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The Stability of a Rotating Cartesian Plume in the Presence of Vertical Boundaries

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Abstract

The effect of two fixed vertical boundaries, a finite distance apart, on the dynamics of a column of buoyant fluid rising in a less buoyant fluid is investigated in the presence of vertical rotation. It is shown that the presence of the boundaries introduces two main effects on a rotating plume. They tend to stabilise the plume but succeed only reducing the value of the growth rate and the plume remains unstable for all finite values of the distance between the boundaries and the plume. In the absence of the sidewalls, two modes of the instability were found known as the sinuous mode and the varicose mode. The influence of the boundaries is such that it reduces the growth rate of the varicose mode more than that of the sinuous mode and consequently the modified sinuous mode is always preferred in the presence of the boundaries.

Keywords

1. Introduction

The study of the dynamics of compositional plumes is important for many real life applications in industry ([1]-[6]), geophysics ([7]-[25]) and environment ([26]-[34]). While the presence of the compositional plumes can be harmful (e.g., in iron bars), it is useful in geophysics (e.g., the hot compositional plumes, that rise from the inner core boundary of the Earth into the outer core interact with the rotation and magnetic field of the Earth and may contribute to the Geodynamo). Such a wide range of applications has motivated many studies on various aspects of the dynamics of compositional plumes. These studies are experimental and theoretical. The experimental works on the dynamics of compositional plumes observed that the plume flow seems to be stable (Sample and

Hellawell [35]; Chen and Chen [36]; Hellawell *et al*. [37]). The laboratory studies by Hellawell *et al*. [37] find that the plumes are thin and long, and its top part tends to break up and disappears. Classen *et al*. [38] experimentally studied the dynamics of compositional plumes under the influence of vertical rotation to find that the plumes are unstable and break into blobs. On the other hand, the theoretical works on the stability of the plumes showed that the Cartesian plume is always unstable in the absence of rotation (Eltayeb and Loper [16]) and in the presence of rotation (Eltayeb and Hamza [18]). These studies assumed that the plume rises vertically in a fluid of unbounded domains. While the experimental studies were conducted in bounded regions to show that the plume was stable, the theoretical models were conducted in unbounded domains to find that the plume was unstable. Thus it is of interest to examine the influence of the vertical boundaries on the dynamics of the plumes. The mathematical model by Al Mashrafi and Eltayeb [6] investigated the influence of the two fixed vertical boundaries on the dynamics of the plumes. They tested the stability of non-rotating Cartesian plumes in a bounded domain to find that the presence of two vertical boundaries affects the stability, but the plumes remain unstable. Moreover, they found that the plume was stable when it was close to the boundary but had a large thickness and the material diffusion is potent in the thin layer between the plume and the nearest boundary.

Motivated by real life applications and laboratory results, we study here the influence of vertical rotation on the dynamics of bounded Cartesian plume. In general, the purpose of this study is to extend the theoretical model by Al Mashrafi and Eltayeb [6] on the dynamics of a Cartesian compositional plume in bounded regions to include the action of vertical rotation. The model by Al Mashrafi and Eltayeb [6] consisted of a column of buoyant fluid of finite thickness, $2x_0$, rising vertically in another less buoyant fluid bounded by two fixed vertical walls located at $x = a_1$ and $x = -a_2$. The system was infinite in the y and z directions. In the current study, we consider that the whole system rotates about the vertical with a uniform angular speed, ω (see **Figure 1**).

In Section 2, we formulate the model mathematically and state the boundary conditions of the system. The presence of rotation introduces an additional parameter, τ, which is a measure of the Coriolis force relative to the viscous force, and this parameter referred to hereinafter as the rotation parameter, defined by

$$\tau\left(= \sqrt{Ta}\right) = \frac{\omega L^2}{\nu}, \tag{1}$$

where Ta is the Taylor number, ν is the kinematic viscosity and L is the unit of length (see Equation (6) below). In Section 3, we investigate the influence of the vertical boundaries on the linear stability of a rotating
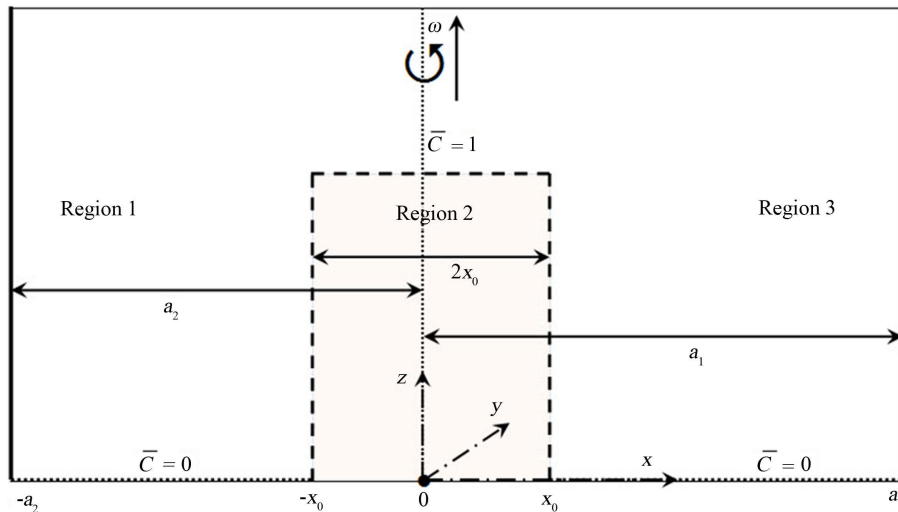
Figure 1. The geometry of the problem showing the profile of the basic state concentration of light material, $\overline{C}(x)$, representing a plume of width, $2x_0$, and concentration, 1, rising vertically in a finite fluid of width, $d\left(= a_1 + a_2\right)$, and concentration, 0. The system rotates uniformly about the vertical with angular speed, ω. The plume divided the system into three regions: region 2 represents the plume whereas regions 1 and 3 represent the surrounding fluid.

bounded Cartesian plume. The problem of the rotating plume was studied by Eltayeb and Hamza [18] in the absence of boundaries. In Section 4, we discuss the effect of the boundaries on the stability of a rotating plume. The growth rate is maximised over the wave numbers plane (m, n) in the parameter space. In Section 5, we make some concluding remarks.

2. Formulation of the Problem

We consider a two-component fluid, in which the concentration of the solvent component (light material) is C and the temperature is T, rotating uniformly about the vertical with angular velocity ω. The fluid has kinematic viscosity, ν, and thermal diffusivity, κ, and material diffusion is negligible. The dimensionless equations of the system have been derived by Eltayeb and Hamza [18]. They are

$$R\left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\right] + \tau \hat{\boldsymbol{z}} \times \boldsymbol{u} = -\nabla\left(p + \frac{z}{\beta \tilde{C}}\right) + \nabla^2 \boldsymbol{u} + (T - T_r + C - C_r)\hat{\boldsymbol{z}}, \tag{2}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{3}$$

$$R\sigma\left[\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T\right] = \nabla^2 T, \tag{4}$$

$$\frac{\partial C}{\partial t} + \boldsymbol{u} \cdot \nabla C = 0. \tag{5}$$

Here R, σ and τ are the Grashoff number, the Prandtl number and the rotation parameter, respectively, defined by

$$R = \beta \tilde{C}\left(\frac{g\kappa^3}{\alpha^3 \gamma^3 \nu^5}\right)^{\frac{1}{4}}\left(=\frac{UL}{\nu}\right), \quad \sigma = \frac{\nu}{\kappa}, \quad \tau = \frac{\omega L^2}{\nu}\left(=\sqrt{Ta}\right), \tag{6}$$

where Ta is the Taylor number and U, L and $\tilde{C}$ are characteristic units of velocity, length and concentration, respectively (Al Mashrafi and Eltayeb [6]), and $\boldsymbol{u}$, p, g, $\hat{\boldsymbol{z}}$, t, α, β, ρ, γ are the velocity vector, the pressure, the constant gravitational acceleration, the upward unit vector, the time, the thermal expansion coefficient, the compositional expansion coefficient, the density, the uniform temperature gradient and the subscript "r" in the Equation (2) refers to reference values.

We consider a basic concentration profile

$$\overline{C}(x) = \begin{cases} 0, & -a_2 \leq x < -x_0, \ x_0 < x \leq a_1 \\ 1, & -x_0 \leq x \leq x_0 \end{cases}, \tag{7}$$

which defines a plume of thickness, $2x_0$, rising with velocity $\overline{w}(x)$ in the presence of mean temperature $\overline{T}(x)$ and mean pressure $\overline{p}(x)$ such that

$$-\frac{d\overline{p}}{dx}\hat{\boldsymbol{x}} + \left(\frac{d^2\overline{w}}{dx^2} + \overline{C} + \overline{T}\right)\hat{\boldsymbol{z}} = 0, \tag{8}$$

$$\frac{d^2\overline{T}}{dx^2} = \overline{w}(x), \tag{9}$$

which are the same equations obtained in the absence of rotation. The presence of rotation does not affect the basic state, and the solution of (8) and (9) is the same as in the absence of rotation. We include it here for easy reference:

$$F(x) = \begin{cases} -A\sinh(ka_1)\sinh[k(x+a_2)]; & -a_2 \leq x < -x_0 \\ -A\sinh(ka_1)\sinh[k(x+a_2)] + \cosh[k(x+x_0)] - 1; & -x_0 \leq x \leq x_0 \\ A\sinh(ka_2)\sinh[k(x-a_1)]; & x_0 < x \leq a_1 \end{cases}, \tag{10}$$

where $F(x)$, A and k are defined by

$$F(x) = \overline{T}(x) - i\overline{w}(x), \quad A = \frac{2\sinh(kx_0)}{\sinh(kd)}, \quad k = \frac{1}{\sqrt{2}}(1+i), \quad i^2 = -1. \tag{11}$$

3. The Stability Analysis

In this section, we use the perturbation Equations (14)-(17) to investigate the linear stability of the basic state solution given by (10). We assume that the interface at the plane $x = x_0$ is given a small harmonic disturbance of the form

$$x = x_0 + \varepsilon E + c.c., \quad E = \exp\{\Omega t + i(my - nz)\}, \tag{12}$$

where m and n are the wavenumber components in the (y,z) plane, Ω is the growth rate and $c.c.$ refers to the complex conjugate.

The disturbance (12) will propagate into the fluid, and affect the second interface and the variables of the system to produce the perturbations. Consequently, the interface at $x = -x_0$ can be written in the form

$$x = -x_0 + \varepsilon \eta_1 E + c.c., \tag{13}$$

where η_1 is a measure of the amplitude of the interface and it is evaluated as a part of the solution. The perturbations introduced into the system are governed by the dimensionless equations (see Al Mashrafi and Eltayeb [6])

$$R\left[\frac{\partial \boldsymbol{u}^\dagger}{\partial t} + \overline{w}\hat{z} \cdot \nabla \boldsymbol{u}^\dagger + \left(\boldsymbol{u}^\dagger \cdot \nabla \overline{w}\right)\hat{z}\right] + \tau\hat{z} \times \boldsymbol{u}^\dagger = -\nabla p^\dagger + \nabla^2 \boldsymbol{u}^\dagger + \left(T^\dagger + C^\dagger\right)\hat{z}, \tag{14}$$

$$\nabla \cdot \boldsymbol{u}^\dagger = 0, \tag{15}$$

$$\sigma R\left[\frac{\partial T^\dagger}{\partial t} + \overline{w}\frac{\partial T^\dagger}{\partial z} + \boldsymbol{u}^\dagger \cdot \nabla \overline{T}\right] + \boldsymbol{u}^\dagger \cdot \hat{z} = \nabla^2 T^\dagger, \tag{16}$$

$$\frac{\partial C^\dagger}{\partial t} + \overline{w}\frac{\partial C^\dagger}{\partial z} + \boldsymbol{u}^\dagger \cdot \nabla \overline{C} = 0, \tag{17}$$

where $\boldsymbol{u}^\dagger$, $p^\dagger$, $T^\dagger$ and $C^\dagger$ are the perturbations in velocity, pressure, temperature and concentration, respectively.

The perturbation variables take the form

$$\{\boldsymbol{u}^\dagger, C^\dagger, T^\dagger, p^\dagger\} = \{-inu(x), nmv(x), w(x), C(x), T(x), -inp(x)\}E + c.c., \tag{18}$$

in which the factors $-in$, nm, and $-in$ are introduced in the variables $u(x)$, $v(x)$ and $p(x)$, respectively, for convenience.

Substituting the variables (18) into the Equations (14)-(17) and after some arrangements we get the following equations

$$\Delta w + T + n^2 p = R\left(\overline{\Omega}w - inuD\overline{w}\right), \tag{19}$$

$$\Delta T - w = \sigma R\left(\overline{\Omega}T - inuD\overline{T}\right), \tag{20}$$

$$\Delta p - T = 2inRuD\overline{w} + im\tau\varsigma, \tag{21}$$

$$\Delta\varsigma = R\left(-invD\overline{w} + \overline{\Omega}\varsigma\right) + \frac{i}{m}\tau w, \tag{22}$$

$$\left(D^2 - m^2\right)v = D\varsigma - w, \tag{23}$$

$$n^2 u = -D(p + w) + m^2\varsigma - i\tau mv - R\overline{\Omega}u, \tag{24}$$

$$C = 0, \tag{25}$$

subject to the boundary conditions

$$D(w+p) = v = w = T = \varsigma = 0 \quad \text{at } x = a_1, \ x = -a_2, \tag{26}$$

$$v, \ w, \ T, \ p, \ \varsigma, \ Dv, \ DT, \ D(w+p) \quad \text{are continuous across } x = \pm x_0, \tag{27}$$

$$\langle Dw(x_0) \rangle = \langle \overline{C}(x_0) \rangle, \quad \langle Dw(-x_0) \rangle = \eta_1 \langle \overline{C}(-x_0) \rangle, \tag{28}$$

$$-inu(x_0) = \Omega - in\overline{w}(x_0), \quad -inu(-x_0) = \left[\Omega - in\overline{w}(-x_0) \right] \eta_1, \tag{29}$$

where the variable ς is related to the vertical component of vorticity and is defined as

$$\varsigma = Dv - u, \tag{30}$$

and we have introduced the notation

$$D \equiv \frac{d}{dx}, \quad \Delta \equiv D^2 - m^2 - n^2, \quad \langle f(\alpha) \rangle = f(\alpha^-) - f(\alpha^+), \quad \overline{\Omega} = \Omega - in\overline{w}(x). \tag{31}$$

(cf. Al Mashrafi and Eltayeb [6]).

We use the same method adopted in Al Mashrafi and Eltayeb [6] and expand the perturbation variables and the growth rate in the small parameter R, thus

$$f(x,y,z,t) = \sum_{s=0}^{\infty} f_s(x,y,z,t) R^s, \quad \Omega = \sum_{s=1}^{\infty} \Omega_s R^{s-1}, \quad R \ll 1, \tag{32}$$

where $f(x,y,z,t)$ indicates any of the perturbation variables u, v, w, p, ς and T.

It turned out that the leading order terms in the equations determines the stability of the system. The relevant equations and the boundary conditions are then obtained from (19)-(24) and (26)-(29) by neglecting the terms with R. In order to facilitate comparison with the results of the non-rotating case, we shall maintain the subscript 0. The equations are given by

$$\Delta w_0 + T_0 + n^2 p_0 = 0, \tag{33}$$

$$\Delta T_0 - w_0 = 0, \tag{34}$$

$$\Delta p_0 - T_0 - im\tau \varsigma_0 = 0, \tag{35}$$

$$\Delta \varsigma_0 - \frac{i}{m} \tau w_0 = 0, \tag{36}$$

$$\left(D^2 - m^2 \right) v_0 - D\varsigma_0 + w_0 = 0, \tag{37}$$

$$n^2 u_0 = -D(p_0 + w_0) + m^2 \varsigma_0 - i\tau m v_0. \tag{38}$$

The associated boundary conditions are obtained from (26)-(29) by introducing the subscript 0 to all the variables and the subscript 1 to Ω.

The system (33)-(38) together with the boundary conditions poses an eigenvalue problem for the growth rate, Ω, which determines the stability of the system. The real part Ω_r governs the variations of the amplitude of the disturbance with time, and hence it determines the stability of the disturbance. If it is negative for all possible values of the wavenumbers m and n, then the plume is stable, while if at least one pair of m and n gives a positive value, then the plume is unstable. If Ω_r vanishes for all values of the wavenumbers, the plume is neutrally stable. If the preferred mode occurs for m, n both non-zero, it is referred to as a 3-dimensional mode (oblique), and if $m = 0$, it is called 2-dimensional (vertical). The case $n = 0$ and $m \neq 0$ is found not to occur. The imaginary part Ω_i determines the phase speed of the disturbance. The vertical phase speed u_z and the horizontal phase speed u_h are defined as

$$u_z = \frac{\Omega_i}{n}, \qquad u_h = \frac{\Omega_i}{m} = \frac{nu_z}{m}. \tag{39}$$

We note that u_h is defined only if $m \neq 0$.

We operate on Equation (33) with Δ^2, and use Equations (34)-(36) to get

$$\Delta^3 w_0 + \Delta w_0 + n^2 \left(1 - \tau^2\right) w_0 = 0. \tag{40}$$

The general solution of the differential Equation (40) can be written in the form

$$w_0^{(i)}(x) = \sum_{j=1}^{3} \mu_j^3 \left[A_j^{(i)} \cosh\left(\lambda_j x\right) + B_j^{(i)} \sinh\left(\lambda_j x\right) \right], \tag{41}$$

where the superscript (i) refers to the three regions of the system (see **Figure 1**), $A_j^{(i)}$, $B_j^{(i)}$ $(i, j = 1, 2, 3)$ are constants, and μ_j are the roots of the cubic equation

$$\mu_j^3 + \mu_j + n^2 \left(1 - \tau^2\right) = 0, \tag{42}$$

with λ_j is given by

$$\lambda_j = \left(\mu_j + b^2\right)^{1/2}. \tag{43}$$

We use the Equations (34)-(36) and the solution (41) to obtain

$$\left\{T_0^{(i)}, p_0^{(i)}, \varsigma_0^{(i)}\right\}(x) = \sum_{j=1}^{3} \left\{\mu_j^2, \mu_j \left(1 - \tau^2\right), \frac{i}{m} \tau \mu_j^2 \right\} \left[A_j^{(i)} \cosh\left(\lambda_j x\right) + B_j^{(i)} \sinh\left(\lambda_j x\right) \right]. \tag{44}$$

The Equation (37) can be solved in the form a complementary function and particular solution to find

$$v_0^{(i)}(x) = \frac{1}{m} \sum_{j=1}^{3} \frac{\mu_j^2}{\mu_j + n^2} \left\{-m\mu_j C_j^{(i)} + i\tau\lambda_j D_j^{(i)}\right\} + \frac{1}{m} \left\{C^{(i)} \cosh\left(mx\right) + D^{(i)} \sinh\left(mx\right)\right\}, \tag{45}$$

where $C^{(i)}$, $D^{(i)}$ are constants, and $C_j^{(i)}$, $D_j^{(i)}$ are given by

$$\left. \begin{aligned} C_j^{(i)} &= A_j^{(i)} \cosh\left(\lambda_j x\right) + B_j^{(i)} \sinh\left(\lambda_j x\right) \\ D_j^{(i)} &= B_j^{(i)} \cosh\left(\lambda_j x\right) + A_j^{(i)} \sinh\left(\lambda_j x\right) \end{aligned} \right\}. \tag{46}$$

Now we apply the boundary conditions at $x = \pm x_0$, $x = a_1$, $x = -a_2$ and solve the resulting algebraic equations for the constants $A_j^{(i)}$, $B_j^{(i)}$, $C^{(i)}$, $D^{(i)}$ for $i, j = 1, 2, 3$. The details of the solution are given in the **Appendix A**.

The growth rate is given by the quadratic equation

$$\left(\Omega_1/in\right)^2 + S_1 \left(\Omega_1/in\right) + S_2 = 0, \tag{47}$$

where S_1 and S_2 are given in the **Appendix A**.

Solving Equation (47) yields

$$\Omega_1 = \frac{in}{2} \left(-S_1 \pm \sqrt{S_1^2 - 4S_2}\right), \tag{48}$$

and the displacement η_1 is given by

$$\eta_1 = \frac{F_{1-}}{\Omega_1 - in\overline{w}\left(-x_0\right) - F_{e-}}, \tag{49}$$

in which F_{1-} and F_{e-} are defined in the **Appendix A**.

We note that, as in the absence of rotation, the system has two modes. The upper sign in the expression (48) corresponds to the modified varicose mode (MV) and the lower sign refers to the modified sinuous mode (MS).

4. Discussions of the Results

The growth rate (48) is evaluated numerically in the (m, n) plane as a function of the parameters x_0, a_2, d and Ta for both modes. The contours of Ω_1 in the (m, n) plane are plotted for sample values of the parameters x_0, a_2, Ta and $d = 10$ in the **Figures 2-4**.

In **Figure 2**, we present a comparison between the contours of the two modes MV and MS for different values of the Taylor number, Ta, when $x_0 = 0.5$, $a_2 = 1$ and $d = 10$. We note that the MV mode always possesses a minimum with negative growth rate and one or two maxima with positive growth rates while the MS mode has

Figure 2. Contours of the growth rate , Ω_1, of the modes MV, as in (a), (c), (e) and MS, as in (b), (d), (f) for the rotating bounded Cartesian plume where $x_0 = 0.5$, $a_2 = 1$, $d = 10$. Here $Ta = 0.5$ for (a), (b); $Ta = 2$ for (c), (d); and $Ta = 5$ for (e), (f). Note that the preferred mode is MS.

Figure 3. Contours of the growth rate, Ω_1, of the modes MV, as in (a), (c), (e) and MS, as in (b), (d), (f) for $x_0 = 0.5$, $Ta = 0.8$, $d = 10$. Here $a_2 = 1$ for (a), (b); $a_2 = 3$ for (c), (d); and $a_2 = 5$ for (e), (f). Note that the preferred mode of the instability is 2-dimensional when $a_2 = 5$.

a minimum with negative growth rate only when Ta is small and one maximum with positive growth rate which is larger than that of the corresponding MV mode. This indicates that the MS mode is preferred for this set of parameters. The local maxima correspond to 3-dimensional modes and the largest maximum always increases with the increase in the rotation parameter Ta.

Figure 4. Isolines of the growth rate , Ω_1^{MV} , as in (a), (c), (e) and Ω_1^{MS} as in (b), (d), (f) for $a_2 = 5$, $Ta = 7$, $d = 10$. Here $x_0 = 1$ for (a), (b); $x_0 = 2$ for (c), (d); and $x_0 = 3$ for (e), (f). Note that at fixed point (m,n), the values of Ω_1^{MV} and Ω_1^{MS} are nearly same but there different in the sign.

Figure 3 presents the influence of the distance a_2 between the plume and the nearest sidewall on the stability of the plume when the thickness of the plume and the rotation parameter are held fixed at $x_0 = 0.5$ and $Ta = 0.8$. The figure is plotted for three different values of a_2: $a_2 = 1$ for subfigures (a), (b); $a_2 = 3$ for subfigures (c), (d); and $a_2 = 5$ for subfigures (e), (f). When the plume is situated half-way between the sidewalls $(a_2 = 5)$, the MV mode has a negative growth rate everywhere and hence stable while the MS mode has a growth rate that is positive everywhere and hence is preferred. The local maximum possesses a vanishing horizontal wave number and hence propagates vertically. As the plume moves towards a wall, the growth rates of both modes increase and the MV mode develops local maxima with positive growth rates but they are not preferred because the MS mode local maximum increases as well and becomes 3-dimenaional.

Figure 4 illustrates the influence of the plume thickness on the contours of the growth rate for fixed values of the rotation parameter and the distance between the two sidewalls when the plume is situated half-way between the two sidewalls. As x_0 increases from $x_0 = 1$, the growth rate for MS mode, which is positive everywhere, increases and that of the MV mode, which is negative everywhere, decreases until x_0 reaches a critical value, x_{0c} , when the growth rate of the MS mode decreases and that of the MV mode increases.

The contours of **Figures 2-4** indicate that the preferred mode of instability is the modified sinuous (MS) mode. This indication is quantified by calculating the maximum growth rate and the associated wave numbers and phase speeds for different values of the parameters. For fixed values of x_0, a_2, d and Ta, we maximize over the wave numbers m and n by demanding that

$$\frac{\partial \Omega_1^{(k)}}{\partial n} = \frac{\partial \Omega_1^{(k)}}{\partial m} = 0, \quad k = \{MV, MS\}. \tag{50}$$

The solution of (50) gives the values m_c and n_c at maximum growth rate, Ω_{1c} , calculated from (48) for these values of m and n to give, together with the corresponding value U_c of the phase speed, the parameters $\left(\Omega_{1c}, m_c, n_c, U_c\right)$ of the preferred mode. For fixed parameters x_0, a_2, d and Ta, and particular mode, all possible local maxima of $\Omega_1^{(k)}$ are identified and the largest value taken together with the corresponding wavenumbers and phase speeds as defining the preferred mode for that set of parameters for that mode. This is carried out for both modified varicose (MV) and modified sinuous (MS) modes, and the largest is chosen as the preferred mode. A sample of the results is given in **Figures 5-9**.
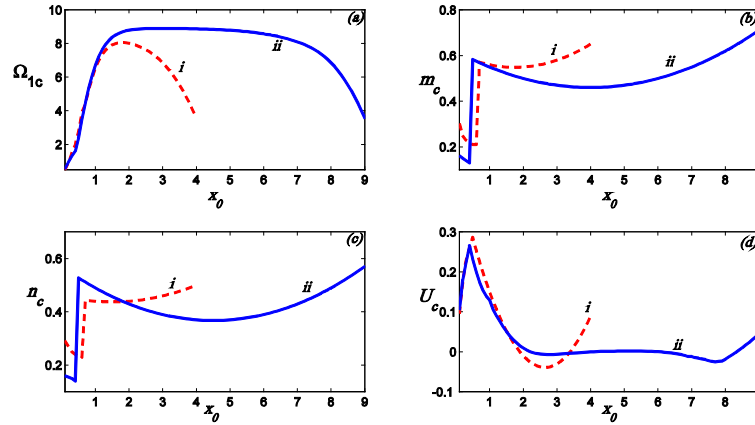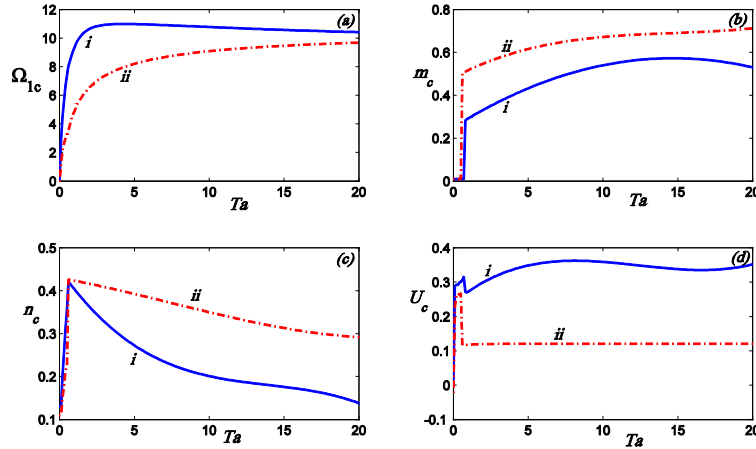
Figure 5. The preferred mode parameters $\Omega_{1c}, m_c, n_c, U_c$ are plotted in (a), (b), (c), (d) as a function of Ta, for $x_0 = 2$, and two different values of d and the plume half-way between the sidewalls. The roman numbers i and ii refer to $d = 10, a_2 = 5$ and $d = 20, a_2 = 10$, respectively.
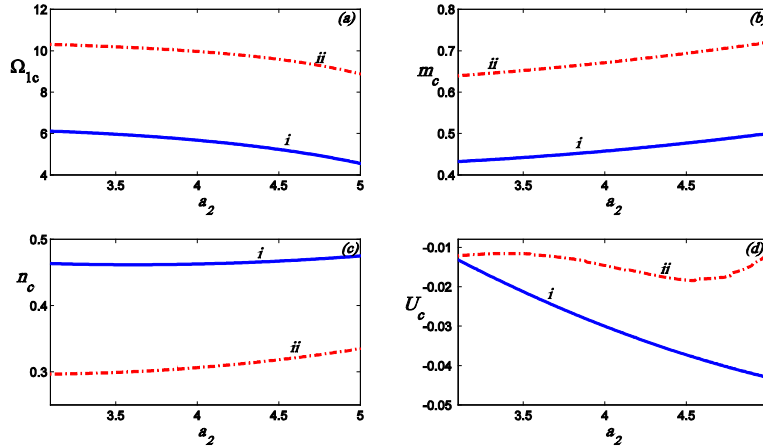

Figure 6. The preferred mode parameters $\Omega_{1c}, m_c, n_c, U_c$ are plotted in (a), (b), (c), (d) as a function of x_0, for $T_a = 2$, and two different values of d and the plume half-way between the sidewalls. The roman numbers i and ii refer to $d = 10, a_2 = 5$ and $d = 20, a_2 = 10$, respectively.

Figure 5 and **Figure 6** illustrate the influence of the distance between the boundaries on the preferred mode of instability of the rotating plume. In **Figure 5**, the preferred mode parameters are plotted as a function of Ta for fixed x_0 and two values of the distance between the sidewalls, d, for a plume situated half-way between the sidewalls. It is found that the preferred mode is the MS mode and the influence of the boundaries tends to stabilise the plume but only reduces the growth rate slightly. The horizontal and vertical wavenumbers are increased while the phase speeds are reduced as the distance, d, between the two walls is reduced.

Figure 6 presents the preferred mode of instability as a function of x_0 and fixed Ta where the plume is equidistant from the two walls and d takes the values 10 and 20. The growth rate increases rapidly as x_0 increases from zero until it reaches a maximum when the plume occupies the middle half of the region between the sidewalls. As the thickness of the plume increases further, the growth rate decreases if d is small but stays at the maximum value for increasing x_0 if the distance d is large. In both case, the growth rate drops to zero as the wall is approached and the plume nearly fill the whole region between the sidewalls. The wavenumbers decrease as x_0 increases from zero but they soon jump to larger values and increase with x_0. In both cases the MS mode is preferred.
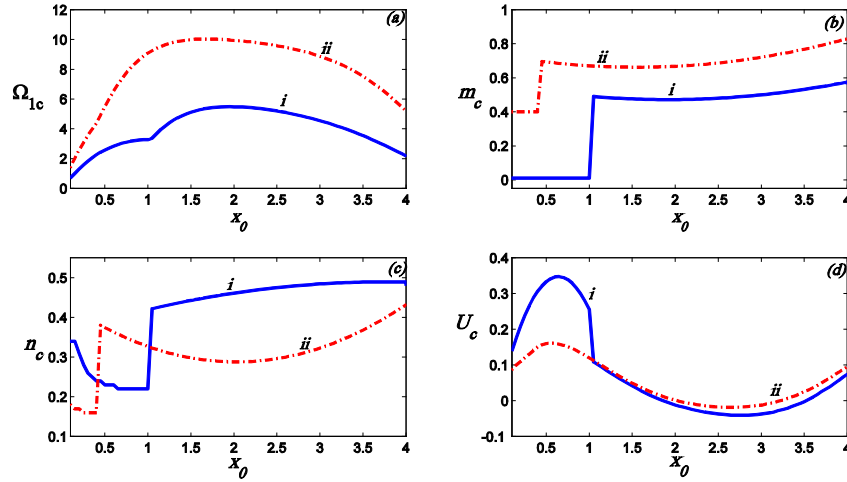
Figure 7. The preferred mode parameters $\Omega_{1c}, m_c, n_c, U_c$ are plotted in (a), (b), (c), (d), respectively, as a function of Ta, for $x_0 = 1$, $d = 10$ and two different values of a_2. The solid curves refer to $a_2 = 2$ and the broken curves refer to $a_2 = 5$. Note that preferred mode is MS in all cases.

Figure 8. The preferred mode parameters $\Omega_{1c}, m_c, n_c, U_c$ are plotted in (a), (b), (c), (d), respectively, as a function of a_2, for $x_0 = 3$, $d = 10$ and for two different values of Ta. The roman numbers i and ii refer to $Ta = 0.5$ and $Ta = 10$, respectively. Note that the maximum growth decreases as the plume moves away from the wall to the centre.

Figure 7 illustrates the dependence of the preferred mode on Ta for $x_0 = 1$, $d = 10$ and a_2 takes two values: $a_2 = 2$ and $a_2 = 5$. As the Taylor number increases from $Ta = 0$, the maximum growth rate Ω_{1c} increases rapidly until Ta reaches a certain value that increases as the distance between plume and the nearest sidewall increases, after which the growth rate varies much more slowly. The wavenumbers of the preferred mode show sudden changes at a small value of Ta indicating a change of local maximum as Ta, increases through a value, Ta_e. The horizontal wavenumber, m_c, is zero for small Ta corresponding to 2-dimensional motions, but as Ta reaches Ta_e, m_c jumps to a nonzero value and increases thereafter. The vertical wavenumber increases rapidly as Ta increase to Ta_e and then decreases as Ta increases further. The vertical phase speed of the preferred mode also suffers a change at $Ta = Ta_e$ and the jump depends on how far the plume is from the nearest sidewall.

In **Figure 8**, we present the dependence of the preferred mode on the distance a_2 between the plume and the nearest sidewall. We note that the growth rate decreases when the plume moves towards the location half-way

Figure 9. The preferred mode parameters $\Omega_{1c}, m_c, n_c, U_c$ are depicted in (a), (b), (c), (d) as a function of x_0, for $a_2 = 5$ and $d = 10$. The solid (broken) curves refer to $Ta = 0.5\,(10)$. Note that the preferred mode of instability is modified sinuous mode in all cases. When the plume is wide, the growth rate decreases whenever the plume moves to the wall.

between the two sidewalls. Both wavenumber components increase steadily as the distance between plume and the nearest sidewall increases. The vertical phase speed however behaves differently as a_2 increases; it decreases steadily with increasing a_2 if Ta is small but when Ta is large, it decreases slowly reaching a minimum before it increases at a moderate rate. In all cases, it is the MS mode that provides the preferred mode.

Figure 9 shows the dependence of the critical modes $(\Omega_{1c}, m_c, n_c, U_c)$ on x_0. The growth rate increases gradually until x_0 reaches a critical thickness, x_{0c}, and then it decreases. This indicates that there is a critical thickness representing the most unstable plume. The behaviour of the wavenumber components and phase speed with x_0 is quite complicated. The horizontal wavenumber vanishes for small x_0 when Ta is small. For large, m_c, behaves in a similar way when x_0 is close to x_{0c}. The vertical wavenumber, on the other hand decreases as x_0 increases from zero and jumps to a larger value when the local maximum changes, and then varies very slowly until x_0 is almost 4, when $d = 10$.

5. Conclusions

The dynamics of a fully developed plume of buoyant fluid, in the form of a channel of finite width, $2x_0$, rising in a less buoyant fluid contained between two parallel vertical walls, a distance d apart, and two fluids rotate uniformly about the vertical have been investigated.

In the absence of boundaries ([18]), it was found that the stability problem depended on the parameters: the Grashoff number, R, the Taylor number, Ta, and the thickness of the plume, $2x_0$. The magnitude of the growth rate was of the order $O(1)$ for $R \ll 1$ and the instability took one of the two types: sinuous mode or varicose mode.

The presence of the vertical boundaries here introduces two dimensionless parameters: the distance between the plume and the nearest wall, a_2, and the distance between the two vertical sidewalls, d. The introduction of the boundaries modifies the two modes to be the modified sinuous mode (MS) and the modified varicose mode (MV). It is shown here that the boundaries introduce two main effects to the rotating Cartesian plume studied by Eltayeb and Hamza [18]. First, the sidewalls tends to stabilise the plume but succeed only reducing the growth rate and the plume remains unstable for all values of the Taylor number and the distance from the nearest sidewall. Second, the presence of the sidewalls suppresses the modified varicose mode and allows the modified sinuous only to be unstable. The preferred mode can be 3-dimensional or 2-dimensional depending on the values of the parameters of the system. When the preferred mode is 2-dimensional, propagation can be upwards or downwards.

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Appendix A: The Derivation of the Dispersion Relation

In this appendix, we apply the boundary conditions (26)-(28) when the subscript is 0 for the variables (41), (44) and (45) to evaluate the constants of the solution. Application of the boundary conditions at the interface $x = x_0$ gives

$$\sum_{j=1}^{3} \mu_j^3 E_j = 0, \qquad \sum_{j=1}^{3} \mu_j^2 E_j = 0, \qquad \sum_{j=1}^{3} \left(1 - \tau^2\right) \mu_j E_j = 0, \tag{A:1}$$

$$\sum_{j=1}^{3} \lambda_j \mu_j^3 F_j = 1, \qquad \sum_{j=1}^{3} \lambda_j \mu_j^2 F_j = 0, \qquad \sum_{j=1}^{3} \lambda_j \mu_j \left\{ \mu_j^2 + \left(1 - \tau^2\right) \right\} F_j = 0, \tag{A:2}$$

$$\sum_{j=1}^{3} \frac{\mu_j^2}{\mu_j + n^2} \left\{ -m \mu_j E_j + i \tau \lambda_j F_j \right\} + E = 0, \tag{A:3}$$

$$\sum_{j=1}^{3} \frac{\lambda_j \mu_j^2}{\mu_j + n^2} \left\{ -m \mu_j F_j + i \tau \lambda_j E_j \right\} + m F = 0, \tag{A:4}$$

where E_j, F_j, E and F are defined by

$$E_j = \left(A_j^{(2)} - A_j^{(3)} \right) \cosh\left(\lambda_j x_0 \right) + \left(B_j^{(2)} - B_j^{(3)} \right) \sinh\left(\lambda_j x_0 \right), \tag{A:5}$$

$$F_j = \left(A_j^{(2)} - A_j^{(3)} \right) \sinh\left(\lambda_j x_0 \right) + \left(B_j^{(2)} - B_j^{(3)} \right) \cosh\left(\lambda_j x_0 \right), \tag{A:6}$$

$$E = \left(C^{(2)} - C^{(3)} \right) \cosh\left(m x_0 \right) + \left(D^{(2)} - D^{(3)} \right) \sinh\left(m x_0 \right), \tag{A:7}$$

$$F = \left(C^{(2)} - C^{(3)} \right) \sinh\left(m x_0 \right) + \left(D^{(2)} - D^{(3)} \right) \cosh\left(m x_0 \right). \tag{A:8}$$

We use the Equations (A:5)-(A:8) to obtain

$$E_j \cosh\left(\lambda_j x_0 \right) - F_j \sinh\left(\lambda_j x_0 \right) = A_j^{(2)} - A_j^{(3)}, \tag{A:9}$$

$$E_j \sinh\left(\lambda_j x_0 \right) - F_j \cosh\left(\lambda_j x_0 \right) = B_j^{(3)} - B_j^{(2)}, \tag{A:10}$$

$$E \cosh\left(m x_0 \right) - F \sinh\left(m x_0 \right) = C^{(2)} - C^{(3)}, \tag{A:11}$$

$$E \sinh\left(m x_0 \right) - F \cosh\left(m x_0 \right) = D^{(3)} - D^{(2)}. \tag{A:12}$$

Similarly, at the interface $x = -x_0$, we find

$$\sum_{j=1}^{3} \mu_j^3 K_j = 0, \qquad \sum_{j=1}^{3} \mu_j^2 K_j = 0, \qquad \sum_{j=1}^{3} \left(1 - \tau^2\right) \mu_j K_j = 0, \tag{A:13}$$

$$\sum_{j=1}^{3} \lambda_j \mu_j^3 L_j = -\eta_1, \qquad \sum_{j=1}^{3} \lambda_j \mu_j^2 L_j = 0, \qquad \sum_{j=1}^{3} \lambda_j \mu_j \left\{ \mu_j^2 + \left(1 - \tau^2\right) \right\} L_j = 0, \tag{A:14}$$

$$\sum_{j=1}^{3} \frac{\mu_j^2}{\mu_j + n^2} \left\{ -m \mu_j K_j + i \tau \lambda_j L_j \right\} + K = 0, \tag{A:15}$$

$$\sum_{j=1}^{3} \frac{\lambda_j \mu_j^2}{\mu_j + n^2} \left\{ -m \mu_j L_j + i \tau \lambda_j K_j \right\} + m L = 0, \tag{A:16}$$

in which K_j, L_j, K and L are given by

$$K_j = \left(A_j^{(1)} - A_j^{(2)} \right) \cosh\left(\lambda_j x_0 \right) - \left(B_j^{(1)} - B_j^{(2)} \right) \sinh\left(\lambda_j x_0 \right), \tag{A:17}$$

$$L_j = -\left(A_j^{(1)} - A_j^{(2)} \right) \sinh\left(\lambda_j x_0 \right) + \left(B_j^{(1)} - B_j^{(2)} \right) \cosh\left(\lambda_j x_0 \right), \tag{A:18}$$

$$K = \left(C^{(1)} - C^{(2)}\right)\cosh\left(mx_0\right) - \left(D^{(1)} - D^{(2)}\right)\sinh\left(mx_0\right), \tag{A:19}$$

$$L = -\left(C^{(1)} - C^{(2)}\right)\sinh\left(mx_0\right) + \left(D^{(1)} - D^{(2)}\right)\cosh\left(mx_0\right). \tag{A:20}$$

Equations (A:17)-(A:20) give

$$K_j \cosh\left(\lambda_j x_0\right) + L_j \sinh\left(\lambda_j x_0\right) = A_j^{(1)} - A_j^{(2)}, \tag{A:21}$$

$$K_j \sinh\left(\lambda_j x_0\right) + L_j \cosh\left(\lambda_j x_0\right) = B_j^{(1)} - B_j^{(2)}, \tag{A:22}$$

$$K \cosh\left(mx_0\right) + L\sinh\left(mx_0\right) = C^{(1)} - C^{(2)}, \tag{A:23}$$

$$K \sinh\left(mx_0\right) + L\cosh\left(mx_0\right) = D^{(1)} - D^{(2)}. \tag{A:24}$$

The boundary conditions at the boundaries $x = a_1$ and $x = -a_2$ give

$$\sum_{j=1}^{3}\mu_j^3 P_j^{(3)} = 0, \quad \sum_{j=1}^{3}\mu_j^2 P_j^{(3)} = 0, \quad \sum_{j=1}^{3}\mu_j^3 P_j^{(1)} = 0, \quad \sum_{j=1}^{3}\mu_j^2 P_j^{(1)} = 0, \tag{A:25}$$

$$\sum_{j=1}^{3}\frac{\mu_j^2}{\mu_j + n^2}\left[-m\mu_j P_j^{(3)} + i\tau\lambda_j Q_j^{(3)}\right] + C^{(3)}\cosh\left(ma_1\right) + D^{(3)}\sinh\left(ma_1\right) = 0, \tag{A:26}$$

$$\sum_{j=1}^{3}\frac{\mu_j^2}{\mu_j + n^2}\left[-m\mu_j P_j^{(1)} + i\tau\lambda_j Q_j^{(1)}\right] + C^{(1)}\cosh\left(ma_2\right) - D^{(1)}\sinh\left(ma_2\right) = 0, \tag{A:27}$$

$$\sum_{j=1}^{3}\lambda_j\mu_j\left\{\mu_j^2 + 1 - \tau^2\right\}Q_j^{(3)} = 0, \quad \sum_{j=1}^{3}\lambda_j\mu_j\left\{\mu_j^2 + 1 - \tau^2\right\}Q_j^{(1)} = 0, \tag{A:28}$$

such that $P_j^{(3)}$, $P_j^{(1)}$, $Q_j^{(3)}$ and $Q_j^{(1)}$ are defined by

$$P_j^{(3)} = A_j^{(3)}\cosh\left(\lambda_j a_1\right) + B_j^{(3)}\sinh\left(\lambda_j a_1\right), \tag{A:29}$$

$$P_j^{(1)} = A_j^{(1)}\cosh\left(\lambda_j a_2\right) - B_j^{(1)}\sinh\left(\lambda_j a_2\right), \tag{A:30}$$

$$Q_j^{(3)} = A_j^{(3)}\sinh\left(\lambda_j a_1\right) + B_j^{(3)}\cosh\left(\lambda_j a_1\right), \tag{A:31}$$

$$Q_j^{(1)} = -A_j^{(1)}\sinh\left(\lambda_j a_2\right) + B_j^{(1)}\cosh\left(\lambda_j a_2\right). \tag{A:32}$$

The equations (A:1) and (A:2) can be solved to find

$$E_j = 0, \quad F_j = \frac{-\left(\mu_j + n^2\right)}{\lambda_j\mu_j\left\{2\mu_j + 3n^2\left(1 - \tau^2\right)\right\}}, \quad j = 1, 2, 3, \tag{A:33}$$

since the roots μ_j are distinct. Hence Equations (A:3)-(A:5) give

$$\left(A_j^{(2)} - A_j^{(3)}\right)\cosh\left(\lambda_j x_0\right) + \left(B_j^{(2)} - B_j^{(3)}\right)\sinh\left(\lambda_j x_0\right) = 0, \tag{A:34}$$

$$i\tau\sum_{j=1}^{3}\frac{\lambda_j\mu_j^2}{\mu_j + n^2}F_j + E = 0, \tag{A:35}$$

$$-\sum_{j=1}^{3}\frac{\lambda_j\mu_j^3}{\mu_j + n^2}F_j + F = 0. \tag{A:36}$$

Using the expression for F_j in (A:33), and the properties of the 3 roots of the cubic Equation (42) to simplify the expressions (A:35), (A:36). The simplifications lead to

$$E = F = 0. \tag{A:37}$$

Hence the expressions (A:9)-(A:12) reduce to

$$A_j^{(2)} - A_j^{(3)} = -F_j \sinh\left(\lambda_j x_0\right), \tag{A:38}$$

$$B_j^{(2)} - B_j^{(3)} = F_j \cosh\left(\lambda_j x_0\right), \tag{A:39}$$

$$C^{(2)} = C^{(3)}, \quad D^{(2)} = D^{(3)}. \tag{A:40}$$

The same method can be applied to the conditions at $x = -x_0$. The Equations (A:13)-(A:16) leads to

$$K_j = 0, \quad L_j = -\eta_1 F_j, \quad K = L = 0, \tag{A:41}$$

thus the expressions (A:21)-(A:24) give

$$A_j^{(1)} - A_j^{(2)} = L_j \sinh\left(\lambda_j x_0\right), \tag{A:42}$$

$$B_j^{(1)} - B_j^{(2)} = L_j \cosh\left(\lambda_j x_0\right), \tag{A:43}$$

$$C^{(1)} = C^{(2)}, \quad D^{(1)} = D^{(2)}. \tag{A:44}$$

The expressions $P_j^{(3)}$, $P_j^{(1)}$, $Q_j^{(3)}$ and $Q_j^{(1)}$, given in (A:29)-(A:32), can be simplified by using the relations (A:38), (A:39) and (A:42), (A:43) to find

$$P_j^{(3)} = V_j + F_j \sinh\left\{\lambda_j\left(x_0 - a_1\right)\right\}, \tag{A:45}$$

$$Q_j^{(3)} = W_j - F_j \cosh\left\{\lambda_j\left(x_0 - a_1\right)\right\}, \tag{A:46}$$

$$P_j^{(1)} = R_j + L_j \sinh\left\{\lambda_j\left(x_0 - a_2\right)\right\}, \tag{A:47}$$

$$Q_j^{(1)} = S_j + L_j \cosh\left\{\lambda_j\left(x_0 - a_2\right)\right\}, \tag{A:48}$$

where we have defined the notation

$$V_j = A_j^{(2)} \cosh\left(\lambda_j a_1\right) + B_j^{(2)} \sinh\left(\lambda_j a_1\right), \tag{A:49}$$

$$W_j = A_j^{(2)} \sinh\left(\lambda_j a_1\right) + B_j^{(2)} \cosh\left(\lambda_j a_1\right), \tag{A:50}$$

$$R_j = A_j^{(2)} \cosh\left(\lambda_j a_2\right) - B_j^{(2)} \sinh\left(\lambda_j a_2\right), \tag{A:51}$$

$$S_j = -A_j^{(2)} \sinh\left(\lambda_j a_2\right) + B_j^{(2)} \cosh\left(\lambda_j a_2\right). \tag{A:52}$$

The Equations (A:49)-(A:52) give

$$A_j^{(2)} = \frac{1}{\sinh\left(\lambda_j d\right)}\left\{R_j \sinh\left(\lambda_j a_1\right) + V_j \sinh\left(\lambda_j a_2\right)\right\}, \tag{A:53}$$

$$B_j^{(2)} = \frac{-1}{\sinh\left(\lambda_j d\right)}\left\{R_j \cosh\left(\lambda_j a_1\right) - V_j \cosh\left(\lambda_j a_2\right)\right\}, \tag{A:54}$$

$$W_j = \frac{1}{\sinh\left(\lambda_j d\right)}\left\{-R_j + V_j \cosh\left(\lambda_j d\right)\right\}, \tag{A:55}$$

$$S_j = \frac{-1}{\sinh\left(\lambda_j d\right)}\left\{R_j \cosh\left(\lambda_j d\right) - V_j\right\}. \tag{A:56}$$

The boundary conditions at $x = a_1$ and $x = -a_2$, given in (A:25) and (A:28), can be written by using

(A:45)-(A:48) and (A:55) and (A:56) to obtain

$$\sum_{j=1}^{3}\left\{\mu_j^3;\mu_j^2;\bar{d}_j\left(\cosh\left(\lambda_j d\right)-1\right)\right\}\left(R_j+V_j\right)=\left\{x_1;x_2;x_3\right\},\tag{A:57}$$

$$\sum_{j=1}^{3}\left\{\mu_j^3;\mu_j^2;\bar{d}_j\left(\cosh\left(\lambda_j d\right)+1\right)\right\}\left(R_j-V_j\right)=\left\{x_4;x_5;x_6\right\},\tag{A:58}$$

in which x_1, x_2, x_3, x_4, x_5 and x_6 are defined by

$$x_1=Y_3-\eta_1 Y_1,\qquad x_2=Y_4-\eta_1 Y_2,\qquad x_3=Y_6-\eta_1 Y_5,\tag{A:59}$$

$$x_4=-Y_3-\eta_1 Y_1,\qquad x_5=-Y_4-\eta_1 Y_2,\qquad x_6=-Y_6-\eta_1 Y_5,\tag{A:60}$$

and we have defined $\bar{d}_j$ and Y_i, $i=1$ - 6 as

$$\bar{d}_j=\frac{\lambda_j\mu_j}{\sinh\left(\lambda_j d\right)}\left\{\mu_j^2+1-\tau^2\right\},\qquad j=1,2,3,\tag{A:61}$$

$$Y_1=\sum_{j=1}^{3}\mu_j^3 F_{2j}\sinh\left[\lambda_j\left(a_2-x_0\right)\right],\qquad Y_2=\sum_{j=1}^{3}\mu_j^2 F_{2j}\sinh\left[\lambda_j\left(a_2-x_0\right)\right],\tag{A:62}$$

$$Y_3=\sum_{j=1}^{3}\mu_j^3 F_{2j}\sinh\left[\lambda_j\left(a_1-x_0\right)\right],\qquad Y_4=\sum_{j=1}^{3}\mu_j^2 F_{2j}\sinh\left[\lambda_j\left(a_1-x_0\right)\right],\tag{A:63}$$

$$Y_5=\sum_{j=1}^{3}\bar{d}_j F_{2j}\sinh\left(\lambda_j d\right)\cosh\left[\lambda_j\left(a_2-x_0\right)\right],\quad Y_6=\sum_{j=1}^{3}\bar{d}_j F_{2j}\sinh\left(\lambda_j d\right)\cosh\left[\lambda_j\left(a_1-x_0\right)\right].\tag{A:64}$$

The algebraic system (A:57) and (A:58) can be solved by elimination to find

$$R_j=R_{j0}+\eta_1 R_{je},\qquad V_j=V_{j0}+\eta_1 V_{je},\quad\text{for }j=1,2,3,\tag{A:65}$$

in which R_{j0}, R_{je}, V_{j0} and V_{je} are given by

$$R_{10}=\frac{-\left(f_{2+}g_{10-}+f_{2-}g_{10+}\right)}{2f_{2-}f_{2+}},\qquad R_{1e}=\frac{-\left(f_{2+}g_{1e-}+f_{2-}g_{1e+}\right)}{2f_{2-}f_{2+}},\tag{A:66}$$

$$R_{20}=\frac{\left(f_{2+}f_{10-}+f_{2-}f_{10+}\right)}{2f_{2-}f_{2+}},\qquad R_{2e}=\frac{\left(f_{2+}f_{1e-}+f_{2-}f_{1e+}\right)}{2f_{2-}f_{2+}},\tag{A:67}$$

$$R_{30}=\frac{-\left(f_{2+}A_{10-}+f_{2-}A_{10+}\right)}{2f_{2-}f_{2+}},\qquad R_{3e}=\frac{-\left(f_{2+}A_{1e-}+f_{2-}A_{1e+}\right)}{2f_{2-}f_{2+}},\tag{A:68}$$

$$V_{10}=\frac{-\left(f_{2+}g_{10-}-f_{2-}g_{10+}\right)}{2f_{2-}f_{2+}},\qquad V_{1e}=\frac{-\left(f_{2+}g_{1e-}-f_{2-}g_{1e+}\right)}{2f_{2-}f_{2+}},\tag{A:69}$$

$$V_{20}=\frac{\left(f_{2+}f_{10-}-f_{2-}f_{10+}\right)}{2f_{2-}f_{2+}},\qquad V_{2e}=\frac{\left(f_{2+}f_{1e-}-f_{2-}f_{1e+}\right)}{2f_{2-}f_{2+}},\tag{A:70}$$

$$V_{30}=\frac{-\left(f_{2+}A_{10-}-f_{2-}A_{10+}\right)}{2f_{2-}f_{2+}},\qquad V_{3e}=\frac{-\left(f_{2+}A_{1e-}-f_{2-}A_{1e+}\right)}{2f_{2-}f_{2+}},\tag{A:71}$$

and we have defined the following notation

$$g_{10\pm}=\mp\mu_2^2 T_{3\pm}\left(\mu_2 Y_4-Y_3\right)\mp\mu_3^2 T_{2\pm}\left(Y_3-\mu_3 Y_4\right)\pm\mu_2^2\mu_3^2\left(\mu_2-\mu_3\right)Y_6,\tag{A:72}$$

$$g_{1e\pm}=\mu_2^2 T_{3\pm}\left(-\mu_2 Y_2+Y_1\right)+\mu_3^2 T_{2\pm}\left(-Y_1+\mu_3 Y_2\right)+\mu_2^2\mu_3^2\left(\mu_2-\mu_3\right)Y_5,\tag{A:73}$$

$$f_{10\pm}=\mp\mu_1^2 T_{3\pm}\left(\mu_1 Y_4-Y_3\right)\mp\mu_3^2 T_{1\pm}\left(\mp Y_3-\mu_3 Y_4\right)\pm\mu_1^2\mu_3^2\left(\mu_1-\mu_3\right)Y_6,\tag{A:74}$$

$$f_{1e\pm} = \mu_1^2 T_{3\pm}\left(-\mu_1 Y_2 + Y_1\right) + \mu_3^2 T_{1\pm}\left(-Y_1 + \mu_3 Y_2\right) + \mu_1^2\mu_3^2\left(\mu_1 - \mu_3\right)Y_5, \tag{A:75}$$

$$A_{10\pm} = \mp\mu_1^2 T_{2\pm}\left(\mu_1 Y_4 - Y_3\right) \mp \mu_2^2 T_{1\pm}\left(Y_3 - \mu_2 Y_4\right) \pm \mu_1^2\mu_2^2\left(\mu_1 - \mu_2\right)Y_6, \tag{A:76}$$

$$A_{1e\pm} = \mu_1^2 T_{2\pm}\left(-\mu_1 Y_2 + Y_1\right) + \mu_1^2\mu_2^2\left(\mu_1 - \mu_2\right)Y_5 + \mu_2^2 T_{1\pm}\left(-Y_1 + \mu_2 Y_2\right), \tag{A:77}$$

$$f_{2\pm} = \mu_2^2\mu_3^2\left(\mu_2 - \mu_3\right)T_{1\pm} - \mu_1^2\mu_3^2\left(\mu_1 - \mu_3\right)T_{2\pm} + \mu_1^2\mu_2^2\left(\mu_1 - \mu_3\right)T_{3\pm}, \tag{A:78}$$

$$T_{j\pm} = \overline{d}_j\left\{\cosh\left(\lambda_j d\right)\pm 1\right\}. \tag{A:79}$$

The constants $A_j^{(i)}$ and $B_j^{(i)}$ for $i,j = 1,2,3$ can easily evaluated using (A:38), (A:39), (A:42), (A:43), (A:53), (A:54). The next step is to find the expressions of the constants $C^{(i)}$ and $D^{(i)}$. The Equations (A:26), (A:27), (A:40) and (A:44) give

$$C^{(1)} = C^{(2)} = C^{(3)} = \frac{-1}{\sinh\left(md\right)}\left\{\sinh\left(ma_2\right)\sum_{j=1}^{3}H_j^{(3)} + \sinh\left(ma_1\right)\sum_{j=1}^{3}H_j^{(1)}\right\}, \tag{A:80}$$

$$D^{(1)} = D^{(2)} = D^{(3)} = \frac{-1}{\sinh\left(md\right)}\left\{\cosh\left(ma_2\right)\sum_{j=1}^{3}H_j^{(3)} - \cosh\left(ma_1\right)\sum_{j=1}^{3}H_j^{(1)}\right\}, \tag{A:81}$$

where $H_j^{(1)}$ and $H_j^{(3)}$ are defined by

$$H_j^{(1)} = \frac{\mu_j^2}{\mu_j + n^2}\left\{-m\mu_j P_j^{(1)} + i\lambda_j\tau Q_j^{(1)}\right\}, \qquad H_j^{(3)} = \frac{\mu_j^2}{\mu_j + n^2}\left\{-m\mu_j P_j^{(3)} + i\lambda_j\tau Q_j^{(3)}\right\}. \tag{A:82}$$

The application of the boundary condition (29) gives

$$\Omega_1 - in\overline{w}\left(x_0\right) = F_{1+} + \eta_1 F_{e+}, \qquad \left\{\Omega_1 - in\overline{w}\left(-x_0\right)\right\}\eta_1 = F_{1-} + \eta_1 F_{e-}, \tag{A:83}$$

such that $F_{1\pm}$ and $F_{e\pm}$ are given by

$$F_{1\pm} = G_\pm\left(R_{j0}, V_{j0}\right) + R_{0\pm}, \qquad F_{e\pm} = G_\pm\left(R_{je}, V_{je}\right) + V_{0\pm}, \tag{A:84}$$

where $G_\pm\left(R_{j\alpha}, V_{j\alpha}\right)$ for $\alpha = \{0, e\}$, $R_{0\pm}$ and $V_{0\pm}$ are defined by

$$G_\pm\left(R_{j\alpha}, V_{j\alpha}\right) = \frac{i}{n}\sum_{j=1}^{3}\overline{d}_j\left[-R_{j\alpha}\cosh\left\{\lambda_j\left(a_1 \mp x_0\right)\right\} + V_{j\alpha}\cosh\left\{\lambda_j\left(a_2 \pm x_0\right)\right\}\right] + \frac{\tau}{n}Z_\pm, \tag{A:85}$$

$$R_{0\pm} = \frac{\tau\sinh\left[m\left(a_2 \pm x_0\right)\right]}{n\sinh\left(md\right)}\sum_{j=1}^{3}\frac{\mu_j^2 a_{1j}}{\mu_j + n^2}F_j, \qquad V_{0\pm} = \frac{\tau\sinh\left[m\left(a_1 \mp x_0\right)\right]}{n\sinh\left(md\right)}\sum_{j=1}^{3}\frac{\mu_j^2 a_{2j}}{\mu_j + n^2}F_j, \tag{A:86}$$

and we have used the notation $Z_\pm$, a_{1j}, a_{2j}, $a_{3j\pm}$, $a_{4j\pm}$, $E_{1s\pm}$ and $E_{2s\pm}$

$$Z_\pm = m\sum_{j=1}^{3}\frac{\mu_j^2}{\sinh\left(\lambda_j d\right)}\left\{R_j\sinh\left[\lambda_j\left(a_1 \mp x_0\right)\right] + V_j\sinh\left[\lambda_j\left(a_2 \pm x_0\right)\right]\right\}$$
$$+ \sum_{j=1}^{3}\frac{\mu_j^2\left(-a_{3j\mp}R_j + a_{4j\pm}V_j\right)}{\left(\mu_j + n^2\right)\sinh\left(\lambda_j d\right)} + \frac{1}{\sinh\left(md\right)}\sum_{j=1}^{3}\frac{\mu_j^2\left(E_{1s\mp}R_j + E_{2s\pm}V_j\right)}{\mu_j + n^2}, \tag{A:87}$$

$$a_{1j} = -m\mu_j\sinh\left\{\lambda_j\left(a_1 - x_0\right)\right\} + i\tau\lambda_j\cosh\left\{\lambda_j\left(a_1 - x_0\right)\right\}, \tag{A:88}$$

$$a_{2j} = m\mu_j\sinh\left\{\lambda_j\left(a_2 - x_0\right)\right\} + i\tau\lambda_j\cosh\left\{\lambda_j\left(a_2 - x_0\right)\right\}, \tag{A:89}$$

$$a_{3j\pm} = m\mu_j\sinh\left\{\lambda_j\left(a_1 \pm x_0\right)\right\} + i\tau\lambda_j\cosh\left\{\lambda_j\left(a_1 \pm x_0\right)\right\}, \tag{A:90}$$

$$a_{4j\pm} = -m\mu_j\sinh\left\{\lambda_j\left(a_2 \pm x_0\right)\right\} + i\tau\lambda_j\cosh\left\{\lambda_j\left(a_2 \pm x_0\right)\right\}, \tag{A:91}$$

$$E_{1s\pm} = E_{j+} \sinh\{m(a_1 \pm x_0)\} + i\tau\lambda_j \frac{\sinh\{m(a_2 \mp x_0)\}}{\sinh\{\lambda_j d\}}, \tag{A:92}$$

$$E_{2s\pm} = E_{j-} \sinh\{m(a_2 \pm x_0)\} - i\tau\lambda_j \frac{\sinh\{m(a_1 \mp x_0)\}}{\sinh\{\lambda_j d\}}, \tag{A:93}$$

and $E_{j\pm}$ has the form

$$E_{j\pm} = m\mu_j \pm i\tau\lambda_j \coth(\lambda_j d). \tag{A:94}$$

Now the growth rate Ω_1 and the displacement η_1 can be evaluated by solving the Equation (A:83) to find

$$\eta_1 = \frac{F_{1-}}{\Omega_1 - in\bar{w}(-x_0) - F_{e-}}, \tag{A:95}$$

$$(\Omega_1/in)^2 + S_1(\Omega_1/in) + S_2 = 0, \tag{A:96}$$

where S_1 and S_2 are defined by

$$S_1 = \frac{i}{n}(F_{1+} + F_{e-}) - \bar{w}(x_0) - \bar{w}(-x_0), \tag{A:97}$$

$$S_2 = \left(\frac{i}{n}F_{1+} - \bar{w}(x_0)\right)\left(\frac{i}{n}F_{e-} - \bar{w}(-x_0)\right) + \frac{1}{n^2}F_{e+}F_{1-}. \tag{A:98}$$

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