi}(z+y)\mathrm{d}z-\int_0^\infty \varphi(y)\mathrm{d}y\int_0^\infty \bar{V}_g(y,z)\bar{F}(z+y)\mathrm{d}z\Bigg)\\
&=2\rho_0\Bigg(\int_0^\infty \bar{G}(t)F(t)\mathrm{d}t+M\beta\bar{\Phi}(0)+M\beta\int_0^\infty \bar{\Phi}(y)\bar{F}(y)h_g(y)\mathrm{d}y-\int_0^\infty \bar{\Phi}(y)\bar{F}(y)\mathrm{d}y\\
&-\int_0^\infty f(y)\mathrm{d}y\int_0^\infty \bar{V}_g(y,z)\bar{\Phi}(z+y)\mathrm{d}z-\int_0^\infty \varphi(y)\mathrm{d}y\int_0^\infty \bar{V}_g(y,z)\bar{F}(z+y)\mathrm{d}z\Bigg),\end{aligned} \tag{11}$$

$$
\begin{aligned}
\int_E m(e)\rho(\mathrm{d}e) &= \int_{E_2} m(e)\rho(\mathrm{d}e) + \int_{E_1} m(e)\rho(\mathrm{d}e) + \int_{E_0} m(e)\rho(\mathrm{d}e) \\
&= 2\rho_0 M\beta\left(1+\bar{\Phi}(0)+\int_0^\infty \bar{\Phi}(x)\bar{F}(x)h_g(x)\mathrm{d}x\right) \\
&= 2\rho_0 M\beta\left(1+\int_0^\infty \bar{\Phi}(x)\bar{F}(x)\mathrm{d}\hat{H}_g(x)\right).
\end{aligned} \tag{12}
$$

By substituting the determined expressions in Formulas (10), we get Formulas (8).

Let us define the stationary probability of request loss. We will consider the subset of states:

$E_{\text{loss}} = \{311x_1x_2\}$ —a received request was lost.

We will find $P_{\text{loss}} = \lim_{t\to\infty}\Phi(t,e,E_{\text{loss}})$.

$$
\begin{aligned}
\int_{E_{loss}} m(e)\rho(\mathrm{d}e) = \rho_0\Bigg(&\int_0^\infty \mathrm{d}x_1\int_0^{x_1}\mathrm{d}x_2\int_0^{x_2}\bar{G}(t)\mathrm{d}t\int_0^\infty \varphi(x_2+z)f(x_1+z)h_g(z)\mathrm{d}z \\
&+\int_0^\infty \mathrm{d}x_1\int_0^{x_1}\mathrm{d}x_2\int_0^{x_2}\bar{G}(t)\mathrm{d}t\int_0^\infty \varphi(x_1+z)f(x_2+z)h_g(z)\mathrm{d}z \\
&+\int_0^\infty \mathrm{d}x_1\int_{x_1}^\infty\mathrm{d}x_2\int_0^{x_1}\bar{G}(t)\mathrm{d}t\int_0^\infty \varphi(x_2+z)f(x_1+z)h_g(z)\mathrm{d}z \\
&+\int_0^\infty \mathrm{d}x_1\int_{x_1}^\infty\mathrm{d}x_2\int_0^{x_1}\bar{G}(t)\mathrm{d}t\int_0^\infty \varphi(x_1+z)f(x_2+z)h_g(z)\mathrm{d}z\Bigg) \\
= 2\rho_0\int_0^\infty \bar{\Phi}(t)\bar{F}(t)G(t)\mathrm{d}t.&
\end{aligned}
$$

Therefore, the stationary probability of request loss equals:

$$
P_{\text{loss}} = \frac{1}{C}\int_0^\infty \bar{\Phi}(t)\bar{F}(t)G(t)\mathrm{d}t. \tag{13}
$$

Important characteristics of the QS under consideration are average stationary sojourn times $T(E_i)$ of the system in the selected subsets of states $E_i, i=\overline{0,2}$. To determine them we will use Formulas [5] [6]:

$$
T(E_i) = \frac{\int_{E_i} m(e)\rho(\mathrm{d}e)}{\int_{E\setminus E_i} P(e,E_i)\rho(\mathrm{d}e)},\quad i=\overline{0,2}. \tag{14}
$$

Let us find the values of the expressions in the denominators of Formulas (14).

$$
\begin{aligned}
\int_{E\setminus E_2} P(e,E_2)\rho(\mathrm{d}e) = 2\rho_0\Bigg(&\int_0^\infty \mathrm{d}x\int_0^\infty g(t)f(x+t)\mathrm{d}t + \int_0^\infty \mathrm{d}z\int_z^\infty \mathrm{d}x\int_0^\infty \varphi(x+y)f(y)v_g(y,z)\mathrm{d}y \\
&+\int_0^\infty \mathrm{d}z\int_z^\infty \mathrm{d}x\int_0^\infty \varphi(y)f(x+y)v_g(y,z)\mathrm{d}y\Bigg) \\
= 2\rho_0\bar{\Phi}(0).&
\end{aligned} \tag{15}
$$

The transformations used the following formula:

$$
\int_0^\infty \bar{G}(t)f(t)\mathrm{d}t + \int_0^\infty f(y)\mathrm{d}y\int_0^\infty \varphi(x+y)\bar{V}_g(y,x)\mathrm{d}x + \int_0^\infty \varphi(y)\mathrm{d}y\int_0^\infty f(x+y)\bar{V}_g(y,x)\mathrm{d}x = 1, \tag{16}
$$

which results from the first equation of the system (4),

$$\begin{aligned}\int\limits_{E\setminus E_1} P(e,E_1)\rho(\mathrm{d}e) = 2\rho_0 \Bigg(& \int_0^\infty \mathrm{d}z \int_0^\infty f(m) g(z+m)\mathrm{d}m + \int_0^\infty \mathrm{d}z \int_0^\infty \mathrm{d}m \int_0^\infty \varphi(m+y) f(y) v_g(y,z+m)\mathrm{d}y \\ & + \int_0^\infty \bar{F}(x)\bar{G}(x)\varphi(x)\mathrm{d}x + \int_0^\infty \varphi(x)\mathrm{d}x \int_0^x \bar{G}(y) f(y)\mathrm{d}y \\ & + \int_0^\infty \bar{G}(x_1)\mathrm{d}x_1 \int_{x_1}^\infty \mathrm{d}x_2 \int_0^\infty \varphi(x_2+t) f(x_1+t) h_g(t)\mathrm{d}t \\ & + \int_0^\infty \bar{G}(x_1)\mathrm{d}x_1 \int_{x_1}^\infty \mathrm{d}x_2 \int_0^\infty \varphi(x_1+t) f(x_2+t) h_g(t)\mathrm{d}t \\ & + \int_0^\infty \mathrm{d}x_1 \int_0^{x_1} \bar{G}(x_2)\mathrm{d}x_2 \int_0^\infty \varphi(x_2+t) f(x_1+t) h_g(t)\mathrm{d}t \\ & + \int_0^\infty \mathrm{d}x_1 \int_0^{x_1} \bar{G}(x_2)\mathrm{d}x_2 \int_0^\infty \varphi(x_1+t) f(x_2+t) h_g(t)\mathrm{d}t \Bigg) \\ = {} & 2\rho_0\left(1+\bar{\Phi}(0)\right),\end{aligned} \tag{17}$$

$$\begin{aligned}\int\limits_{E\setminus E_0} P(e,E_0)\rho(\mathrm{d}e) = 2\rho_0 \Bigg(& \int_0^\infty \mathrm{d}z \int_0^\infty f(t) g(z+t)\mathrm{d}t + \int_0^\infty \mathrm{d}z \int_0^z \mathrm{d}x \int_0^\infty \varphi(x+y) f(y) v_g(y,z)\mathrm{d}y \\ & + \int_0^\infty \mathrm{d}z \int_0^z \mathrm{d}x \int_0^\infty \varphi(y) f(x+y) v_g(y,z)\mathrm{d}y \Bigg) = 2\rho_0.\end{aligned} \tag{18}$$

In the derivations of equalities (17), (18) Formula (16) was used in the same way.

Having placed the determined values of the denominators into Formulas (14), we obtain:

$$\begin{aligned} T(E_0) = & \int_0^\infty \bar{G}(t) F(t)\mathrm{d}t + \int_0^\infty M\beta_y \bar{\Phi}(y) f(y)\mathrm{d}y + \int_0^\infty M\beta_y \bar{F}(y)\varphi(y)\mathrm{d}y \\ & - \int_0^\infty f(y)\mathrm{d}y \int_0^\infty \bar{V}_g(y,z)\bar{\Phi}(z+y)\mathrm{d}z + \int_0^\infty \varphi(y)\mathrm{d}y \int_0^\infty \bar{V}_g(y,z)\bar{F}(z+y)\mathrm{d}z, \end{aligned} \tag{19}$$

$$T(E_1) = \frac{\int_0^\infty \bar{F}(t)\bar{G}(t)\mathrm{d}t + \int_0^\infty f(y)\mathrm{d}y \int_0^\infty \bar{\Phi}(x+y)\bar{V}_g(y,x)\mathrm{d}x + \int_0^\infty \varphi(y)\mathrm{d}y \int_0^\infty \bar{F}(x+y)\bar{V}_g(y,x)\mathrm{d}x}{1+\bar{\Phi}(0)},$$

$$T(E_2) = \frac{\int_0^\infty \bar{\Phi}(t)\bar{F}(t)\mathrm{d}t}{\bar{\Phi}(0)}.$$

5. Particular Cases of QS $GI/G/2/0$

Let us look at particular cases of QS *GI*/*G*/2/0.

1) We find the stationary characteristics of QS $M/M/2/0$. In this case,

$$f(t) = \lambda \mathrm{e}^{-\lambda t}, \quad g(t) = \mu \mathrm{e}^{-\mu t}, \quad h_g(t) = \mu, \quad v_g(t,x) = \mu \mathrm{e}^{-\mu x},$$

$$\tilde{\gamma}(y) = \frac{\lambda\mu}{\mu-\lambda}\left(\mathrm{e}^{-\lambda y} - \mathrm{e}^{-\mu y}\right), \quad \beta(x,t) = \frac{\lambda\mu}{\lambda+\mu}\mathrm{e}^{-\lambda(x+t)}, \quad h(y) = \frac{\lambda\mu}{\lambda+\mu}\left(1-\mathrm{e}^{-(\lambda+\mu)y}\right),$$

$$\tilde{h}(t) = \frac{\mu}{\lambda+\mu}\left(\lambda + \mu \mathrm{e}^{-(\lambda+\mu)t}\right), \quad \gamma(x,t) = \frac{2\lambda\mu}{2\lambda+\mu}\mathrm{e}^{-\lambda(x+t)}, \quad \pi(x,y) = \mu \mathrm{e}^{-\lambda(x+y)},$$

$$\varphi(x)=\mu e^{-\lambda x},\quad \bar{\Phi}(x)=\frac{\mu}{\lambda}e^{-\lambda x},\quad \bar{\Pi}(x,y)=\frac{\mu}{\lambda}e^{-\lambda(x+y)},$$

$$\varphi_4(x)=\varphi_5(x)=\rho_0\mu e^{-\lambda x},\quad \varphi_1(x_1,x_2)=\rho_0\mu^2 e^{-\lambda x_1}e^{\lambda x_2},$$

$$\varphi_2(x,z)=\varphi_3(x,z)=\rho_0\mu^2 e^{-\mu z}e^{-\lambda x},\quad \varphi_6(z)=\varphi_7(z)=\rho_0\mu e^{-\mu z}.$$

Using Formulas (8), (13), (19), we obtain:

$$P_0=\frac{2\lambda^2}{2\lambda^2+2\lambda\mu+\mu^2},\quad P_1=\frac{2\lambda\mu}{2\lambda^2+2\lambda\mu+\mu^2},\quad P_2=\frac{\mu^2}{2\lambda^2+2\lambda\mu+\mu^2},\quad P_{\text{loss}}=\frac{\mu^3}{(2\lambda+\mu)(2\lambda^2+2\lambda\mu+\mu^2)},$$

$$T(E_0)=\frac{1}{\mu},\quad T(E_1)=\frac{1}{\lambda+\mu},\quad T(E_2)=\frac{1}{2\lambda}.$$

2) Let us examine QS *M*/*G*/2/0, $g(t)=\mu e^{-\mu t}$, $F_1(t)=F_2(t)=F(t)$.

The direct substitution into the system (4) can show that the stationary distribution of EMC is determined by the formulas:

$$\varphi_4(x)=\varphi_5(x)=\rho_0\mu\bar{F}(x),\quad \varphi_1(x_1,x_2)=\rho_0\mu^2\bar{F}(x_1)\bar{F}(x_2),$$
$$\varphi_2(x,z)=\varphi_3(x,z)=\rho_0\mu^2 e^{-\mu z}\bar{F}(x),\quad \varphi_6(z)=\varphi_7(z)=\rho_0\mu e^{-\mu z}.$$

Functions (5) in this case are as follows:

$$h_g(t)=\mu,\ v_g(y,x)=\mu e^{-\mu x},\ \tilde{\gamma}(y)=(g*f)(y),\ \beta(x,t)=\int_0^\infty f(x+y+t)g(y)\mathrm{d}y,$$

$h(y)=\sum_{n=1}^{\infty}(f*g)^{*(n)}(y)$—is the density of function 1: of renewals [11];

$\tilde{h}(t)=g(t)+\sum_{n=1}^{\infty}g*(f*g)^{*(n)}(t)$—is the density of function 0: of renewals [11];

$$\gamma(x,t)=\int_0^\infty f(y+x+t)\tilde{h}(y)\mathrm{d}y,$$

$$\varphi(x)=\int_0^\infty f(x+t)\tilde{h}(t)\mathrm{d}t+\int_0^\infty \pi(x,y)\mathrm{d}y\int_0^\infty f(y+t)\tilde{h}(t)\mathrm{d}t=\pi(x,0).$$

Consequently,

$$\varphi_4(x)=\varphi_5(x)=\rho_0\pi(x,0)=\rho_0\mu\bar{F}(x).$$

Using Formulas (8), (13), (19), we obtain that the stationary characteristics of QS *M*/*G*/2/0 are written as:

$$P_0=\frac{2M^2\beta}{M^2\alpha+2M\alpha M\beta+2M^2\beta},\quad P_1=\frac{2M\alpha M\beta}{M^2\alpha+2M\alpha M\beta+2M^2\beta},$$

$$P_2=\frac{M^2\alpha}{M^2\alpha+2M\alpha M\beta+2M^2\beta},\quad P_{\text{loss}}=\frac{M^2\alpha-2\int_0^\infty\bar{F}(x)\mathrm{d}x\int_0^x\bar{F}(t)\bar{G}(t)\mathrm{d}t}{M^2\alpha+2M\alpha M\beta+2M^2\beta},$$

$$T(E_0)=M\beta,\quad T(E_1)=\frac{M\alpha M\beta}{M\alpha+M\beta},\quad T(E_2)=\frac{1}{2}M\alpha.$$

Thus, in this case, as shown in [4], stationary probabilities $P_i, i=\overline{0,2}$ are invariant under the laws of distribution of service time.

The semi-Markov model of QS $GI/M/2/0$ is considered in [14].

In the paper [5], a similar approach to the building of QS model under consideration is used. To find the stationary distribution of EMC, a method based on the usage of taboo-probabilities is applied.

In monograph [13], the semi-Markov model of QS $M/\vec{G}/N/0$ is considered and stationary characteristics are defined.

Using built semi-Markov model, limiting theorems and Markov renewal equations [5]-[7], one can find other stationary and non-stationary characteristics of QS $GI/G/2/0$.

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