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On the Full Transitivity of a Cotorsion Hull of the Pierce Group

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Abstract

The paper considers the problem of full transitivity of a cotorsion hull $G^{\bullet}$ of a separable primary group G when a ring of endomorphisms $E(G)$ of the group G has the form $E_s(G) \oplus J_p$, where $E_s(G)$ is a subring of small endomorphisms of the ring $E(G)$, whereas J_p is a ring of integer p-adic numbers. Investigation of the issue of full transitivity of a group is essentially helpful in studying its fully invariant subgroups as well as the lattice formed by these subgroups. It is proved that in the considered case, the cotorsion hull is not fully transitive. A lemma is proposed, which can be used in the study of full transitivity of a group and in other cases.

Keywords

Full Transitivity of a Group; Cotorsion Hull; Fully Invariant Subgroup

1. Introduction

The groups discussed in the paper are abelian and the operation is written in additive terms. We use here the notation and terminology of the monographs [1] [2].

The symbol p denotes a fixed prime number. Z and Q are respectively the groups of integer and rational numbers. A subgroup B of the group A is called fully invariant if it is self-mapped for any endomorphism of the group A.

The knowledge of the construction of fully invariant subgroups of an abelian group and their lattice is essentially helpful in the study of the properties of the group itself and also in the investigation of the properties of its rings of endomorphisms and quasi-endomorphisms, the group of automorphisms and other algebraic systems connected with the initial group.

For a sufficiently wide class of p-groups these topics were studied by R. Baer, I. Kaplansky, P. Linton, R.

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Pierce, D. Moore, E. Hewett and others. The works of A. Mader, R. Göbel, P. A. Krylov, S. Ya. Grinshpon, A. I. Moskalenko and other authors are dedicated to the investigation of these topics in torsion-free and mixed groups.

However little is known about the results obtained in this area for the class of cotorsion groups. A group A is called a cotorsion group if its extension by means of any torsion-free group C splits as follows: $Ext(C,A)=0$. The importance of the class of cotorsion groups in the theory of abelian groups is due to two factors: for any groups A, B, the group $Ext(A,B)$ is a cotorsion one and any reduced group G is isomorphically embeddable in the group $G^{\bullet}=Ext(Q/Z,G)$ called the cotorsion hull of the group G. If the torsion part of the group G is denoted by tG, then

$$Ext(Q/Z,tG)\cong\prod_{p}Ext\left(Z\left(p^{\infty}\right),T_p\right)$$

where $tG=\oplus T_p$. Thus the study of cotorsion groups essentially reduces to the study of groups of the form $Ext\left(Z\left(p^{\infty}\right),T\right)=T^{\bullet}$, where T is a p-primary group.

It is noteworthy that endomorpohisms in cotorsion groups are completely defined by their action on the torsion part and, as shown by W. May and E. Toubassi [3], for a mixed group A the ring of endomorphisms $E(A)$ is isomorphic to $E(tA)$ if and only if A is a fully invariant subgroup of the cotorsion hull $(tA)^{\bullet}$.

The notion of full transitivity of a group plays an essential role in describing the lattice of fully invariant subgroups.

By the p-indicator of an element a of the group A we mean an increasing sequence of ordinal numbers

$$H(a)=\left(h(a),h(pa),\cdots,h\left(p^{n}a\right),\cdots\right)$$

where h is the generalized p-height of an element, *i.e.* for $h(a)=\sigma$ if $a\in p^{\sigma}A\setminus p^{\sigma+1}A$ and $h(0)=\infty$. Now for the set of indicators we can introduce the order

$$H(a)\le H(b)\Leftrightarrow h\left(p^{i}a\right)\le h\left(p^{i}b\right),\ i=0,1,\cdots$$

A reduced p-group is called fully transitive if for arbitrary elements a and b, when $H(a)\le H(b)$ there exists an endomorphism φ of the group such that $\varphi a=b$. The class of fully transitive groups includes such important groups as separable p-groups, algebraically compact groups and quasi-pure injective groups.

Using the indicators of fully transitive groups we can describe the lattice of fully invariant subgroups (see [4]-[11]).

For a module over a commutative ring, A. Mader formulated a general scheme that can be used to describe the lattice of fully invariant submodules of the module (see [10], Theorem 2.1 or [12], Theorem 1.1).

In the same way as we did for a p-group we define the notion of full transitivity for the group $T^{\bullet}=Ext\left(Z\left(p^{\infty}\right),T\right)$. According to A. Mader [10], an algebraically compact group is fully transitive and described with the aid of indicators the lattice of fully invariant subgroups of this group. This means to describe the lattice of fully invariant subgroups of the group $T^{\bullet}=Ext\left(Z\left(p^{\infty}\right),T\right)$ when T is a torsion-complete group. When T is the direct sum of cyclic p-groups, A. Moskalenko [11] proved that $T^{\bullet}$ is also fully transitive and described by means of indicators the lattice of fully invariant subgroups of the group $T^{\bullet}$. In general, for the separable primary group T, the cotorsion hull $T^{\bullet}$ is not fully transitive. In particular if T is an infinite direct sum of torsion-complete groups, then, as shown by the author [13], the group $T^{\bullet}$ is not fully transitive and in that case the lattice of fully invariant subgroups of the group $T^{\bullet}$ cannot be described by means of indicators (see [12]).

R. Pierce [14] considered the primary group G, a ring of whose endomorphisms has the form

$$E(G)=E_s(G)\oplus J_p \tag{1.1}$$

where $E_s(G)$ is the ring of small endomorphisms of the group G which is the ideal of the ring of endomorphisms $E(G)$ of the group G, whereas J_p is the ring of integer p-adic numbers. A small endomorphism of the group G is defined as follows (see [14]).

For all $k \geq 0$ there exists an integer n such that

$$e(x) \leq k \text{ and } h(x) \geq n \text{ imply } \varphi(x) = 0. \tag{1.2}$$

The Pierce group G is important when studying the ring of endomorphisms of abelian groups (see [15]). The aim of the present paper consists in elucidating the full transitivity of the cotorsion hull $G^\bullet$ and also in finding the conditions, under which the cotorsion hull is not fully transitive.

2. Full Transitivity of the Cotorsion Hull of the Pierce Group

As mentioned above, R. Pierce [14] considered the separable primary group G with a standard basic subgroup $B = \bigoplus_{n=1}^{\infty} B_n$, $B_n \cong Z(p^n)$, $n = 1, 2, \cdots$, $B \subset G \subset \bar{B}$, where $\bar{B}$ is a torsion-complete group, *i.e.* the torsion part of a p-adic completion of the group B. The cardinality is $|G| = 2^{\aleph_0}$ and the ring of endomorphisms of the group G has form (1.1).

To study the full transitivity of the group $G^\bullet$, we use the following representation of elements of the cotorsion hull of $T^\bullet$ given by A. Moskalenko [11] for the separable p-group T

$$T^\bullet = \left\{ (a_0, a_1 + T, \cdots, a_i + T, \cdots) \middle| a_i \in \hat{T},\ pa_{i+1} - a_i \in T, i = 0, 1, \cdots \right\}. \tag{2.1}$$

Representation of elements in this form makes it easy to calculate the height and the indicator. In particular, if $a = (a_0, a_1 + T, \cdots)$, then

$$H_{T^\bullet}(a) = \begin{cases} H_{\hat{T}}(a_0) & \text{if } \mathcal{O}(a_0) = \infty, \\ \left(h_{\hat{T}}(a_0), h_{\hat{T}}(pa_0), \cdots, h_{\hat{T}}(p^{n-1}a_0), \omega + m, \omega + m + 1, \cdots\right) & \\ \qquad \text{if } a_0 \in \hat{T} \setminus T,\ \mathcal{O}(a_0) = p^n,\ \mathcal{O}(a_0 + T) = p^{n-m}, & \\ \left(h_T(a_0), h_T(pa_0), \cdots, h_T(p^{n-1}a_0), \omega + n + k, \omega + n + k + 1, \cdots\right) & \\ \qquad \text{if } \mathcal{O}(a_0) = p^n,\ a_0, a_1, \cdots, a_k \in T,\ a_{k+1} \notin T, & \\ H_T(a_0) & \text{if } a_i \in T \text{ for any } i, \end{cases} \tag{2.2}$$

where ω is the smallest infinite ordinal number.

Let B be a basic subgroup of the reduced separable p-group T lying between B and $\bar{B}$. Elements $a_0, b_0 \in \hat{T}$, $\mathcal{O}(a_0) = \mathcal{O}(a_0) = p$, $a_0 \notin T$. As is know, an endomorphism φ of the group T extends uniquely to an endomorphism of $\hat{T}$.

The following lemma is true.

Lemma 2.1. *If $h_{\hat{T}}(a_0) \leq h_{\hat{T}}(b_0)$ and there exists no endomorphism φ of the group T for which $\varphi a_0 = b_0$, then a cotorsion hull $T^\bullet$ is not fully transitive.*

Proof. Consider two elements

$$a = (a_0, a_1 + T, a_2 + T, \cdots), \quad b = (b_0, b_1 + T, b_2 + T, \cdots)$$

of the group $T^\bullet$. Then by the condition of the theorem and (2.2) we have $H_{T^\bullet}(a) \leq H_{T^\bullet}(b)$. As is known, each endomorphism of the group T extends uniquely to an endomorphism of the group $T^\bullet$. We will show that if for an endomorphism φ, $\varphi a = b$, then $\varphi a_0 = b_0$. Let

$$E: 0 \longrightarrow T \xrightarrow{\mu} G \xrightarrow{\gamma} \mathbb{Z}(p^\infty) \longrightarrow 0 \tag{2.3}$$

be the element of the group $T^\bullet$ defined by the sequence $a = (a_0, a_1 + T, \cdots)$. For an endomorphism φ of the group T let us show that $\varphi a = (\varphi a_0, \varphi a_1 + T, \cdots)$. According to ([1], Section 50), the extension of φE is defined from the commutative diagram

$$\begin{array}{ccccccccc} E: 0 & \longrightarrow & T & \xrightarrow{\mu} & G & \xrightarrow{\gamma} & \mathbb{Z}(p^\infty) & \longrightarrow & 0 \\ & & {\scriptstyle\varphi}\downarrow & & {\scriptstyle\eta}\downarrow & & \| & & \\ \varphi E: 0 & \longrightarrow & T & \xrightarrow[\mu^*]{} & G^* & \xrightarrow[\gamma^*]{} & \mathbb{Z}(p^\infty) & \longrightarrow & 0 \end{array} \tag{2.4}$$

where μ is the identical inclusion,

$$G^* = (T \oplus G)/H, \quad H = \{(-\varphi t, \mu t) \quad t \in T|\},$$

$$\mu^* : t \to (t,0) + H, \quad \eta : g \to (0,g) + H,$$

$$\gamma^* : (t,g) + H \to \gamma g, \quad t \in T, \quad g \in G.$$

The commutativity of diagram (2.4) immediately follows from the definition of these homomorphisms.

To extension (2.3) there corresponds the sequence $(a_0, a_1 + T, \cdots)$, where elements are defined as follows: fix a system of generators $\{\bar{g}_n : n \in N\}$ of the group $\mathbb{Z}(p^\infty)$, $\bar{g}_0 = 0$, $p\bar{g}_{n+1} = \bar{g}_n$. Let $\{g_n\}$, $g_0 = 0$, be a system of representatives of the adjacent classes $\bar{g}_n$ of the group G, $\bar{g}_n = g_n + T$, $pg_{n+1} = g_n + c_n$, $c_n \in T$. Denote

$$a_i = \lim_{n\to\infty}\left(c_i + pc_{i+1} + \cdots + p^n c_{i+n}\right), \ a_i \in \hat{T}, i = 0,1,\cdots.$$

Then for each i,

$$pa_{i+1} - a_i = -c_i \in T.$$

For an endomorphism φ of the group T we have $\varphi c_n \in T$, $n \in N$, and can define

$$\varphi a_i = \lim_{n\to\infty}\left(\varphi c_i + p\varphi c_{i+1} + \cdots + p^n \varphi c_{i+n}\right) \tag{2.5}$$

It is obvious that the right-hand part of equality (2.5) defines the extension of an endomorphism φ on $\hat{T}$ and if φ^* is some other endomorphism of the group $\hat{T}$, which induces φ on T, then $Ker(\varphi - \varphi^*)$ contains $T \supset B$ and $\varphi = \varphi^*$ ([1], Proposition 34.1). From (2.5) we have

$$p\varphi a_{i+1} - \varphi a_i = -\varphi c_i \in T.$$

Now we can consider an element

$$(\varphi a_0, \varphi a_1 + T, \varphi a_2 + T, \cdots) \tag{2.6}$$

of the group $T^\bullet$ and with its aid define the corresponding short exact sequence.

Let G' be the group defined by a system of generators $T \cup \{g_i', i \in N\}$ which are defined by the relations of the group T and the equalities $pg_{i+1}' = g_i' + \varphi c_i$, $g_0' = 0$, $i = 1,2,\cdots$. Then

$$E' : 0 \longrightarrow T \xrightarrow{\mu'} G' \xrightarrow{\gamma'} \mathbb{Z}(p^\infty) \longrightarrow 0 \tag{2.7}$$

where μ' is the identical inclusion and, for each element $c + kg_i'$, $c \in T$, $k \in Z$, $\gamma'(c + kg_i') = k\bar{g}_i'$ is a short exact sequence. To extension (2.7) there corresponds sequence (2.6) (see [11], Proof of Theorem 1). Let us show that by using extensions (2.3) and (2.7) we can compose the commutative diagram

$$\begin{array}{ccccccccc} E: 0 & \longrightarrow & T & \xrightarrow{\mu} & G & \xrightarrow{\gamma} & \mathbb{Z}(p^\infty) & \longrightarrow & 0 \\ & & {\scriptstyle\varphi}\downarrow & & {\scriptstyle\eta}\downarrow & & \| & & \\ E': 0 & \longrightarrow & T & \xrightarrow[\mu']{} & G' & \xrightarrow[\gamma']{} & \mathbb{Z}(p^\infty) & \longrightarrow & 0 \end{array} \tag{2.8}$$

where φ is the above-mentioned endomorphism and $\eta'(c + kg_i) = \varphi c + kg_i'$, $c \in T$, $k \in Z$, $g_i \in G$. Indeed, from the definition of a triple $(\varphi, \eta', =)$ we immediately conclude that (2.8) is a commutative diagram.

Thus we have shown that (2.4) and (2.8) are commutative diagrams. Then, according to ([1], Section 50), φE and E' are equivalent extensions and thereby define one and the same sequence from $T^{\bullet}$. But, by virtue of our construction, $(\varphi a_0, \varphi a_1 + T, \cdots)$ is the sequence corresponding to the extension E'; therefore it corresponds to the extension φE, too. Thus $\varphi a = (\varphi a_0, \varphi a_1 + T, \cdots)$. Therefore if the endomorphism φ maps the element $a = (a_0, a_1 + T, \cdots)$ into $b = (b_0, b_1 + T, \cdots)$, then $\varphi a_0 = b_0, \varphi a_1 + T = b_1 + T, \cdots$, *i.e.* we have proved more than what has been mentioned at the beginning of the proof of the lemma. Thus it obviously follows that if there exists no endomorphism φ of the group T for which $\varphi a_0 = b_0$, then there exists no endomorphism φ of the group $T^{\bullet}$ which maps the element a into b, *i.e.* $T^{\bullet}$ is not fully transitive. The lemma is proved.

For the Pierce group G the following statement is true.

Theorem 2.1. *The cotorsion hull* $G^{\bullet}$ *of the group* G *is not fully transitive.*

Proof. We use representation (2.1) of cotorsion hull elements and assume that $a = (a_0, a_1 + G, \cdots)$ and $c = (c_0, c_1 + G, \cdots)$ are elements of the group $G^{\bullet}$, where $\mathcal{O}(a_0) = \mathcal{O}(c_0) = p$, $a_0, c_0 \in \overline{B} \setminus G$. By virtue of (11, Item 2), elements a_0 and b_0 can be written in the form

$$a_0 = b_0 + pb_1 + p^2 b_2 + \cdots,$$
$$c_0 = b_0' + pb_1' + p^2 b_2' + \cdots,$$

where $b_i, b_i' \in B_i$, $i = 0, 1, \cdots$. Since $B \subset G \subset \overline{B}$ and B is infinite, taking into account ([14]: Lemma 15.1, Theorem 15.4) we can assume that

$$a_0 - c_0 \notin G \tag{2.9}$$

By (2.2) we have $H_{T^{\bullet}}(a) = H_{T^{\bullet}}(c) = (0, \omega, \omega, \cdots)$, *i.e.* the following condition is fulfilled

$$H_{T^{\bullet}}(a) \leq H_{T^{\bullet}}(c).$$

Let φ be an endomorphism of the group G. Using (1.1) we have $\varphi = \psi + \alpha$, where ψ is a small endomorphism of the group G and α is the p-adic number $\alpha = \alpha_0 + \alpha_1 p + \alpha_2 p^2 + \cdots$ $(\alpha_n = 0, 1, \cdots, p-1)$. As is known ([1], Section 39), the endomorphism φ uniquely extends to the endomorphism $\overline{\varphi} = \overline{\psi} + \overline{\alpha}$ of the group $\overline{B}$

$$\overline{\varphi} a_0 = (\overline{\psi} + \overline{\alpha}) a_0 = \overline{\psi} a_0 + \overline{\alpha} a_0$$
$$= \overline{\psi}(b_0 + pb_1 + p^2 b_2 + \cdots) + \overline{\alpha}(b_0 + pb_1 + p^2 b_2 + \cdots).$$

Since

$$\mathcal{O}(p^i b_i) \leq p, \ i = 0, 1, \cdots \tag{2.10}$$

and φ is a small endomorphism of the group G (see (1.2)), starting with some k we have

$$\overline{\psi}(p^k b_k) = \psi(p^k b_k) = 0.$$

Therefore

$$\overline{\psi} a_0 = \psi(b_0 + pb_1 + p^2 b_2 + \cdots + p^{k-1} b_{k-1}) = g \in G,$$
$$b_0 + pb_1 + p^2 b_2 + \cdots + p^{k-1} b_{k-1} \in B \subset G.$$

On the other hand, from (2.10) we obtain

$$\overline{\alpha} a_0 = (\alpha_0 + \alpha_1 p + \alpha_2 p^2 + \cdots)(b_0 + pb_1 + \cdots)$$
$$= \alpha_0 b_0 + \alpha_0 pb_1 + \cdots = \alpha_0 (b_0 + pb_1 + p^2 b_2 + \cdots) = \alpha_0 a_0.$$

Therefore

$$\bar{\varphi}a_0 = g + \alpha_0 a_0, \quad g \in G$$

But $g + \alpha_0 a_0 \neq c_0$ since $\mathcal{O}(a_0) = p$, $\alpha_0 \in \{0,1,\cdots,p-1\}$, and in that case the equality $g + \alpha_0 a_0 = c_0$ would contradict condition (2.9). Therefore

$$\bar{\varphi}a_0 \neq c_0 .$$

Thus there exists no endomorphism φ of the group G which extends to the endomorphism $\bar{\varphi}$ of the group $\bar{B}$ and $\bar{\varphi}a_0 = c_0$. Then from Lemma 1.1 it follows that Theorem 2.1 is valid.

Note that one more example of a separable primary group, the cotorsion hull of which is not fully transitive, can be found in ([11], item 3).

As mentioned above, if the separable primary group T is a direct sum of cyclic p-groups or a cotorsion-complete group, then the cotorsion hull $T^\bullet$ is fully transitive. In 1993, at Professor A. Fomin's seminar A. Moskalenko made a conjecture that $T^\bullet$ is fully transitive only in these two cases. The proved lemma and theorem may serve as a positive argument in favor of this conjecture.

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