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On Finite Rank Operators on Centrally Closed Semiprime Rings

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Abstract

We prove that the multiplication ring of a centrally closed semiprime ring R has a finite rank operator over the extended centroid C iff R contains an idempotent q such that qRq is finitely generated over C and, for each $x \in qRq$, there exist $z \in qRq$ and e an idempotent of C such that $xz = eq$.

Keywords

Ring, Semiprime Ring, Extended Centroid, Minimal Idempotent

1. Introduction

The symmetric ring of quotients $Q_s(R)$ of a semiprime ring R is probably the most comfortable ring of quotients of R. This notion was first introduced by W.S. Martindale [1] for prime rings and extended to the semiprime case by Amitsur [2]. Recall that a ring R is said to be *semiprime* (resp. *prime*) if $I^2 \neq 0$ for every nonzero ideal I of R (resp. if $IJ \neq 0$ for all nonzero ideals I, J of R). The center C of $Q_s(R)$ is called the *extended centroid* of R, and the C-subring $Q_R := RC$ of $Q_s(R)$ generated by R is called the *central closure* of R. A semiprime R is said to be *centrally closed* whenever $R = RC$. For every $q \in R$, we will denote L_q and R_q the left and right multiplication operators, respectively, by q on R. The multiplication ring of R, $M(R)$, is defined as the subring of $L(R)$ generated by the identity operator Id_R and the set $\{L_q, R_q \mid q \in R\}$. The goal of this paper is to give a semiprime extension of the following well-known result (see for instance [3], Theorem A.9):

"*If the multiplication ring of a centrally closed prime ring R has a finite rank operator over C then R contains an idempotent q such that qRq is a division algebra finitely generated over C*".

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It is also well know that the extended centroid of a prime ring is a field, however, for a semiprime ring, we can only assert that said extended centroid is a von Neumann regular ring. This is the cause of the difficulty of extending this result. The starting point of this path relies on the fact that each subset S of $Q_s(R)$ has an associated idempotent $e_{[S]}$ of the extended centroid C (see [4], Theorem 2.3.9) and on a consequence (see [4], Theorem 2.3.3 and Proposition 1.1 below) of the Weak Density Theorem ([4], Theorem 1.1.5).

2. Tools

We shall assume throughout this paper that R is a centrally closed semiprime ring. First of all, we recall that if $\mathcal{B}_R$ is the set of all idempotents in C has a partial order given by $e \le f$ iff $e = ef$. Moreover, $\mathcal{B}_R$ is a Boolean algebra for the operations

$$e \wedge f = ef, \quad e \vee f = e + f - ef, \quad \text{and} \quad e^* = 1 - e.$$

In fact, [5], Theorem 1.8 remains valid in case that $A = R$ is a ring, and so this Boolean algebra is complete, that is, every subset of $\mathcal{B}_R$ admits supremum and infimum. We will use the properties of the idempotent associated to a subset referred to in ([4], Theorem 2.3.9 (i) and (ii)) without notice.

Given a C-submodule M of R, we will say that M is C-*finitely generated* if there exist $q_1, q_2, \cdots, q_n \in R$ such that $M \subseteq \sum_{i=1}^{n} Cq_i$.

Next, we establish our main tool.

Proposition 1.1 *Let* N *be a* C-*finitely generated* C-*submodule of* R, *and let* $q \in R$. *Then there exists* $f_0 \in \mathcal{B}_R$ *such that*: *a)* $f_0 \le e_{[N]}$, *b)* $f_0 q \in N$ *and c)* $N + Cq = N \oplus (1 - f_0) q$.

Proof. We denote $e = e_{[N]}$. If $q \in N$, then $f_0 = e_{[N]}$. Suppose that $q \in R \setminus N$. If $N \cap Cq = 0$, then we take $f_0 = 0$. In other case, take $e\lambda q \in N \cap Cq$, for some $\lambda \in C$. By ([4] Theorem 2.3.9), there exists $\mu \in C$ such that $\lambda\mu\lambda = \lambda$ and $\lambda\mu \in \mathcal{B}_R$. In particular, $\lambda\mu e q \in N$, and $\lambda\mu e \le e$. Thus, the family $\{f_i\} \subseteq \mathcal{B}_R$ of all non-zero idempotents satisfying $f_i \le e$ and $f_i q \in N$ is not empty. Let $f_0 = \vee f_i$. Note that $f_0 \in \mathcal{B}_R$ because of completeness of $\mathcal{B}_R$, and, of course, $f_0 \le e$. If $f_0 q \notin N$, then, by ([4], Theorem 2.3.3), there exists $F \in M(R)$ such that $F(f_0 q) \ne 0$ and $F(N) = 0$. But, since $F(f_i q) = 0$, we have $f_i e_{[\{F(q)\}]} = 0$ and so $f_i \le 1 - e_{[\{F(q)\}]}$ for all i. Hence $f_0 \le 1 - e_{[\{F(q)\}]}$, that is, $f_0 e_{[\{F(q)\}]} = 0$, which is a contradiction with $F(f_0 q) \ne 0$. Therefore $f_0 q$ belongs to N. Take $m = (1 - f_0) q$. Let us see that $N + Cq = N \oplus Cm$. Indeed, for every $p \in N + Cq$, we can write:

$$p = m' + \lambda q = m' + \lambda f_0 q + \lambda m \in M + Cm. \tag{1}$$

Moreover, if there exists $m_0 \in N$ and $\lambda \in C$ such that

$$m_0 = \lambda e m = \lambda e (1 - f_0) q,$$

then $\lambda e q = m_0 + \lambda e f_0 q \in N$. Take $\mu \in C$ such that $\lambda^2 \mu = \lambda$ and $\mu\lambda$ is an idempotent in C. It is clear that $\mu\lambda e q \in M$, and so $\mu\lambda e \le f_0$ by maximality. Thus, $\mu\lambda e(1 - f_0) = 0$ and $\mu m_0 = 0$. Finally, note that:

$$0 = \lambda\mu m_0 = \lambda^2 \mu e (1 - f_0) q = \lambda e (1 - f_0) q = m_0.$$

Thus, the sum is direct. Note that $f_0 \in \mathcal{B}_R$ verifies properties a), b) and c). □

As a consequence, we have the following:

Corollary 1.2 *Let* M *be a nonzero C-submodule of* R *and* $q \in R$ *such that* $M \subseteq Cq$. *Then there exists* $e \in \mathcal{B}_R$ *such that* $M = Ceq$.

Proof. If $q \in M$ take $e = 1$. In other case, $M + Cq = Cq$. By Proposition 1.1, there is $e \in \mathcal{B}_R$ such that $eq \in M$ and $Cq = M \oplus C(1 - e) q$. Thus, $Ceq \oplus C(1 - e) q = M \oplus C(1 - e) q$, and so, $Ceq = M$. □

Note that if $p, q \in R$ then it may be that $p \in Cq$ but $q \notin Cp$. This forces us to make a convenient definition of set C-linearly independent. We will say that n nonzero elements $q_1, q_2, \cdots, q_n$ of R are *C-linearly independent* (or that the set $S := \{q_1, q_2, \cdots, q_n\}$ is C-*linearly independent*) if, for all $\lambda_1, \lambda_2, \cdots, \lambda_n \in C$, $\sum \lambda_i q_i = 0$ implies $\lambda_i q_i = 0$ for all $i \in \{1, \cdots, n\}$, or equivalently, if the C-linear envelope M of the subset S satisfies: $M = \bigoplus_{i=1}^{n} Cq_i$. Note that for every $0 \ne q \in R$ and $e \in \mathcal{B}_R$, if eq and $(1 - e) q$ are nonzero, then the sets $\mathcal{S} := \{q\}$ and $\mathcal{S}_1 := \{eq, (1-e)q\}$ are C-linearly independent and both generate the C-module Cq. In general, any C-finitely generated C-module M can be obtained as the C-linear envelope of C-linearly independent sets with different cardinal. In this sense, in ([4] Theorem 2.3.9. (iv)) is asserted that one can select a C-linearly independent set with a minimal number of generators under certain conditions. In any case, certain properties of the vector spaces remain true for the C-submodules: the next results, probably well-known, are obtained as a consequence of Proposition 1.1.

Corollary 1.3 *Let* $\{q_1, q_2, \cdots, q_n\}$ *be a subset of* R *and* $N \subsetneq M$ *two C-finitely generated C-submodules of* R *such that* $M = N + \sum_{i=1}^{n} Cq_i$. *Then there are* $e_1, e_2, \cdots, e_n \in \mathcal{B}_R$ *such that the subset of* R

$$\{p_1, p_2, \cdots, p_m\} = \left\{(1-e_1)q_1, (1-e_2)q_2, \cdots, (1-e_n)q_n \middle| (1-e_i)q_i \neq 0\right\}$$

is C-linearly independent, and $M = N \bigoplus_{j=1}^{m} Cp_j$.

Proof. If $q_1 \in N$, we take $e_1 = 1$. In other case, by Proposition 1.1, there exists $e_1 \in \mathcal{B}_R$ such that $N + Cq_1 = N \oplus C(1-e_1)q_1$. Now, if $q_2 \in N \oplus C(1-e_1)q_1$ then take $e_2 = 1$, and if $q_2 \notin N \oplus C(1-e_1)q_1$ then, by Proposition 1.1, there exists $e_2 \in \mathcal{B}_R$ such that $N \oplus C(1-e_1q_1) + Cq_2 = N \oplus C(1-e_1)q_1 \oplus C(1-e_2)q_2$. To conclude, it is enough to repeat this procedure n times. □

Corollary 1.4 *If* N *is a C-finitely generated C-submodule then there exist* $m \leq n$ *and* $p_1, p_2, \cdots, p_m \in N$ *such that* $N = \bigoplus_{i=1}^{m} Cp_i$.

Proof. Let $q_1, q_2, \cdots, q_n \in R$ such that $N \subseteq \sum_{i=1}^{n} Cq_i$. By Corollary 1.3 we can assume that the set $\{q_1, q_2, \cdots, q_n\}$ is C-linearly independent.

It is clear that $N + \sum_{i=1}^{n} Cq_i = \bigoplus_{i=1}^{n} Cq_i$. By Proposition 1.1, there exist $e_1, e_2, \cdots, e_n \in \mathcal{B}_R$ such that, for every $1 \leq j \leq n$, $e_j q_j \in N \oplus \bigoplus_{i=1}^{j-1} Cq_i$ and

$$\bigoplus_{i=1}^{n} Cq_i = N \oplus \bigoplus_{i=1}^{n} C(1-e_i)q_i.$$

Hence,

$$\bigoplus_{i=1}^{n-1} Cq_i \oplus Ce_n q_n \oplus C(1-e_n)q_n = N \oplus \bigoplus_{i=1}^{n-1} C(1-e_i)q_i \oplus C(1-e_n)q_n.$$

Therefore, $\bigoplus_{i=1}^{n-1} Cq_i \oplus Ce_n q_n = N \oplus \bigoplus_{i=1}^{n-1} C(1-e_i)q_i$. Analogously, since $e_n q_n = r_n^{n-2} + s$ with $r_n^{n-2} \in N \oplus \bigoplus_{i=1}^{n-2} C(1-e_i)q_i$ and $s \in C(1-e_{n-1})q_{n-1}$, we have

$$\left[\bigoplus_{i=1}^{n-2} Cq_i + Cr_n^{n-2} + Ce_{n-1}q_{n-1}\right] \oplus C(1-e_{n-1})q_{n-1} = \left[N \oplus \bigoplus_{i=1}^{n-2} C(1-e_i)q_i\right] \oplus C(1-e_{n-1})q_{n-1},$$

and so, $\bigoplus_{i=1}^{n-2} Cq_i + Cr_n^{n-2} + Ce_{n-1}q_{n-1} = N \oplus \bigoplus_{i=1}^{n-2} C(1-e_i)q_i$.

By repeating this procedure, there are $r_n^1, r_{n-1}^1, \cdots, r_2^1 \in N \oplus C(1-e_1)q_1$ such that

$$\left[Cq_1 + Cr_3^1 + \cdots + Cr_n^1 + Ce_2 q_2\right] \oplus C(1-e_2)q_2 = N \oplus (1-e_1)q_1 \oplus (1-e_2)q_2,$$

and hence, $Cq_1 + Ce_2 q_2 + Cr_3^1 + \cdots + Cr_n^1 = N \oplus C(1-e_1)q_1$. Therefore, since, $e_2 q_2 = r_2 + s_2$ with $r_2 \in N$ and $s_2 \in (1-e_1)q_1$, and, for each $j > 2$, $r_j^1 = r_j + s_j$ with $r_j \in N$ and $s_j \in (1-e_1)q_1$, we deduce that

$$\left[Ce_1 q_1 + Cr_2^1 + \cdots + Cr_n^1\right] \oplus C(1-e_1)q_1 = N \oplus (1-e_1)q_1,$$

and so, $Ce_1 q_1 + Cr_2 + \cdots + Cr_n = N$. Again, by Corollary 1.3, we obtain $p_1, p_2, \cdots, p_m$ C-linear independent elements of R such that $N = \bigoplus_{i=1}^{m} Cp_i$. □

Let $I \neq 0$ be a right ideal of R. We say that I is a $\mathcal{B}_R$-*minimal right ideal* if for every nonzero right ideal J of R contained in I, there exists some $e \in \mathcal{B}_R$ such that $0 \neq eJ = eI$. Note that if R is prime then, since C is a field, $\mathcal{B}_R = \{1\}$, and so, the concepts of $\mathcal{B}_R$-minimal right ideal and minimal right ideal agree.

Recall that for a subset S of R the *left annihilator* $\{x \in R : xS = 0\}$ will be denoted by $l(S)$. The *right annihilator* $r(S)$ is similarly defined.

Proposition 1.5 *Let* I *be a* $\mathcal{B}_R$*-minimal right ideal of* R*. Then there exists an idempotent* $0 \neq q \in R$ *and* $e \in \mathcal{B}_R$ *such that* $eI = qR$*. As a consequence* qR *is a* $\mathcal{B}_R$*-minimal ideal of* R*.*

Proof. Since $I \neq 0$ and *R* is semiprime, $0 \neq I^2 \subseteq I$, and hence there exists $0 \neq q' \in I$ such that $0 \neq q'I \subseteq I$. Note that this implies the existence of some $f \in \mathcal{B}_R$ such that $0 \neq fq'I = fI$. Since $q' \in I$, there exists $p \in I$ such that $0 \neq fq'p = fq'$. Note that $fq'p^2 = fq'p$, and then: $fq'\left(p^2 - p\right) = 0$, that is, $f\left(p^2 - p\right) \in r\left(fq'\right) \cap fI$. Since $r\left(fq'\right)$ is a right ideal of R, if $r\left(fq'\right) \cap fI \neq 0$, by minimality there exists $g \in \mathcal{B}_R$ such that $0 \neq gr\left(fq'\right) \cap gfI = gI$. But, since $gN \subseteq gfI$, we have $gI = fgI = fq'gI = 0$, a contradiction. Hence, $fp^2 = fp$ ($0 \neq fp$ because $fq'p \neq 0$). Then $0 \neq fp = fp^2 \in fpR \subseteq fI \subseteq I$. Since I is $\mathcal{B}_R$-minimal, there exists some $e \in \mathcal{B}_R$ such that $efpR = eI$. □

We finalized this section with a desirable result, which is similar to the well-known result for minimal right ideals (see for instance [4], Proposition 4.3.3).

Proposition 1.6 *Let* q *be an idempotent of* R*. The following assertion are equivalent*:

1) qR is $\mathcal{B}_R$-minimal right ideal of R.

2) For every $x \in qRq \setminus \{0\}$ there exist $z \in qRq$ and $e \in \mathcal{B}_R$ such that $xz = eq$.

Proof. (1) $\Rightarrow$ (2). Since q is an idempotent, it is clear that q is the unit of qRq. Take $x \in qRq \setminus \{0\}$. It is clear that $0 \neq xR = qxR \subseteq qR$, and so, since *xR* is right ideal of *R*, there exists $f \in \mathcal{B}_R$ such that $fxR = fqR$. In particular, there is $z' \in R$ such that $fxz' = fq$. Therefore $xfqz'q = fxz'q = fq$.

(2) $\Rightarrow$ (1)

Let *I* be a nonzero right ideal of *R* such that $I \subseteq qR$. Let us see that there exists $f \in \mathcal{B}_R$ such that $fq \in I$. Indeed, if we take $0 \neq p \in I$, by semiprimeness of *R*, there exists $q' \in R$ such that $0 \neq pq'p$. Note that $qp' = p'$ for every $p' \in I \subseteq qR$. Consequently, $pq'q = qpq'q$ is a nonzero element of qRq, and hence there are $z \in R$ and $e \in \mathcal{B}_R$ such that $\left(pq'q\right)\left(qzq\right) = eq$. Therefore $eq \in pR \subseteq I$, and so, $eqRq \subseteq eI \subseteq eqRq$. Thus $eI = eqRq$. □

A nonzero idempotent *q* of *R* is said to be $\mathcal{B}_R$*-minimal* when the above assertions are fulfilled.

3. Theorem

In this section we will prove a semiprime extension of [3], Theorem A.9. Concretely,

Theorem 2.1 *Let R be a centrally closed semiprime ring. Then* $M(R)$ *has a C-finite rank operator if, and only if,* R *contains a* $\mathcal{B}_R$*-minimal idempotent* q *such that* qRq *is* C*-finitely generated.*

We begin this proof with an another consequence of Proposition 1.1,which is an improvement of Corollary 1.2 to case $n > 1$. Given a nonzero *C*-module *M C*-finitely generated, we will say that $\dim_{\mathcal{B}_R}(M) = n$ whenever

$$n = \mathrm{Min}\left\{k \in \mathbb{N} : \exists p_i, p_2, \cdots, p_k \in R \setminus \{0\} \text{ such that } M \subseteq \sum_{i=1}^{k} Cp_i\right\}.$$

Lemma 2.2 *Let* M *be a nonzero* C*-submodule of* R *and suppose that, for every* $f \in \mathcal{B}_R$ *such that* $fM \neq 0$, $\dim_{\mathcal{B}_R}(fM) = n > 1$*. If* $M \subseteq \bigoplus_{i=1}^{n} Cq_i$ *for some* $q_i \in R \setminus \{0\}$ *then there exists* $e \in \mathcal{B}_R$ *such that* $0 \neq eM = \bigoplus_{i=1}^{n} Ceq_i$.

Proof. It is clear that $M + \sum_{i=1}^{n} Cq_i = \bigoplus_{i=1}^{n} Cq_i$. By Proposition 1.1, there exist $f_n \in \mathcal{B}_R$ such that

$$\bigoplus_{i=1}^{n} Cq_i = \left[M + \bigoplus_{i=1}^{n-1} Cq_i\right] \oplus C\left(1 - f_n\right) q_n$$

and $f_n q_n \in M + \bigoplus_{i=1}^{n-1} Cq_i$, in fact, $f_n q_n \in f_n M + \bigoplus_{i=1}^{n-1} Cf_n q_i$. Moreover,

$$\bigoplus_{i=1}^{n-1} Cq_i \oplus Cf_n q_n \oplus C\left(1 - f_n\right) q_n = \left[M + \bigoplus_{i=1}^{n-1} Cq_i\right] \oplus C\left(1 - f_n\right) q_n.$$

Hence,

$$\bigoplus_{i=1}^{n-1} Cq_i \oplus Cf_n q_n = M + \bigoplus_{i=1}^{n-1} Cq_i .$$

If $f_n q_n = 0$, then

$$\bigoplus_{i=1}^{n-1} Cq_i = M + \bigoplus_{i=1}^{n-1} Cq_i ,$$

that is, $M \subseteq \bigoplus_{i=1}^{n-1} Cq_i$, and this is a contradiction. Thus, $f_n q_n \neq 0$ and

$$\bigoplus_{i=1}^{n} Cf_n q_i = f_n M + \bigoplus_{i=1}^{n-1} Cf_n q_i .$$

Note that if $f_n M = 0$ then $0 \neq f_n q_n \in \bigoplus_{i=1}^{n-1} Cf_n q_i$, which is a contradiction. By Proposition 1.1, there exist $f_{n-1} \in \mathcal{B}_R$ such that

$$\bigoplus_{i=1}^{n} Cf_n q_i = \left[f_n M + \bigoplus_{i=1}^{n-2} Cf_n q_i \right] \oplus C\left(1 - f_{n-1}\right) f_n q_{n-1}$$

and $f_{n-1} f_n q_{n-1} \in f_n M + \bigoplus_{i=1}^{n-2} Cf_n q_i$. Therefore, since $f_n q_n = p + p'$ with $p \in f_n M + \bigoplus_{i=1}^{n-2} Cf_n q_i$ and $p' \in C\left(1 - f_{n-1}\right) f_n q_{n-1}$, it is clear that

$$\left[\bigoplus_{i=1}^{n-2} Cf_n q_i + Cp + Cf_{n-1} f_n q_{n-1} \right] \oplus C\left(1 - f_{n-1}\right) f_n q_{n-1} = \left[f_n M + \bigoplus_{i=1}^{n-2} Cf_n q_i \right] \oplus C\left(1 - f_{n-1}\right) f_n q_{n-1} .$$

Hence,

$$\bigoplus_{i=1}^{n-2} Cf_n q_i + Cp + Cf_{n-1} f_n q_{n-1} = f_n M + \bigoplus_{i=1}^{n-2} Cf_n q_i .$$

If $f_{n-1} f_n q_{n-1} = 0$, then $f_n M$ is contained in $n-1$ summands, which is a contradiction. Hence, since $f_{n-1} p = f_{n-1} f_n q_n$, we have

$$\bigoplus_{i=1}^{n} Cf_{n-1} f_n q_i = f_{n-1} f_n M + \bigoplus_{i=1}^{n-2} Cf_{n-1} f_n q_i .$$

Note that if $f_{n-1} f_n M = 0$, then $0 \neq f_{n-1} f_n q_{n-1} \in \bigoplus_{i=1}^{n-2} Cf_{n-1} f_n q_i$, which is a contradiction. By repeating this procedure, we find $f_2, \cdots, f_n \in \mathcal{B}_R$ such that, $f_2 \cdots f_n q_2 \in f_2 \cdots f_n M + Cf_2 \cdots f_n q_1$, $0 \neq f_2 \cdots f_n M$, and

$$\oplus_{i=1}^{n} Cf_2 \cdots f_n q_i = f_2 \cdots f_n M \oplus Cf_2 \cdots f_n q_1 .$$

Therefore, denoting $e_2 = f_2 \cdots f_n$, again by Proposition 1.1, there exists $f_1 \in \mathcal{B}_R$ such that $f_1 e_2 q_1 \in e_2 M$ and,

$$\left[Cf_1 e_2 q_1 + Ce_2 q_2 + \cdots + e_2 q_n \right] \oplus C\left(1 - f_1\right) e_2 q_1 = e_2 M \oplus C\left(1 - f_1\right) e_2 q_1 ,$$

and hence,

$$Cf_1 e_2 q_1 + Ce_2 q_2 + \cdots + Ce_2 q_n = e_2 M ,$$

or even

$$Cf_1 e_2 q_1 + Cf_1 e_2 q_2 + \cdots + Cf_1 e_2 q_n = f_1 e_2 M .$$

Of course, $0 \neq f_1 e_2 q_1$ because $\dim_{\mathcal{B}_R}\left(e_2 M\right) = n$, and so, $0 \neq f_1 e_2 M$. Thus, take $e = f_1 e_2$. □

The next result is an immediate consequence of the Weak Density (see [4], Theorem 2.3.3). We will denote by $M_{p,q}$ the operator $L_p R_q$ for all $p, q \in R$.

Lemma 2.3 *Let* $p_1, \cdots, p_n, q_1, \cdots, q_n \in R$. *Assume that* $\{p_1, \cdots, p_n\}$ *or* $\{q_1, q_2, \cdots, q_n\}$ *are C-linearly independent sets such that* $\sum_{i=1}^{n} M_{p_i, q_i} \neq 0$. *Then there are* $1 \leq j \leq n$ *and* $G \in M(R)$ *such that*

$0 \neq M_{p_j, G(q_j)} = \sum_{i=1}^{n} M_{p_i, G(q_i)}$.

Proof. Assume that $q_1, q_2, \cdots, q_n \in R$ are *C*-linearly independent. If $e_{[p_i]} e_{[q_i]} = 0$ for all $i \in \{1, \cdots, n\}$ then, since $\sum_{i=1}^{n} M_{p_i, q_i} = \sum_{i=1}^{n} M_{p_i, e_{[p_i]} e_{[q_i]} q_i}$, we deduce that $\sum_{i=1}^{n} M_{p_i, q_i} = 0$, is a contradiction. For simplicity, we can suppose that $e_{[p_1]} e_{[q_1]} \neq 0$. By [4] (Theorem 2.3.3), there exists $G = \sum_{j=1}^{m} M_{s_j, t_j}$ with $s_j, t_j \in R$, such that $G\left(e_{[p_1]} q_1\right) \neq 0$ and $G(q_i) = 0$ for all $i \in \{2, \cdots, n\}$. Put $q_{1'} = G\left(e_{[p_1]} q_1\right) \neq 0$, and note that, for every $q' \in R$, we have:

$$\sum_{j=1}^{m} \left(\sum_{i=1}^{n} p_i q' M_{s_j, t_j}(q_i) \right) = \sum_{i=1}^{n} p_i q' G\left(e_{[p_i]} q_i\right) = p_1 q' q_1'.$$

As a consequence: $\sum_{i=1}^{n} M_{p_i, G(q_i)} = M_{p_1, q_{1'}} = M_{p_1, G(q_1)}$. Moreover, by [4] (Corollary 2.3.10), $0 \neq M_{p_1, G(q_1)}$. □

First step in the proof of Theorem

Proposition 2.4 *If* $M(R)$ *has a* C*-finite rank operator then there are* $p, q \in R$ *such that* pRq *is* C*-finitely generated.*

Proof. First of all, given a nonzero operator $G \in M(R)$ with *C*-finite rank we can find an operator of the form $\sum_{i=1}^{n} M_{p_i, q_i}$, which has also *C*-finite rank. In fact, the most general form of *G* is: $\sum L_{r_i} R_{s_i} + L_r + R_s + \alpha Id_R$ for some $\alpha \in \mathbb{K}$, and $r_i, s_i, r, s \in R$. We can take an element $q \in R$ such that $L_p G \neq 0$, because in other case we would have $G(R) \subseteq r(R) = 0$, a contradiction. Analogously, there exists some $q \in R$ such that $R_q L_p G \neq 0$. Now, $F = M_{p,q} G$ is a nonzero operator with the desired form. Moreover, if $G(R)$ is C-finitely generated then $F(R)$ is also C-finitely generated. Secondly, taking in mind Corollary 1.3, we can assume without loss of generality that the set $\{p_1, p_2, \cdots, p_n\}$ is *C*-linearly independent. Finally, by Lemma 2.3 there are $p, q \in R$ and $H \in M(R)$ such that $0 \neq M_{p,q} = \sum_{i=1}^{n} M_{p_i, H(q_i)}$, and so, pRq is also C-finitely generated. □

Second step in the proof of Theorem is a consequence of Lemma 2.2, and its proof can be obtained from a careful reading of the proof of [4] (Lemma 6.1.4).

Proposition 2.5 *Let* $p, q \in R$ *such that* $0 \neq pRq$ *is* C*-finitely generated. Then there exist a* $\mathcal{B}_R$*-minimal idempotent* $q_e \in R$ *such that* $q_e R q_e$ *is* C*-finitely generated.*

Proof. Without loss of generality we can assume that $p = q$. Since, in other case, if we take $0 \neq r \in pRq$ then $0 \neq rRr \subseteq pRq$. Suppose further that $qRq = \sum_{i=1}^{n} Cr_i$, for $r_i \in R$. By Corollary 1.3, we can assume that the sum is direct. Consider the set

$$H := \left\{ k \in \mathbb{N} : k \leq n; \exists q, q_1, \cdots, q_k \in R \setminus \{0\}\ s.t\ qRq = \bigoplus_{i=1}^{k} Cq_i \right\}.$$

It is clear that $n \in H$. Take *m* as the minimum of *H* and $q \in R$ such that $qRq = \oplus_{i=1}^{m} Cq_i$ for some $q_i \in R$. Let $I = qRqR$. If $I = 0$, then $qRq \subseteq l(R) = 0$, which is a contradiction because of semiprimeness of R. Thus $I \neq 0$. Let $0 \neq J \subseteq I$ be a right ideal of R and $0 \neq z = \sum_i q x_i q y_i \in J$, where $x_i, y_i \in R$. Setting $u = \sum_i x_i q y_i$ we note that $z = qu$. Note that if $zRq = 0$ then $0 = quRqu$, a contradiction with the semi-Primeness. Take $0 \neq q' \in zRq$, it is clear that $q'Rq' \subseteq zRq \subseteq qRq$. Note that $M = q'Rq'$ satisfies the hypothesis either of the Corollary 1.2 (if $m = 1$) or of the Proposition 2.2 (if $m > 1$), in any case, there is $e \in \mathcal{B}_R$ such that $0 \neq eq'Req' = \oplus_{i=1}^{m} Ceq_i = e(qRq)$. In particular, $eI = eq'Req'R \subseteq ezR \subseteq J$. Therefore, $0 \neq eJ = eI$, that is, I is a $\mathcal{B}_R$-minimal right ideal of R. By Proposition 1.5, there exist $e \in \mathcal{B}_R$, and $q_e \in R$ such that $eI = q_e R$. Clearly $q_e = q_e^2 \in eM$, and so $q_e = \sum_{i=1}^{n} q u_i q v_i$ where $u_i, v_i \in R$. Hence $q_e R q_e \subseteq \sum_{i=1}^{n} qRqv_i$ and so $q_e R q_e$ is C-finitely generated. □

Finally, the converse is obvious.

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References

[1] Martindale 3rd, W.S. (1969) Prime Rings Satisfying a Generalized Polynomial Identity. *Journal of Algebra*, **12**, 576-584. http://dx.doi.org/10.1016/0021-8693(69)90029-5

[2] Amitsur, S.A. (1972) On Rings of Quotiens. *Symposia Mathematica*, **8**, 149-164.

[3] Bresar, M., Chebotar, M.A. and MartindalWe III, W.S. (2007) Functional Identities. Birkhauser Verlag, Basel-Boston-Berlin.

[4] Beidar, K.I., Martindale III, W.S. and Mikhalev, A.V. (1996) Rings with Generalized Identities. Marcel Dekker, New York.

[5] Cabello, J.C., Cabrera, M., Rodríguez, A. and Roura, R. (2013) A Characterization of π-Complemented Rings. *Communications in Algebra*, **41**, 3067-3079. http://dx.doi.org/10.1080/00927872.2012.672604