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Stratification Analysis of Certain Nakayama Algebras

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Abstract

Our purpose in these notes is to present a result for a specific Nakayama Algebra. In essence, it affirms that for any order of simple modules, the cyclic Nakayama Algebras with relations $rad^r KQ = 0$ (*i.e.* Λ_n^r) are not standardly stratified or costandardly stratified.

Keywords

Standardly Stratified, Cyclical Nakayama Algebras, Infinite Projective Dimension

1. Introduction

The aim of this paper is to present a series of results obtained in relation to a particular class of Nakayama algebras. We will begin by recalling the fundamental notions and results of standarly stratified and almost hereditary algebras theory, which will be our main tool.

The concept of standardly stratified algebras emerged as a natural generalization of quasi-hereditary algebras. The class of quasi-hereditary algebras was introduced by Cline, Parshall and Scott in connection with their study of highest weight categories arising in the representation theory of semisimple complex Lie algebras and algebraic groups.

We present our first result, which allows to obtain the main theorem of the article as an immediate consequence.

Theorem 1. *Let Λ be an algebra, such that all non trivial quotient of indecomposable projective has infinite projective dimension, then Λ is not a standardly stratified algebra for any order of simple modules, unless $\mathcal{F}(\Delta)$ is the subcategory* $Proj(\Lambda)$.

Later, we introduce some notions of uniserial algebras and uniserial modules. In section 4, we introduce Nakayama algebras, also known as uniserial generalised Algebras that are studied by Tadasi Nakayama in [1].

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In his **short notes**, as Nakayama called his publication, it was proposed to make some observations to his previous publication about Frobeniusians Algebras, whose first part was published in 1939 at Annals of Mathematics.

We conclude by presenting a special class of Nakayama algebras, for which is the main result of this paper that we quote below:

Theorem 2. *There is no simple order of simple modules for which the cyclical Nakayama Algebras with relations* $rad^r KQ = 0$ (*i.e.* Λ_n^r) *are standardly stratified or costandardly stratified.*

2. Projective Dimension, Injective Dimension and Global Dimension

The following concepts will allow us to define the notions of projective dimension, injective dimension and global dimension; which we will be very useful in demonstrating the fundamental result of this paper.

Definition 1. *Let M be an* Λ*-module. A projective resolution of M is a complex P whit* $P_i = 0$ *for* $i < 0$ *in* $mod\Lambda$

$$P:\cdots \longrightarrow P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

where P_n is a projective module for $n \geq 0$. It should also satisfy that the map ϵ is an epimorphism and $Ker\epsilon = Im\partial_1$.

It is possible show that being Λ a *K*-algebra, it follows that every $M \in mod\Lambda$ has a projective resolution in $mod\Lambda$. More generally , if an abelian category *A* has enough projectives, then every object *M* in *A* has a projective resolution.

Definition 2. *Let M be an* Λ*-module. A minimal projective resolution of M is a projective resolution of M such that* $\forall n \geq 1$, *the homomorphism* $\partial_n : P_n \to Ker\partial_{n-1}$ *is a projective cover* P_{n-1} *and* $\epsilon : P_0 \to M$ *is a projective cover of M.*

Dually we define concepts injective resolution and minimal injective resolution. Is possible also prove that if Λ is a finite dimensional algebra then all module in $mod\Lambda$ has a minimal projective resolution and minimal injective resolution in $mod\Lambda$. The concepts of projective dimension, injective dimension and global dimension for a Λ-module *M* are as follows.

Definition 3. *Projective dimension of* Λ*-module M is the number* $\mathbf{pd}M = n \geq 0$ *such that there is a projective resolution,*

$$P_M : 0 \longrightarrow P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

M of length *n* and *M* does not have projective resolution of length $m-1$. If *M* does not admit a finite projective resolution, then by convention the projective dimension is said to be infinite.

Dually it has the injective dimension of a Λ-module *M*.

Definition 4. *Let* Λ *be an finite dimensional K-algebra. The global dimension* $(\mathbf{gld}\Lambda)$ *is the supremum of the set of projective dimensions of all* Λ *modules, i.e.*

$$\mathbf{gld}\Lambda := sup\{\mathbf{pd}M; whit\ M\ a\ \Lambda\text{-module}\} = sup\{\mathbf{id}M; whit\ M\ a\ \Lambda\text{-module}\}.$$

3. Standardly Stratified Algebras and Quasi-Hereditary Algebras

Let *R* be a commutative Artin ring and Λ a basic Artin algebra over *R*. As further we assume full subcategories of $mod\Lambda$, unless otherwise stated. We consider *K*-algebras of finite dimension basic and indecomposable, where *K* is an algebraically closed field and by the Gabriel theorem, $\lambda = \frac{KQ}{I}$, where *Q* is a finite quiver and *I* is an admissible ideal.

The principal results of this section can be find in [2]-[8].

Definition 5. *Let* Λ *be a Artin algebra and* $\theta = \{\theta(1), \cdots, \theta(n)\} \subset mod\Lambda$ *such that* $Ext^1(\theta(j), \theta(i)) = 0$, $\forall j \geq i$. *We denote*:

1. $\mathcal{F}(\theta)$ the class of $M \in mod\mathcal{A}$ for wich there is a chain of submodules with $\frac{M_i}{M_{i-1}} \cong \theta(k) \in \theta$.
2. $\mathcal{X}(\theta)$ the subcategory on $mod\Lambda$ of modules are direct summands of modules in $\mathcal{F}(\theta)$.

In the following we consider that Λ denote an $\mathcal{K}$-algebra together with a fixed ordering on a complete set $e=(e_1,e_2,\cdots,e_n)$ of primitive orthogonal idempotents (given by the natural ordering of indices). Note that consider the system *e* is equivalent to consider an order established of set of all simple Λ-modules not isomorphic to $S_i \cong e_i \dfrac{\Lambda}{rad\Lambda}$ (we know to be Λ an Artin algebra has a finite number of Λ-modules).

Definition 6. *Let M be a Λ-module. A normal series in M is a sequence of submodules*

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_t = M .$$

The number *t* is called the length of the series. The quotients $\dfrac{M_{i+1}}{M_i}$ are called factors of the series. A series of composition is a normal series whose factors are simple modules, *i.e.*, a normal serie which can not be refined to another longest.

If *X* is a Λ-module, we denote by $[X:S_i]$ the number of factors isomorphic to S_i in composition series *X*, ie, the multiplicity of S_i as composition factor of *X*.

For $1 \le i \le n$, let S_i be the simple Λ-module, which is the simple top of the indecomposable projective $\mathcal{P}_i = \Lambda e_i$.

Definition 7. *Standard module* Δ_i, *for* $1 \le i \le n$, *is the maximal factor module of* $\mathcal{P}_i$ *without composition factors* S_j, *for* $j \le i$. *Dually for* $1 \le i \le n$, *module coestndar* ∇_i *is the maximum submodule* $\mathcal{Q}_i$ *without composition factors* S_j, *for* $j \le i$. *Let* Δ *be the full subcategory consisting of all* Δ_i. *In similar way, we introduce* ∇, *and so on.*

Note that the above definition implies that $[\Delta_i : S_j] = 0$ to $j \le i$ and module $\Delta_n = \mathcal{P}_n$. Dually, it has that $[\nabla_i : S_j] = 0$ to $j \le i$ and module $\nabla_n = \mathcal{Q}_n$.

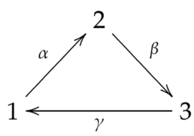
Definition 8. *An algebra Λ is said standardly stratified if* $\Lambda \in \mathcal{F}(\Delta)$. *If in addition to that, the endomorphism ring of each standard module is simple then we say that algebra is quasi-hereditary* (*i.e. standardly stratified algebras generalize the concept of quasi-hereditary algebras where we require the additional condition that the standard modules are Schur modules*). *Dually, if* $D\Lambda \in \mathcal{F}(\nabla)$ *we say that Λ is costandardly stratified.*

Note that if Λ is standardly stratified the projective modules are in $\mathcal{F}(\Delta)$. In addition, if Λ is quasi-hereditary the injective modules are $\mathcal{F}(\nabla)$ and $\Delta_1 = \nabla_1 = S_1$.

The following example will allow us to understand the theory discussed above.

Example 3.1. *Let Λ be the algebra given by the following quiver whit relations* $I = \langle \beta\alpha, \alpha\gamma\beta, \alpha\gamma \rangle$;

$Q:$



We have to:

$$P_1 : \begin{matrix}1\\2\end{matrix} \qquad P_2 : \begin{matrix}2\\3\\1\end{matrix} \qquad P_3 : \begin{matrix}3\\1\end{matrix} \quad \text{and} \quad I_1 : \begin{matrix}2\\3\\1\end{matrix} \qquad I_2 : \begin{matrix}1\\2\end{matrix} \qquad I_3 : \begin{matrix}2\\3\end{matrix}$$

It is not difficult to check that this algebra is standardly stratified and costandardly stratified only at orders for respective simple modules given below:

1. To order $2,1,3 \rightsquigarrow \Delta = \{S_2, \mathcal{P}_1, \mathcal{P}_3\}$ all P_i are filtered.
2. To order $1,2,3 \rightsquigarrow \nabla = \{S_1, \mathcal{I}_2, \mathcal{I}_3\}$ all I_i are filtered.

We denote by $P^{<\infty}(\Lambda)$ the full subcategory of $mod\Lambda$ defined by modules of finite projective dimension. The following result is in [8] which will be of great utility.

Proposition 3. *Let Λ be an standardly stratified algebra, then* $\mathcal{F}(\Delta) \subset P^{<\infty}$.

The following theorem is the first result that present us in this paper. It will allow us to obtain, as an immediate consequence, our main result.

Theorem 4. *Let Λ be an algebra, such that all non trivial quotient of indecomposable projective has infinite projective dimension, then Λ is not standardly stratified algebra for any order of simple modules, unless* $\mathcal{F}(\Delta)$ *is the subcategory* $Proj(\Lambda)$.

Proof. It's clear that $\Delta_i \in \mathcal{F}(\Delta)$. Furthermore Δ_i is quotient of projective P_i. As we assume that all indecomposable projective quotient has infinite projective dimension then $\boldsymbol{pd}\Delta_i = \infty$ therefore $\mathcal{F}(\Delta) \not\subseteq P^{<\infty}$ so Λ is standardly stratified in any order of simple modules.

4. Nakayama Algebras

Throughout, Λ is assumed to be a finite dimensional *K*-algebra, defined over an algebraically closed field *K*.

The principal results of this section can be find in [1] [9] [10].

Definition 9. *Let M be a* Λ*-module. Radical series M is defined as follows*:

$$0 \subset \cdots \subset rad^2 M \subset radM \subset M$$

We agree to denote by $rl(M)$ the radical series length of *M*.

We can define inductively soclo series for module *M* as:

$$soc^0 M := 0,$$

$$soc^{i+1} M := \pi^{-1} soc\left(\frac{M}{soc^i M}\right)$$

where $\pi : M \to \frac{M}{soc^i M}$ is the quotient application, *i.e*.

$$\frac{soc^{i+1} M}{soc^i M} \cong soc\left(\frac{M}{soc^i M}\right).$$

We denote $sl(M)$ the soclo series lenght of *M*.

Note that if $M \neq 0$, $radM \subset M$ and furthermore it has to $dim_k M < \infty$. This clearly implies that the radical series *M* is finite. How $rad^i M = (rad\Lambda)^i M$, then $rad^i \Lambda = (rad\Lambda)^i$ and $rl(M) \leq rl(\Lambda)$.

We can observe that $dim_k M < \infty$, then $socM \neq 0$ if $M \neq 0$ and the soclo series

$$0 \subset socM \subset soc^2 M \subset \cdots \subset M$$

is finit.

Proposition 5. *Let* $M \in mod\Lambda$*, then* $sl(M) = rl(M)$.

Definition 10. *Let* $M \in mod\Lambda$*, the Loewy lenght of M is defined by* $ll(M) := rl(M) = sl(M)$.

Is necessary introduce new notion for define the Nakayama Algebras.

Definition 11. *Let M be an* Λ*-module. We say that M is uniserial if M possesses exactly one composition series*.

Lemma 4.1 *Let M be an* Λ*-module. Next conditions are equivalents*.

1. *M* is uniserial;
2. Radical series of *M* is a composition serie;
3. Soclo series of *M* is a composition serie;
4. $l(M) = ll(M)$.

Definition 12. *Let* Λ *be an K = - algebra.* Λ *is right serial if all right indecomposable projective is a uniserial* Λ*-module. Dually define us the left indecomposable projective notion*.

If Q_Λ denote the underlying quiver of Λ then,

Theorem 6. *A basic K-algebra* Λ *is left serial izquierda if and only if for each vertex* α *in* Q_Λ *there is at most one arrow that starts in* α.

Corollary 1. *A basic K-algebra* Λ *is right serial if and only if for each vertex* α *in* Q_Λ *there is at most one arrow that ends in* α.

Definition 13. *The algebra* Λ *is a Nakayama Algebra if every projective indecomposable and every injective indecomposable* Λ*-module is uniserial*.

It is possible to characterize Nakayama Algebras through its underlying quiver.

Theorem 7. *A basic and connected algebra* Λ *is a Nakayama Algebra if and only if* Q_Λ *it has one of the forms*:

$$1 \to 2 \to \cdots \to n$$

or (cyclical)

$$
\begin{array}{ccccc}
 & 1 & \rightarrow & 2 & \\
\nearrow & & & & \searrow \\
n & & & & 3 \\
Q: \uparrow & & & & \downarrow \\
\vdots & & & & \vdots \\
\nwarrow & & & & \swarrow \\
 & & \cdots & &
\end{array}
$$

Proof. Immediately of Theorem 6 and Corollary 1.

5. Main Result

Let Λ be a Nakayama algebra. We denote Λ_n^r the Nakayama algebra with cyclical underlying quiver Q_Λ with relations $rad^r KQ = 0$.

In [11], it shows that both Λ_n^2 and Λ_n^3 are not standardly stratified or costandardly stratified to any order of the simple modules, which motivates us to prove the following generalization of these results.

Theorem 8. *There is no simple order of simple modules for which the cyclical Nakayama Algebras with relations* $rad^r KQ = 0$ (*i.e.* Λ_n^r) *are standardly stratified or costandardly stratified*.

Proof. It is easy to see that every projective module P_i^0 has the same length and we also know that P_i^0 has an only one composition series. Let M be a quotient of projective module P_i^0 and consider the following short exact sequence,

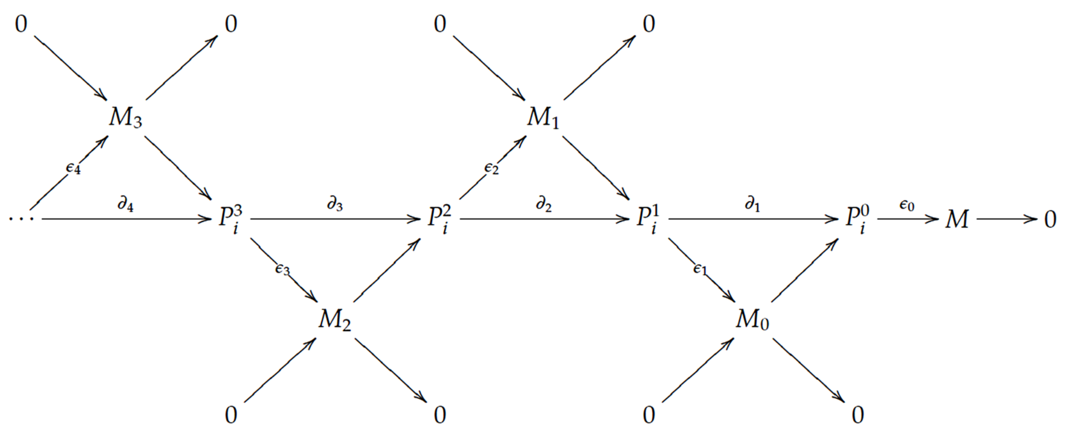
$$0 \longrightarrow M_0 \longrightarrow P_i^0 \xrightarrow{\epsilon_0} M \longrightarrow 0 .$$

Note that the length of Λ-module $M_0 = ker\epsilon_0$ is strict less than P_i^0, therefore M_0 is not projective. Now the following short exact sequence is considered,

$$0 \longrightarrow M_1 \longrightarrow P_i^1 \xrightarrow{\epsilon_1} M_0 \longrightarrow 0$$

in which, again, we note us that the length of Λ-module $M_1 = ker\epsilon_1$ is strict less than P_i^1, therefore M_1 is not projective.

Inductively, given a module M_{n-1} we choose a projective P_i^n and a surjection $\epsilon_n : P_i^n \to M_{n-1}$. Let $M_n = ker\epsilon_n$, and let ∂_n be the composite $P_i^n \xrightarrow{\epsilon_n} M_{n-1} \longrightarrow P_i^{n-1}$. Since $\partial_n \left(P_i^n\right) = M_{n-1} = ker\partial_{n-1}$, this chain complex is a projective resolution of M, $\mathbf{pd}M = \infty$. Then, through Theorem 12, the result is concluded.



□

Commentary 5.1. *Generally Nakayama algebras that are not* Λ_n^r *may be standard stratified or not to be, as we saw in Example* 3.1.

Acknowledgements

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