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A Note on Discontinuous Functions with Continuous Second Iterate

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Abstract

This paper investigates four classes of functions with a single discontinuous point. We give the sufficient and necessary conditions under which the second order iterates are continuous functions. Furthermore, the sufficient conditions for the continuity of the even order iterates with finitely many discontinuous points are obtained.

Keywords

Iteration, Discontinuous Point, Continuous Function

1. Introduction

For a nonempty set X and $n \in \mathbb{N}$, the n-th iterate of a self-mapping $f: X \to X$ is defined by $f^n(x) = f\left(f^{n-1}(x)\right)$ and $f^0(x) = x$ for all $x \in X$ inductively. As a nonlinear operator, iteration usually amplifies the complexity of functions [1]-[7], computing the n-th iterate of functions is complicated, even for simple functions (see [8]-[12]). On the other hand, iteration can turn complex functions into simple ones. Recently, the following problem was first formulated by X. Liu, L. Liu and W. Zhang: what are discontinuous functions whose iterates of a certain order are continuous? This question, together with three classes of discontinuous functions defined on compact interval, was answered in the affirmative in [13]. That is, suppose that $f: [0,1] \to [0,1]$ with a single discontinuous point (removable discontinuous point, jumping discontinuous or oscillating discontinuous), the authors respectively gave the sufficient and necessary conditions under which the second order iterates are continuous functions.

The purpose of this paper is to study the discontinuous functions defined on open interval. For four classes of discontinuous functions with unique discontinuous point, we obtain the sufficient and necessary conditions for functions being continuous ones under second iterate, which are easily verified respectively. As corollaries, the sufficient conditions for the continuity of the even order iterates with finitely many discontinuous points are obtained. Our results are illustrated by examples in Section 3 .

2. Main Results

In this section the main results for the continuity of f^2 are stated. Throughout the paper we let $I := (0,1)$.

Theorem 1. *Suppose that* $f : I \to I$ *has unique removable discontinuous point* x_r. *Let*

$$c_r := f(x_r) \text{ and } y_r := \lim_{x \to x_r} f(x). \tag{1}$$

Then f^2 is continuous on *I* if and only if the following conditions are fulfilled:

(A$_1$) $f(c_r) = \lim_{x \to y_r} f(x)$,

(A$_2$) $x_r \notin f(I \setminus x_r)$.

Proof. ($\Rightarrow$) Assume that f^2 is continuous on *I*, the removable discontinuous point x_r of *f* is continuous point of f^2 under iteration. Whether y_r defined by (1) is continuous point of *f* or not, we have

$$\lim_{x \to x_r} f^2(x) = \lim_{x \to y_r} f(x). \tag{2}$$

On the other hand, using the definition of c_r and the continuity of f^2,

$$f(c_r) = f^2(x_r) = \lim_{x \to x_r} f^2(x). \tag{3}$$

Thus (2) and (3) lead to (A$_1$). For an indirect proof of (A$_2$), assume that $x_r = f(\tilde{x})$ for $\tilde{x} \in I \setminus x_r$. Then

$$\lim_{x \to \tilde{x}} f^2(x) = \lim_{x \to x_r} f(x) \neq f(x_r) = f^2(\tilde{x}),$$

which contradicts the continuity of f^2 on *I* and gives a proof to (A$_2$).

($\Leftarrow$) It follows from (A$_1$)

$$\lim_{x \to x_r} f^2(x) = \lim_{x \to y_r} f(x) = f(c_r) = f^2(x_r),$$

implying that f^2 is continuous at x_r. The condition (A$_2$), *i.e.*, $x_r \notin f(I \setminus x_r)$, shows that all points $x \in I \setminus x_r$ are continuous points of f^2. Therefore f^2 is continuous on *I*. This completes the proof. □

Corollary 1. *Suppose that* $f : I \to I$ *has finitely many removable discontinuous points* $x_1, x_2, \cdots, x_m$. *If the following conditions*

($\bar{\text{A}}_1$) $f(c_i) = \lim_{x \to y_i} f(x)$,

($\bar{\text{A}}_2$) $x_i \notin f(I \setminus \{x_1, x_2, \cdots, x_m\})$,

are fulfilled for all $i = 1, 2, \cdots, m,$ *where* $c_i := f(x_i)$, $y_i := \lim_{x \to x_i} f(x)$. *Then* f^{2n} *is continuous on I for arbitrary integer* $n \geq 1$.

Proof. By using the sufficiency of Theorem 1, the assumption ($\bar{\text{A}}_1$) implies that f^2 is continuous on those points $x_1, x_2, \cdots, x_m$ and ($\bar{\text{A}}_2$) guarantees that all points $x \in I \setminus \{x_1, x_2, \cdots, x_m\}$ are continuous points of f^2. Thus f^2 is continuous on *I*. Since the composition of continuous functions is continuous, we get the continuity of f^{2n} for all integers $n \geq 1$ inductively. This completes the proof. □

Theorem 2. *Suppose that* $f : I \to I$ *has unique jumping discontinuous point* x_j. *Let* $c_j := f(x_j)$ *and*

$$y_{j-0} := \lim_{x \to x_{j-0}} f(x) \text{ and } y_{j+0} := \lim_{x \to x_{j+0}} f(x).$$

Then f^2 is continuous on *I* if and only if the following conditions are fulfilled:

(B$_1$) $f(c_j) = \lim_{x \to y_{j-0}} f(x) = \lim_{x \to y_{j+0}} f(x)$,

(B$_2$) $x_j \notin f(I \setminus x_j)$.

Proof. ($\Rightarrow$) In view of the definitions of c_j, y_{j-0}, y_{j+0} and the continuity of f^2, we get

$$f(c_j) = f^2(x_j) = \lim_{x \to x_{j-0}} f^2(x) = \lim_{x \to y_{j-0}} f(x), \tag{4}$$

and

$$f(c_j) = f^2(x_j) = \lim_{x \to x_{j+0}} f^2(x) = \lim_{x \to y_{j+0}} f(x). \tag{5}$$

Clearly, (4) and (5) yield (B_1). Suppose the contrary to (ii), there is $x_j = f(\overline{x})$ for some $\overline{x} \in I \setminus x_j$. The limit

$$\lim_{x \to \overline{x}} f^2(x) = \lim_{x \to x_j} f(x)$$

is nonexistence since x_j is a jumping discontinuous point of *f*, which contradicts the fact that f^2 is continuous at the point $\overline{x}$. This contradiction proves (B_2).

($\Leftarrow$) The condition (B_1) implies

$$\lim_{x \to x_{j-0}} f^2(x) = \lim_{x \to y_{j-0}} f(x) = f(c_j) = f^2(x_j), \tag{6}$$

and

$$\lim_{x \to x_{j+0}} f^2(x) = \lim_{x \to y_{j+0}} f(x) = f(c_j) = f^2(x_j). \tag{7}$$

Thus, (6) and (7) lead to

$$\lim_{x \to x_j} f^2(x) = f^2(x_j),$$

which implies that the jumping discontinuous point x_j of *f* change into the continuous point of f^2. Using the similar argument as the sufficiency for (B_2) in Theorem 1, we can prove that all points $x \in I \setminus x_j$ are continuous points of f^2. Thus f^2 is continuous on *I*. That is, we prove the sufficiency. This completes the proof. □

Corollary 2. *Suppose that* $f: I \to I$ *has finitely many jumping discontinuous points* $x_1, x_2, \cdots, x_m$. *If the following conditions*

($\overline{\mathbf{B}}_1$) $f(c_i) = \lim_{x \to y_{i-0}} f(x) = \lim_{x \to y_{i+0}} f(x)$,

($\overline{\mathbf{B}}_2$) $x_i \notin f(I \setminus \{x_1, x_2, \cdots, x_m\})$,

are fulfilled for all $i = 1, 2, \cdots, m$, *where* $c_i := f(x_i)$, $y_{i-0} := \lim_{x \to x_i - 0} f(x)$, $y_{i+0} := \lim_{x \to x_i + 0} f(x)$. *Then* f^{2n} *is continuous on I for arbitrary integer* $n \ge 1$.

Proof. The discussion is similar as that of Corollary 1. By using the sufficiency of Theorem 2, the assumption ($\overline{B}_1$) implies that f^2 is continuous on those points $x_1, x_2, \ldots, x_m$ and ($\overline{B}_2$) implies that $x \in I \setminus \{x_1, x_2, \cdots, x_m\}$ are all continuous points of f^2. Thus f^2 is continuous on *I*. Consequently, we obtain the continuity of f^{2n} for all integers $n \ge 1$ inductively. This completes the proof. □

Theorem 3. *Suppose that* $f: I \to I$ *has unique oscillating discontinuous point* x_o. *Then* f^2 *is continuous on I if and only if the following conditions are fulfilled:*

(C_1) $f^2(x) \equiv f^2(x_o)$ *on a neighborhood* $U(x_o)$,

(C_2) $x_o \notin f(I \setminus x_o)$.

Proof. ($\Rightarrow$) We first show that the condition (C_1) holds. Suppose the contrary, for any $\delta > 0$ there exists a corresponding point $y_\delta \in U(x_o, \delta)$ satisfying $f^2(y_\delta) \ne f^2(x_o)$. Put $\varepsilon_0 = \left| f^2(y_\delta) - f^2(x_o) \right|$, then for $\delta > 0$ there is $y_\delta \in U(x_o, \delta)$ such that $\left| f^2(y_\delta) - f^2(x_o) \right| = \varepsilon_0$, implying f^2 is discontinuous at x_o, a contradiction. This gives a proof to (C_1). To prove (C_2), by reduction to absurdity, we assume that $x_o \in f(I \setminus x_o)$, there is $\hat{x} \in I \setminus x_o$ such that $x_o = f(\hat{x})$. Note that x_o is oscillating discontinuous point of *f*, the limit

$$\lim_{x \to \hat{x}} f^2(x) = \lim_{x \to x_o} f(x)$$

is nothingness, which contradicts the continuity of f^2. Therefore, the claim (C_2) is proved.

($\Leftarrow$) From the assumption (C_1) we see that

$$\lim_{x\to x_o} f^2(x) = \lim_{x\to x_o} f^2(x_o) = f^2(x_o),$$

implying the oscillating discontinuous point x_o of f is a continuous point of f^2. On the other hand, one can use the similar argument as the sufficiency for the condition (C_2) in Theorem 1 and prove that all points $x \in I \setminus x_o$ are continuous points of f^2. This completes the proof. □

Corollary 3. *Suppose that* $f: I \to I$ *has finitely many oscillating discontinuous points* $x_1, x_2, \cdots, x_m$. *If the following conditions*

($\bar{\mathrm{C}}_1$) $f^2(x) \equiv f^2(x_i)$ *on a neighborhood* $U(x_i)$,

($\bar{\mathrm{C}}_2$) $x_i \notin f\left(I \setminus \{x_1, x_2, \cdots, x_m\}\right)$,

are fulfilled for all $i = 1, 2, \cdots, m$. *Then* f^{2n} *is continuous on I for arbitrary integer* $n \ge 1$.

Proof. The discussion is similar as that of Corollary 1. By using the sufficiency of Theorem 3, the second iterate f^2 is continuous on $x_1, x_2, \cdots, x_m$ by ($\bar{\mathrm{C}}_1$) and is continuous on all points $x \in I \setminus \{x_1, x_2, \cdots, x_m\}$ from ($\bar{\mathrm{C}}_2$), thus f^2 is continuous on *I*. Consequently, we have the continuity of f^{2n} for all integers $n \ge 1$ inductively. This completes the proof. □

Theorem 4. *Suppose that* $f: \mathbb{R} \to \mathbb{R}$ *has unique infinite discontinuous point* x_i. *Then* f^2 *is continuous on* $\mathbb{R}$ *if and only if the following conditions are fulfilled:*

(D_1) $f^2(x_i) = \lim_{x\to\infty} f(x) = \text{constant}$,

(D_2) $x_i \notin f(\mathbb{R} \setminus x_i)$.

Proof. ($\Rightarrow$) Note that f^2 is continuous on $\mathbb{R}$, then the infinite discontinuous point x_i of f is a continuous point of f^2, *i.e.*,

$$f^2(x_i) = \lim_{x\to x_i} f^2(x) = \lim_{x\to\infty} f(x),$$

which shows the limit $\lim_{x\to\infty} f(x)$ exists and is equivalent to $f^2(x_i)$. This implies the result (D_1). To prove (D_2), suppose the contrary, there exists a point $\breve{x} \in \mathbb{R} \setminus x_i$ such that $x_i = f(\breve{x})$. Since x_i is infinite discontinuous point of *f*, the limit

$$\lim_{x\to\breve{x}} f^2(x) = \lim_{x\to x_i} f(x)$$

is infinite, which contradicts the continuity of f^2. Thus, the necessary proof of (D_2) is completed.

($\Leftarrow$) From the assumption (D_1) and the fact $\lim_{x\to x_i} f^2(x) = \lim_{x\to\infty} f(x)$, one can see that

$$f^2(x_i) = \lim_{x\to x_i} f^2(x) = A,$$

implying the infinite discontinuous point x_i of f is a continuous point of f^2. If (D_2) holds, then all real numbers $x \ne x_i$ are continuous points of f^2. This completes the proof. □

Corollary 4. *Suppose that* $f: (a, +\infty) \to (a, +\infty)$ $\left(\text{or } f: (-\infty, b) \to (-\infty, b)\right)$ *has unique infinite discontinuous point* x_i, *where* $a, b \in \mathbb{R}$. *Then* f^2 *is continuous on* $(a, +\infty)\left(\text{or } (-\infty, b)\right)$ *if and only if the following conditions are fulfilled:*

($\tilde{\mathrm{D}}_1$) $f^2(x_i) = \lim_{x\to+\infty} f(x) = \text{constant}\left(\text{or } f^2(x_i) = \lim_{x\to-\infty} f(x) = \text{constant}\right)$,

($\tilde{\mathrm{D}}_2$) $x_i \notin f\left((a, +\infty) \setminus x_i\right)\left(\text{or } x_i \notin f\left((-\infty, b) \setminus x_i\right)\right)$.

Proceeding similarly as Theorem 4 one can show this corollary.

Corollary 5. *Suppose that* $f: \mathbb{R} \to \mathbb{R}$ *has finitely many infinite discontinuous points* $x_1, x_2, \cdots, x_m$. *If the following conditions*

($\bar{\mathrm{D}}_1$) $f^2(x_i) = \lim_{x\to\infty} f(x) = \text{constant}$,

($\bar{\mathrm{D}}_2$) $x_i \notin f(\mathbb{R})$, *are fulfilled for all* $i=1,2,\cdots,m$. *Then* f^{2n} *is continuous on* $\mathbb{R}$ *for arbitrary integer* $n\geq 1$.

Proof. We obtain the result by using the similar argument as Corollary 1. In view of the sufficiency of Theorem 4, the second iterate f^2 is continuous on those points $x_1,x_2,\cdots,x_m$ from ($\bar{\mathrm{D}}_1$) and is continuous on all points $x\in I\setminus\{x_1,x_2,\cdots,x_m\}$ from ($\bar{\mathrm{D}}_2$), thus f^2 is continuous on *I*. Then we have the continuity of f^{2n} for all integers $n\geq 1$ inductively. This completes the proof. □

3. Examples

In this section we demonstrate our theorems with examples.

Example 1. *Consider the mapping* $F_1:(-3,3)\to(-3,3)$ *defined by*

$$F_1(x)=\begin{cases}\frac{1}{2}x+2, & x\in(-3,-2),\\ -1, & x=-2,\\ x+3, & x\in(-2,-1],\\ 2, & x\in(-1,3).\end{cases}$$

Clearly, $x_1=-2$ is the unique removable discontinuous point of F_1. By simple calculation, we have

$$c_1:=F_1(-2)=-1,\ \ y_1:=\lim_{x\to-2}F_1(x)=1,$$

$$F_1(c_1)=\lim_{x\to y_1}F_1(x)=2.$$

Moreover, the set $F_1\left((-3,3)\setminus\{-2\}\right)\subset\left(\frac{1}{2},2\right]\cup\{-1\}$ is not include the point –2. By using the sufficiency of Theorem 1, we obtain the continuity of F_1^2 on $(-3,3)$.

Example 2. *Consider the mapping* $F_2:(-3,4)\to(-3,4)$ *defined by*

$$F_2(x)=\begin{cases}\frac{1}{2}x+\frac{7}{2}, & x\in(-3,-1],\\ \frac{3}{5}x+\frac{21}{10}, & x\in\left(-1,\frac{3}{2}\right],\\ -6x+12, & x\in\left(\frac{3}{2},2\right],\\ 3x-6, & x\in(2,4).\end{cases}$$

Clearly, $x_2=-1$ is the unique jumping discontinuous point of F_2. By calculating we have

$$c_1:=F_2(-1)=3,\ \ y_{1-0}:=\lim_{x\to-1-0}F_2(x)=3,\ \ y_{1+0}:=\lim_{x\to-1+0}F_2(x)=\frac{3}{2},$$

$$F_2(c_1)=\lim_{x\to y_{1-0}}F_2(x)=\lim_{x\to y_{1+0}}F_2(x)=3,$$

and $F_2\left((-3,4)\setminus\{-1\}\right)=[0,6)$ is not include the points –1. Then F_2^2 is continuous on $(-3,4)$ using the sufficiency of Theorem 2.

Example 3. *Consider the mapping* $F_3:(4,8)\to(4,8)$ *defined by*

$$F_3(x)=\begin{cases}7-\sin 2, & x\in\left(4,\frac{9}{2}\right],\\ 7+\sin\frac{1}{x-5}, & x\in\left(\frac{9}{2},6\right),\\ 7+\sin 1, & x\in[6,8).\end{cases}$$

Clearly, $x_3 = 5$ is an oscillating discontinuous point of F_3. By calculating, $F_3((4,8)\setminus\{5\}) \subset [6,8]$ is not include 5. Moreover, $F_3^2(x) = F_3^2(5) = 7 + \sin 1$ for $x \in \left(\frac{9}{2}, 6\right)$. Thus the function F_3 is continuous on $(4,8)$ by the sufficiency of Theorem 3.

Example 4. *Consider the mapping* $F_4 : (-\infty, +\infty) \to (-\infty, +\infty)$ *defined by*

$$F_4(x) = -\frac{1}{|x-2|}.$$

Clearly, $x_4 = 2$ is an infinite discontinuous point of F_4. By calculating we have

$$F_4^2(2) = \lim_{x\to\infty} F_4(x) = 0$$

and $F_4((-\infty,+\infty)\setminus\{2\}) \subseteq (-\infty, 0)$ is not include 2, then F_4 is continuous on $(-\infty,+\infty)$ by the sufficiency of Theorem 4.

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