Advances in Pure Mathematics, 2015, 5, 683-688
Published Online September 2015 in SciRes. http://www.scirp.org/journal/apm
http://dx.doi.org/10.4236/apm.2015.511062

Scientific Research Publishing

Mean-Value Theorems for Harmonic Functions on the Cube in $\mathbb{R}^n$

Petar Petrov

Faculty of Mathematics and Informatics, Sofia University, Sofia, Bulgaria
Email: peynov@fmi.uni-sofia.bg

Received 4 August 2015; accepted 13 September 2015; published 16 September 2015

Abstract

Let $I_n(r)=\{x\in\mathbb{R}^n \mid |x_i|\le r, i=1,2,\cdots,n\}$ be a hypercube in $\mathbb{R}^n$. We prove theorems concerning mean-values of harmonic and polyharmonic functions on $I_n(r)$, which can be considered as natural analogues of the famous Gauss surface and volume mean-value formulas for harmonic functions on the ball in $\mathbb{R}^n$ and their extensions for polyharmonic functions. We also discuss an application of these formulas—the problem of best canonical one-sided L^1-approximation by harmonic functions on $I_n(r)$.

Keywords

Harmonic Functions, Polyharmonic Functions, Hypercube, Quadrature Domain, Best One-Sided Approximation

1. Introduction

This note is devoted to formulas for calculation of integrals over the n-dimensional hypercube centered at $\mathbf{0}$

$$I_n := I_n(r) := \{\mathbf{x}\in\mathbb{R}^n \mid |x_i|\le r, i=1,2,\cdots,n\}, r>0,$$

and its boundary $P_n := P_n(r) := \partial I_n(r)$, based on integration over hyperplanar subsets of I_n and exact for harmonic or polyharmonic functions. They are presented in Section 2 and can be considered as natural analogues on I_n of Gauss surface and volume mean-value formulas for harmonic functions ([1]) and Pizzetti formula [2], ([3], Part IV, Ch. 3, pp. 287-288) for polyharmonic functions on the ball in $\mathbb{R}^n$. Section 3 deals with the best one-sided L^1-approximation by harmonic functions.

How to cite this paper: Petrov, P. (2015) Mean-Value Theorems for Harmonic Functions on the Cube in $\mathbb{R}^n$. *Advances in Pure Mathematics*, **5**, 683-688. http://dx.doi.org/10.4236/apm.2015.511062

Let us remind that a real-valued function f is said to be *harmonic* (*polyharmonic of degree* $m \geq 2$) in a given domain $\Omega \subset \mathbb{R}^n$ if $f \in C^2(\Omega)$ $\left(f \in C^{2m}(\Omega)\right)$ and $\Delta f = 0$ $\left(\Delta^m f = 0\right)$ on Ω, where Δ is the Laplace operator and Δ^m is its m-th iterate

$$\Delta f := \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}, \quad \Delta^m f := \Delta\left(\Delta^{m-1} f\right).$$

For any set $D \subset \mathbb{R}^n$, denote by $\mathcal{H}(D)$ $\left(\mathcal{H}^m(D), m \geq 2\right)$ the linear space of all functions that are harmonic (polyharmonic of degree m) in a domain containing D. The notation $\mathrm{d}\lambda_n$ will stand for the Lebesgue measure in $\mathbb{R}^n$.

2. Mean-Value Theorems

Let $B_n(r) := \left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \|x\| := \left(\sum_{i=1}^n x_i^2\right)^{1/2} \leq r \right\}$ and $S_n(r) := \left\{ \boldsymbol{x} \in \mathbb{R}_n \,\middle|\, \|x\| = r \right\}$ be the ball and the hypersphere in $\mathbb{R}^n$ with center $\boldsymbol{0}$ and radius r. The following famous formulas are basic tools in harmonic function theory and state that for any function h which is harmonic on $B_n(r)$ both the average over $S_n(r)$ and the average over $B_n(r)$ are equal to $h(\boldsymbol{0})$.

The surface mean-value theorem. *If* $h \in \mathcal{H}\left(B_n(r)\right)$, *then*

$$\frac{1}{\sigma_{n-1}\left(S_n(r)\right)} \int_{S_n(r)} h \,\mathrm{d}\sigma_{n-1} = h(\boldsymbol{0}), \tag{1}$$

where $\mathrm{d}\sigma_{n-1}$ is the $(n-1)$-dimensional surface measure on the hypersphere $S_n(r)$.

The volume mean-value theorem. *If* $h \in \mathcal{H}\left(B_n(r)\right)$, *then*

$$\frac{1}{\lambda_n\left(B_n(r)\right)} \int_{B_n(r)} h \,\mathrm{d}\lambda_n = h(\boldsymbol{0}). \tag{2}$$

The balls are known to be the only sets in $\mathbb{R}^n$ satisfying the surface or the volume mean-value theorem. This means that if $\Omega \subset \mathbb{R}^n$ is a nonvoid domain with a finite Lebesgue measure and if there exists a point $\boldsymbol{x}_0 \in \Omega$ such that $h(\boldsymbol{x}_0) = \frac{1}{\lambda_n(\Omega)} \int_\Omega h \,\mathrm{d}\lambda_n$ for every function h which is harmonic and integrable on Ω, then Ω is an open ball centered at $\boldsymbol{x}_0$ (see [4]). The mean-value properties can also be reformulated in terms of *quadrature domains* [5]. Recall that $\Omega \subset \mathbb{R}^n$ is said to be a quadrature domain for $\mathcal{H}(\Omega)$, if Ω is a connected open set and there is a Borel measure $\mathrm{d}\mu$ with a compact support $K_\mu \subset \Omega$ such that $\int_{\bar{\Omega}} f \,\mathrm{d}\lambda_n = \int_{K_\mu} f \,\mathrm{d}\mu$ for every λ_n-integrable harmonic function f on Ω. Using the concept of quadrature domains, the volume mean-value property is equivalent to the statement that any open ball in $\mathbb{R}^n$ is a quadrature domain and the measure $\mathrm{d}\mu$ is the Dirac measure supported at its center. On the other hand, no domains having "corners" are quadrature domains [6]. From this point of view, the open hypercube I_n° is not a quadrature domain. Nevertheless, it is proved in Theorem 1 below that the closed hypercube I_n is a quadrature set in an extended sense, that is, we find explicitly a measure $\mathrm{d}\mu$ with a compact support K_μ having the above property with Ω replaced by I_n but the condition $K_\mu \subset I_n^\circ$ is violated exactly at the "corners" (for the existence of quadrature sets see [7]). This property of I_n is of crucial importance for the best one-sided L^1-approximation with respect to $\mathcal{H}(I_n)$ (Section 3).

Let us denote by D_n^{ij} the $(n-1)$-dimensional hyperplanar segments of I_n defined by

$$D_n^{ij} := D_n^{ij}(r) := \left\{ \boldsymbol{x} \in I_n \,\middle|\, |x_k| \leq |x_i| = |x_j|, k \neq i, j \right\}, \quad 1 \leq i < j \leq n,$$

(see **Figure 1**). Denote also

$$\omega_k(\boldsymbol{x}) := \frac{\left(r - \max\left\{|x_1|, |x_2|, \cdots, |x_n|\right\}\right)^k}{k!}, \quad k \geq 0,$$

and $\mathrm{d}\lambda_m^{\omega_k} := \omega_k \mathrm{d}\lambda_m$. It can be calculated that

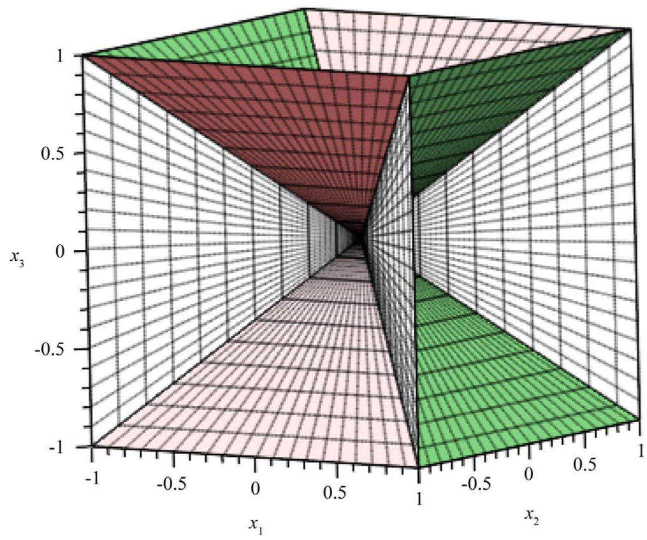

Figure 1. The sets $D_3^{12}(1)$ (white), $D_3^{13}(1)$ (green) and $D_3^{23}(1)$ (coral).

$$\lambda_n^{\omega_k}(I_n) = 2^n n!\frac{r^{n+k}}{(n+k)!}, \quad \lambda_{n-1}^{\omega_k}(P_n) = 2^n n!\frac{r^{n+k-1}}{(n+k-1)!},$$

and

$$\lambda_{n-1}^{\omega_k}(D_n) = 2^{n-1} n!\frac{r^{n+k-1}}{(n+k-1)!}, \quad \text{where } D_n := \bigcup_{1\le i<j\le n} D_n^{ij}.$$

The following holds true.

Theorem 1 *If* $h \in \mathcal{H}(I_n)$, *then h satisfies*:

(i) **Surface mean-value formula for the hypercube**

$$\frac{1}{\lambda_{n-1}(P_n)}\int_{P_n} h\mathrm{d}\lambda_{n-1} = \frac{1}{\lambda_{n-1}(D_n)}\int_{D_n} h\mathrm{d}\lambda_{n-1}, \tag{3}$$

(ii) **Volume mean-value formula for the hypercube**

$$\frac{1}{\lambda_n^{\omega_k}(I_n)}\int_{I_n} h\mathrm{d}\lambda_n^{\omega_k} = \frac{1}{\lambda_{n-1}^{\omega_{k+1}}(D_n)}\int_{D_n} h\mathrm{d}\lambda_{n-1}^{\omega_{k+1}}, \quad k \ge 0. \tag{4}$$

In particular, both surface and volume mean values of *h* are attained on D_n.

Proof. Set

$$M_i := M_i(\boldsymbol{x}) := \max_{j\ne i}|x_j|,$$

and

$$\boldsymbol{x}_t^i := (x_1,\cdots,x_{i-1},t,x_{i+1},\cdots,x_n).$$

Using the harmonicity of *h*, we get for $k \ge 1$

$$\begin{aligned}
0 &= \int_{I_n}\Delta h\mathrm{d}\lambda_n^{\omega_k} = \sum_{i=1}^n\int_{I_n}\omega_k\frac{\partial^2 h}{\partial x_i^2}\mathrm{d}\lambda_n \\
&= -\sum_{i=1}^n\int_{-r}^r\cdots\int_{-r}^r\frac{\partial\omega_k}{\partial x_i}(\boldsymbol{x})\frac{\partial h}{\partial x_i}(\boldsymbol{x})\mathrm{d}x_i\mathrm{d}x_1\cdots\mathrm{d}x_{i-1}\mathrm{d}x_{i+1}\cdots\mathrm{d}x_n \\
&= -\sum_{i=1}^n\int_{-r}^r\cdots\int_{-r}^r\left\{\left(\int_{-r}^{-M_i}+\int_{M_i}^r\right)\operatorname{sign} x_i\omega_{k-1}(\boldsymbol{x})\frac{\partial h}{\partial x_i}(\boldsymbol{x})\mathrm{d}x_i\right\}\mathrm{d}x_1\cdots\mathrm{d}x_{i-1}\mathrm{d}x_{i+1}\cdots\mathrm{d}x_n \\
&= -\sum_{i=1}^n\int_{-r}^r\cdots\int_{-r}^r\left\{\int_{M_i}^r\omega_{k-1}(\boldsymbol{x})\frac{\partial}{\partial x_i}\left[h\left(\boldsymbol{x}_{-x_i}^i\right)+h(\boldsymbol{x})\right]\mathrm{d}x_i\right\}\mathrm{d}x_1\cdots\mathrm{d}x_{i-1}\mathrm{d}x_{i+1}\cdots\mathrm{d}x_n.
\end{aligned}$$

Hence, we have

$$0=-\sum_{i=1}^{n}\int_{-r}^{r}\cdots\int_{-r}^{r}\left\{h\left(\boldsymbol{x}_{-r}^{i}\right)+h\left(\boldsymbol{x}_{+r}^{i}\right)-\left[h\left(\boldsymbol{x}_{-M_i}^{i}\right)+h\left(\boldsymbol{x}_{+M_i}^{i}\right)\right]\right\}\mathrm{d}x_1\cdots\mathrm{d}x_{i-1}\mathrm{d}x_{i+1}\cdots\mathrm{d}x_n \tag{5}$$

if $k=1$ and

$$\begin{aligned}0=&-\sum_{i=1}^{n}\int_{-r}^{r}\cdots\int_{-r}^{r}\int_{M_i}^{r}\omega_{k-2}\left(\boldsymbol{x}\right)\left[h\left(\boldsymbol{x}_{-x_i}^{i}\right)+h\left(\boldsymbol{x}\right)\right]\mathrm{d}x_i\mathrm{d}x_1\cdots\mathrm{d}x_{i-1}\mathrm{d}x_{i+1}\cdots\mathrm{d}x_n\\&+\sum_{i=1}^{n}\int_{-r}^{r}\cdots\int_{-r}^{r}\omega_{k-1}\left(\boldsymbol{x}_{+M_i}^{i}\right)\left[h\left(\boldsymbol{x}_{-M_i}^{i}\right)+h\left(\boldsymbol{x}_{+M_i}^{i}\right)\right]\mathrm{d}x_1\cdots\mathrm{d}x_{i-1}\mathrm{d}x_{i+1}\cdots\mathrm{d}x_n\end{aligned} \tag{6}$$

if $k\geq 2$.

Clearly, (5) is equivalent to (3) and from (6) it follows

$$0=\int_{I_n}\Delta h\mathrm{d}\lambda_n^{\omega_k}=\int_{I_n}h\mathrm{d}\lambda_n^{\omega_{k-2}}-2\int_{D_n}h\mathrm{d}\lambda_{n-1}^{\omega_{k-1}}, \tag{7}$$

which is equivalent to (4). □

Let $M:=M\left(\boldsymbol{x}\right):=\max_{1\leq i\leq n}\left|x_i\right|$. Analogously to the proof of Theorem 1 (ii), Equation (7) is generalized to:

Corollary 1 *If* $h\in\mathcal{H}\left(I_n\right)$ *and* $\varphi\in C^2\left[0,r\right]$ *is such that* $\varphi\left(0\right)=0$ *and* $\varphi'\left(0\right)=0$, *then*

$$0=\int_{I_n}\varphi\left(r-M\right)\Delta h\mathrm{d}\lambda_n=\int_{I_n}\varphi''\left(r-M\right)h\mathrm{d}\lambda_n-2\int_{D_n}\varphi'\left(r-M\right)h\mathrm{d}\lambda_{n-1}. \tag{8}$$

The volume mean-value formula (2) was extended by P. Pizzetti to the following [2] [3] [8].

The Pizzetti formula. *If* $g\in\mathcal{H}^m\left(B_n\left(r\right)\right)$, *then*

$$\int_{B_n(r)}g\mathrm{d}\lambda_n=r^n\pi^{n/2}\sum_{k=0}^{m-1}\frac{r^{2k}}{2^{2k}\Gamma\left(n/2+k+1\right)}\frac{\Delta^k g\left(\mathbf{0}\right)}{k!}.$$

Here, we present a similar formula for polyharmonic functions on the hypercube based on integration over the set D_n.

Theorem 2 *If* $g\in\mathcal{H}^m\left(I_n\right)$, $m\geq 1$, *and* $\varphi\in C^{2m}\left[0,r\right]$ *is such that* $\varphi^{(k)}\left(0\right)=0$, $k=0,1,\cdots,2m-1$, *then the following identity holds true for any* $k\geq 0$:

$$\int_{I_n}\varphi^{(2m)}\left(r-M\right)g\mathrm{d}\lambda_n=2\sum_{s=0}^{m-1}\int_{D_n}\varphi^{(2s+1)}\left(r-M\right)\Delta^{m-s-1}g\mathrm{d}\lambda_{n-1}, \tag{9}$$

where $\varphi^{(j)}\left(t\right)=\dfrac{\mathrm{d}^j\varphi}{\mathrm{d}t^j}\left(t\right)$.

Proof. Equation (9) is a direct consequence from (8):

$$\begin{aligned}0&=\int_{I_n}\varphi\left(r-M\right)\Delta^m g\mathrm{d}\lambda_n\\&=-2\int_{D^n}\varphi^{(1)}\left(r-M\right)\Delta^{m-1}g\mathrm{d}\lambda_{n-1}+\int_{I_n}\varphi^{(2)}\left(r-M\right)\Delta^{m-1}g\mathrm{d}\lambda_n\\&=\cdots=-2\sum_{s=0}^{m-1}\int_{D_n}\varphi^{(2s+1)}\Delta^{m-s-1}g\mathrm{d}\lambda_{n-1}+\int_{I_n}\varphi^{(2m)}g\mathrm{d}\lambda_n.\end{aligned}$$

3. A Relation to Best One-Sided L^1-Approximation by Harmonic Functions

Theorem 1 suggests that for a certain positive cone in $C\left(I_n\right)$ the set D_n is a characteristic set for the best one-sided L^1-approximation with respect to $\mathcal{H}\left(I_n\right)$ as it is explained and illustrated by the examples presented below.

For a given $f\in C\left(I_n\right)$, let us introduce the following subset of $\mathcal{H}\left(I_n\right)$:

$$\mathcal{H}_-\left(I_n,f\right):=\left\{h\in\mathcal{H}\left(I_n\right)\middle|h\leq f\text{ on }I_n\right\}.$$

A harmonic function $h_*^f\in\mathcal{H}_-\left(I_n,f\right)$ is said to be *a best one-sided* L^1*-approximant from below to f with respect to* $\mathcal{H}\left(I_n\right)$ if

$$\left\| f - h_*^f \right\|_1 \le \left\| f - h \right\|_1 \text{ for every } h \in \mathcal{H}_-\left(I_n, f\right),$$

where

$$\|g\|_1 := \int_{I_n} |g| \mathrm{d}\lambda_n.$$

Theorem 1 (ii) readily implies the following ([6] [9]).

Theorem 3 *Let $f \in C\left(I_n\right)$ and $h_*^f \in \mathcal{H}_-\left(I_n, f\right)$. Assume further that the set D_n belongs to the zero set of the function $f - h_*^f$. Then h_*^f is a best one-sided L^1-approximant from below to f with respect to $\mathcal{H}\left(I_n\right)$.*

Corollary 2 *If $f \in C^1\left(I_n\right)$, any solution h of the problem*

$$h_{|D_n} = f_{|D_n}, \ \nabla h_{|D_n} = \nabla f_{|D_n}, \ h \in \mathcal{H}_-\left(I_n, f\right), \tag{10}$$

is a best one-sided L^1-approximant from below to f with respect to $\mathcal{H}\left(I_n\right)$.

Corollary 3 *If $f\left(\boldsymbol{x}\right) = g\left(\boldsymbol{x}\right)\prod_{1\le i<j\le n}\left(x_i^2 - x_j^2\right)^2$, where $g \in C\left(I_n\right)$ and $g \ge 0$ on I_n, then $h_*^f\left(\boldsymbol{x}\right) \equiv 0$ is a best one-sided L^1-approximant from below to f with respect to $\mathcal{H}\left(I_n\right)$.*

Example 1 Let $n = 2$, $r = 1$ and $f_1\left(x_1, x_2\right) = x_1^2 x_2^2$. By Corollary 2, the solution $h_*^{f_1}\left(x_1, x_2\right) = -x_1^4/4 + \frac{3}{2}x_1^2 x_2^2 - x_2^4/4$ of the interpolation problem (10) with $f = f_1$ is a best one-sided L^1-appro-ximant from below to f_1 with respect to $\mathcal{H}\left(I_2\right)$ and $\left\| f_1 - h_*^{f_1} \right\|_1 = 8/45$. Since the function f_1 belongs to the positive cone of the partial differential operator $\mathcal{D}_{2,2}^4 := \frac{\partial^4}{\partial x_1^2 \partial x_2^2}$ (that is, $\mathcal{D}_{2,2}^4 f_1 > 0$), one can compare the best harmonic one-sided L^1-approximation to f_1 with the corresponding approximation from the linear sub-space of $C\left(I_2\right)$:

$$\mathcal{B}^{2,2}\left(I_2\right) := \left\{ b \in C\left(I_2\right) \middle| b\left(x_1, x_2\right) = \sum_{j=0}^{1}\left[a_{0j}\left(x_1\right)x_2^j + a_{1j}\left(x_2\right)x_1^j\right]\right\}.$$

The possibility for explicit constructions of best one-sided L^1-approximants from $\mathcal{B}^{2,2}\left(I_2\right)$, is studied in [10]. The functions $f_1 - b_*^{f_1}$ and $f_1 - b_{f_1}^*$, where $b_*^{f_1}$ and $b_{f_1}^*$ are the unique best one-sided L^1-approximants to f_1 with respect to $\mathcal{B}^{2,2}\left(I_2\right)$ from below and above, respectively, play the role of basic error functions of the canonical one-sided L^1-approximation by elements of $\mathcal{B}^{2,2}\left(I_2\right)$. For instance, $b_*^{f_1}$ can be constructed as the unique interpolant to f_1 on the boundary $\Diamond := \left\{\left(x_1, x_2\right) \in I_2 \middle| \left|x_1\right| + \left|x_2\right| = 1\right\}$ of the inscribed square and $\left\| f_1 - b_*^{f_1} \right\|_1 = 14/45$ (**Figure 2**).

Example 2 Let $n = 2$, $r = 1$ and $f_2\left(x_1, x_2\right) = x_1^8 + 14x_1^4 x_2^4 + x_2^8$. The solution $h_*^{f_2}\left(x_1, x_2\right) = x_1^8 + x_2^8 - 28\left(x_1^6 x_2^2 + x_1^2 x_2^6\right) + 70x_1^4 x_2^4$ of (10) with $f = f_2$ is a best one-sided L^1-approximant from below to f_2 with respect to $\mathcal{H}\left(I_2\right)$ and $\left\| f_2 - h_*^{f_2} \right\| = 8/75$. It can also be verified that $\left\| f_2 - b_*^{f_2} \right\| = 121/900$ (see **Figure 3**).

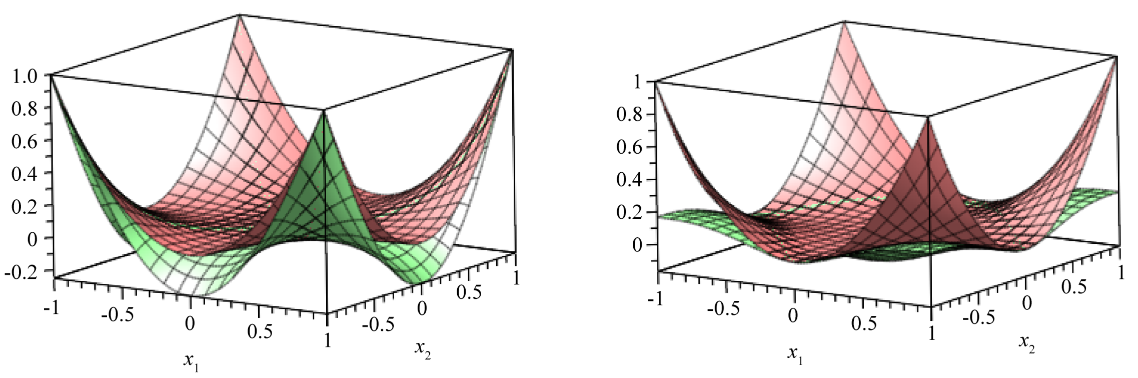

Figure 2. The graphs of the function $f_1\left(x_1, x_2\right) = x_1^2 x_2^2$ (coral) and its best one-sided L^1-approximants from below, $h_*^{f_1}$ with respect to $H\left(I_2\right)$ (left) and $b_*^{f_1}$ with respect to $\mathcal{B}^{2,2}\left(I_2\right)$ (right).

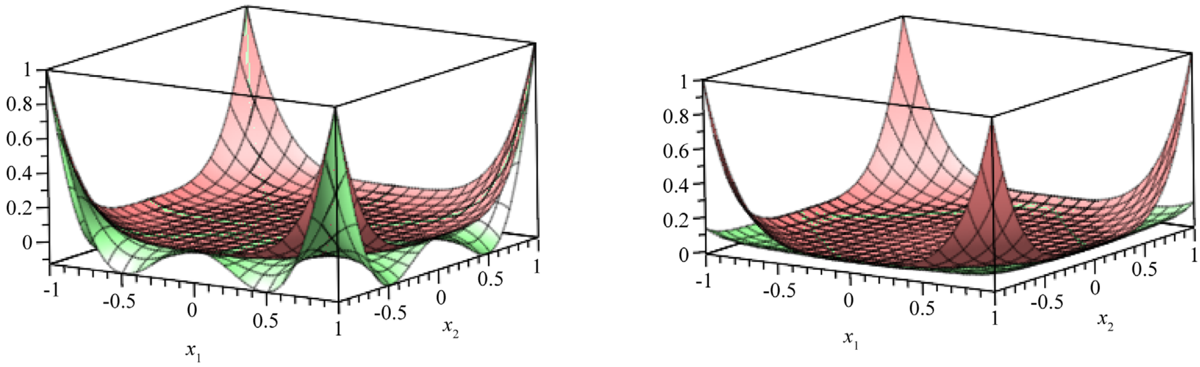

Figure 3. The graphs of the function $f_2(x_1,x_2)=x_1^8+14x_1^4x_2^4+x_2^8$ (coral) and its best one-sided L^1-approximants from below, $h_*^{f_2}$ with respect to $\mathcal{H}(I_2)$ (left) and $b_*^{f_2}$ with respect to $\mathcal{B}^{2,2}(I_2)$ (right).

Remark 1 Let $\varphi\in C^2[0,r]$ is such that $\varphi(0)=0$, $\varphi'(0)=0$, and $\varphi'\ge 0$, $\varphi''\ge 0$ on $[0,r]$. It follows from (8) that Theorem 3 also holds for the best weighted L^1-approximation from below with respect to $\mathcal{H}(I_n)$ with weight $\varphi''(r-M)$. The smoothness requirements were used for brevity and wherever possible they can be weakened in a natural way.

References

[1] Helms, L.-L. (2009) Potential Theory. Springer-Verlag, London. http://dx.doi.org/10.1007/978-1-84882-319-8

[2] Pizzetti, P. (1909) Sulla media dei valori che una funzione dei punti dello spazio assume sulla superficie della sfera. *Rendiconti Linzei—Matematica e Applicazioni*, **18**, 182-185.

[3] Courant, R. and Hilbert, D. (1989) Methods of Mathematical Physics Vol. II. Partial Differential Equations Reprint of the 1962 Original. John Wiley & Sons Inc., New York.

[4] Goldstein, M., Haussmann, W. and Rogge, L. (1988) On the Mean Value Property of Harmonic Functions and Best Harmonic L^1-Approximation. *Transactions of the American Mathematical Society*, **305**, 505-515.

[5] Sakai, M. (1982) Quadrature Domains. Lecture Notes in Mathematics, Springer, Berlin.

[6] Gustafsson, B., Sakai, M. and Shapiro, H.S. (1977) On Domains in Which Harmonic Functions Satisfy Generalized Mean Value Properties. *Potential Analysis*, **71**, 467-484.

[7] Gustafsson, B. (1998) On Mother Bodies of Convex Polyhedra. *SIAM Journal on Mathematical Analysis*, **29**, 1106-1117. http://dx.doi.org/10.1137/S0036141097317918

[8] Bojanov, B. (2001) An Extension of the Pizzetti Formula for Polyharmonic Functions. *Acta Mathematica Hungarica*, **91**, 99-113. http://dx.doi.org/10.1023/A:1010687011674

[9] Armitage, D.H. and Gardiner, S.J. (1999) Best One-Sided L^1-Approximation by Harmonic and Subharmonic Functions. In: Haußmann, W., Jetter, K. and Reimer, M., Eds., *Advances in Multivariate Approximation*, Mathematical Research (Volume 107), Wiley-VCH, Berlin, 43-56.

[10] Dryanov, D. and Petrov, P. (2002) Best One-Sided L^1-Approximation by Blending Functions of Order (2,2). *Journal of Approximation Theory*, **115**, 72-99. http://dx.doi.org/10.1006/jath.2001.3652