Journal of Applied Mathematics and Physics, **2014, 2, 50-59**
Published Online February 2014 (http://www.scirp.org/journal/jamp)
http://dx.doi.org/10.4236/jamp.2014.23006

Classification of Single Traveling Wave Solutions to the Generalized Strong Nonlinear Boussinesq Equation without Dissipation Terms in $P = 1$

Xinghua Du
Department of Mathematics, Northeast Petroleum University, Daqing, China
Email: xinghuadu@126.com

Received January 15, 2014; revised February 15, 2014; accepted February 22, 2014

ABSTRACT

By the complete discrimination system for polynomial method, we obtained the classification of single traveling wave solutions to the generalized strong nonlinear Boussinesq equation without dissipation terms in $p = 1$.

KEYWORDS

Complete Discrimination System for Polynomial; Traveling Wave Solution; Generalized Strong Nonlinear Boussinesq Equation without Dissipation Terms

1. Introduction

There are many methods of obtaining the exact solutions for nonlinear evolution equations, such as the homogeneous balance method [1], the inverse scattering method [2], Hirotas bilinear transformation [3], the extended tanh-function method [4], the sech-function method [5] and so on. Liu introduced complete discrimination system for the polynomial method to obtain the classification of traveling wave solutions to some nonlinear evolution equations [6-8]. In [9], the generalized strong nonlinear Boussinesq equation without dissipation terms was given by

$$u_{tt} + \delta u_{xxxx} - \left(b_1 u + b_2 u^{p+1} + b_3 u^{2p+1}\right)_{xx} = 0, \tag{1}$$

where $\delta > 0$, $p > 0$, b_1, b_2, b_3, p, δ are constants. When $b_3 = 0, p = 1$, Equation (1) becomes

$$u_{tt} + \delta u_{xxxx} - \left(b_1 u + b_2 u^2\right)_{xx} = 0, \tag{2}$$

Equation (2) is an important model equation in physics. It describes the wave propagation in the weakly nonlinear and dispersive media. When $\delta > 0, b_1 = 1$ or $\delta < 0, b_1 = 1$, the Equation (2) becomes good Boussinesq equation [10,11] or bad Boussinesq equation [12,13]. The good Boussinesq equation and bad Boussinesq equation have been studied by many authors [10-17]. But the classification of single traveling wave solutions to these equations hasn't been studied. In the present paper, we consider the following generalized strong nonlinear Boussinesq equation without dissipation terms in $p = 1$:

$$u_{tt} + \delta u_{xxxx} - \left(b_1 u + b_2 u^2 + b_3 u^3\right)_{xx} = 0, \tag{3}$$

where $\delta > 0$, b_1, b_2, b_3, δ are constants. By using Liu's method, the classification of single traveling wave solutions to Equation (3) is obtained.

2. The Traveling Wave Solutions to the Equation (3)

Take wave transformation

$$u(x,t)=u(\xi),\xi=kx-\omega t. \tag{4}$$

Substituting Equation (4) into Equation (3) yields the following nonlinear ordinary difference equation:

$$\omega^2 u''+\delta k^4 u^{(4)}-k^2\left(b_1 u+b_2 u^2+b_3 u^3\right)''=0, \tag{5}$$

Integrating Equation (5) once with respect to ξ, and setting the integration constant to zero yields:

$$\omega^2 u'+\delta k^4 u'''-k^2\left(b_1 u+b_2 u^2+b_3 u^3\right)'=0. \tag{6}$$

Integrating Equation (6) twice with respect to ξ yields:

$$(u')^2=\frac{b_3}{2\delta k^2}u^4+\frac{2b_2}{3\delta k^2}u^3+\frac{k^2 b_1-\omega^2}{\delta k^4}u^2+c_1 u+c_0, \tag{7}$$

where c_1 and c_0 are arbitrary constants.

In order to find the traveling wave solutions to the Equation (3), let us solve Equation (7). In this article, there are two cases to discuss the exact solutions of Equation (7) according to the arbitrary constant c_0.

Case 2.1 $c_0=0$, then Equation (7) becomes

$$(u')^2=u\left(\frac{b_3}{2\delta k^2}u^3+\frac{2b_2}{3\delta k^2}u^2+\frac{k^2 b_1-\omega^2}{\delta k^4}u+c_1\right). \tag{8}$$

Integrating Equation (8) once yields

$$\int\frac{\mathrm{d}u}{\sqrt{\varepsilon u F(u)}}=\pm\sqrt{\frac{\varepsilon b_3}{2\delta k^2}}(\xi-\xi_0), \tag{9}$$

where

$$F(u)=u^3+\frac{4b_2}{3b_3}u^2+\frac{2\left(k^2 b_1-\omega^2\right)}{k^2 b_3}u+\frac{2\delta k^2 c_1}{b_3}. \tag{10}$$

If $\frac{b_3}{\delta}>0$, we take $\varepsilon=1$; if $\frac{b_3}{\delta}<0$, we take $\varepsilon=-1$. The complete discrimination system for the third order polynomial $F(u)$ is given as follows:

$$\begin{aligned}\Delta&=\frac{\left(128b_2^3k^2+1457\delta k^4 c_1 b_3^2-648b_2b_3\left(b_1k^2+\omega^2\right)\right)^2}{19683b_3^6k^4}-\frac{4\left(18k^2b_1-18\omega^2-4k^3b_2\right)^3}{729k^9b_3^3},\\ D_1&=\frac{162k^2b_1b_3-162b_3\omega^2-16k^2b_2^2}{81k^2b_3^2}.\end{aligned} \tag{11}$$

In order to obtain the solutions to the Equation (9), according to the complete discrimination system for the third order polynomial $F(u)$, there are four cases to be discussed.

Case 2.1.1. $\Delta=0, D_1<0$ $F(u)=(u-\alpha)^2(u-\beta)$, where α,β are real constants, $\alpha\neq\beta$, $\beta>0$. If $\varepsilon=1$, when $\alpha>\beta$ and $u>\beta$, from Equation (9), we give the solution of Equation (7) as follows:

$$\pm\sqrt{\frac{b_3}{2\delta k^2}}\alpha(\alpha-\beta)(\xi-\xi_0)=\ln\frac{\left[\sqrt{\alpha(u-\beta)}-\sqrt{u(\alpha-\beta)}\right]^2}{|u-\beta|}, \tag{12}$$

when $\alpha<0$ and $u<\beta$, we have

$$\pm\sqrt{\frac{b_3}{2\delta k^2}}\alpha(\alpha-\beta)(\xi-\xi_0)=\ln\frac{\left[\sqrt{-\alpha(u-\beta)}-\sqrt{u(\beta-\alpha)}\right]^2}{|u-\beta|}, \tag{13}$$

when $\beta > \alpha > 0$, we have

$$\pm\sqrt{\frac{b_3}{2\delta k^2}}\alpha\left(\beta-\alpha\right)\left(\xi-\xi_0\right)=\arcsin\frac{\alpha\left(u-\beta\right)+u\left(\alpha-\beta\right)}{\left|\beta\left(u-\alpha\right)\right|}. \tag{14}$$

If $\varepsilon=-1$, when $\alpha>\beta$ and $u>\beta$, from Equation (9), we give the solutions of Equation (7)

$$\pm\sqrt{-\frac{b_3}{2\delta k^2}}\alpha\left(\beta-\alpha\right)\left(\xi-\xi_0\right)=\ln\frac{\left[\sqrt{\alpha\left(-u+\beta\right)}-\sqrt{u\left(\beta-\alpha\right)}\right]^2}{\left|u-\beta\right|}, \tag{15}$$

when $\alpha<0$, and $u<\beta$, we have

$$\pm\sqrt{-\frac{b_3}{2\delta k^2}}\alpha\left(\beta-\alpha\right)\left(\xi-\xi_0\right)=\ln\frac{\left[\sqrt{-\alpha\left(-u+\beta\right)}-\sqrt{u\left(\alpha-\beta\right)}\right]^2}{\left|u-\beta\right|}, \tag{16}$$

when $\beta>\alpha>0$, we have

$$\pm\sqrt{-\frac{b_3}{2\delta k^2}}\alpha\left(\alpha-\beta\right)\left(\xi-\xi_0\right)=\arcsin\frac{\alpha\left(-u+\beta\right)+u\left(\beta-\alpha\right)}{\left|\beta\left(u-\alpha\right)\right|}. \tag{17}$$

Case 2.1.2. $\Delta=0, D_1=0$ $F\left(u\right)=\left(u-\alpha\right)^3$, where α is real constant. If $\varepsilon=1$, when $u>\alpha$, we have

$$u=\frac{2\delta k^2\alpha}{\alpha^2 b_3\left(\xi-\xi_0\right)^2-2\delta k^2}+\alpha \tag{18}$$

If $\varepsilon=-1$, when $u<\alpha$, we have

$$u=\frac{2\delta k^2\alpha}{-\alpha^2 b_3\left(\xi-\xi_0\right)^2-2\delta k^2}+\alpha \tag{19}$$

Case 2.1.3. $\Delta>0, D_1<0$ $F\left(u\right)=\left(u-\alpha\right)\left(u-\beta\right)\left(u-\gamma\right)$, where α,β,γ are different real constants. If $\varepsilon=1$, when $u<\gamma$, we have

$$u=\frac{-\gamma\alpha\,\mathrm{sn}^2\left(\sqrt{-\frac{\beta b_3}{2\delta k^2}\left(\alpha-\beta\right)}\left(\xi-\xi_0\right),m\right)}{-\gamma\,\mathrm{sn}^2\left(\sqrt{-\frac{\beta b_3}{2\delta k^2}\left(\alpha-\gamma\right)}\left(\xi-\xi_0\right),m\right)-\left(\alpha-\gamma\right)}, \tag{20}$$

$$u=\frac{-\gamma\left(\alpha-\beta\right)\mathrm{sn}^2\left(\sqrt{-\frac{\beta b_3}{2\delta k^2}\left(\alpha-\beta\right)}\left(\xi-\xi_0\right),m\right)-\beta\left(\alpha-\gamma\right)}{-\gamma\,\mathrm{sn}^2\left(\sqrt{-\frac{\beta b_3}{2\delta k^2}\left(\alpha-\gamma\right)}\left(\xi-\xi_0\right),m\right)-\left(\alpha-\gamma\right)}, \tag{21}$$

where $m^2=\dfrac{\gamma\left(\alpha-\beta\right)}{\beta\left(\alpha-\gamma\right)}$. If $\varepsilon=-1$, when $\gamma<u^2<\beta$, we have

$$u=\frac{-\beta\alpha\,\mathrm{sn}^2\left(\sqrt{-\frac{\beta b_3}{2\delta k^2}\left(\alpha-\beta\right)}\left(\xi-\xi_0\right),m\right)+\alpha\beta}{-\alpha\,\mathrm{sn}^2\left(\sqrt{-\frac{\beta b_3}{2\delta k^2}\left(\alpha-\gamma\right)}\left(\xi-\xi_0\right),m\right)+\beta}, \tag{22}$$

$$u=\frac{-\gamma\beta}{\left(\beta-\gamma\right)\mathrm{sn}^2\left(\sqrt{-\frac{\beta b_3}{2\delta k^2}\left(\alpha-\gamma\right)}\left(\xi-\xi_0\right),m\right)-\beta}, \tag{23}$$

where $m^2 = \dfrac{\alpha(\beta-\gamma)}{\beta(\alpha-\gamma)}$.

Case 2.1.4. $\Delta < 0$, $F(u) = (u-\alpha)\left[(u-l)^2 + s^2\right]$, where α, l, s are all real constants, and $\alpha > 0, l, s > 0$, we have

$$u = \frac{a\,\mathrm{cn}\left(\dfrac{\sqrt{\mp\dfrac{sm_1\alpha b_3}{\delta k^2}}}{mm_1}(\xi-\xi_0), m\right) + b}{c\,\mathrm{cn}\left(\dfrac{\sqrt{\mp\dfrac{sm_1\alpha b_3}{\delta k^2}}}{mm_1}(\xi-\xi_0), m\right) + d}, \tag{24}$$

where

$$a = \frac{1}{2}\alpha(c-d), b = \frac{1}{2}\alpha(d-c), c = \alpha - l - \frac{s}{m_1},\ d = -l - sm_1, E = \frac{s^2 - l(\alpha - l)}{s\alpha}, m_1 = E \pm \sqrt{E^2+1}, m^2 = \frac{1}{1+m_1^2}.$$

Case 2.2 $c_0 \neq 0$ In order to solve Equation (7), when $\dfrac{b_3}{\delta} > 0$, we take the transformation as follows

$$w = \left(\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\left(u + \frac{b_2}{3b_3}\right), \xi_1 = \left(\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\xi. \tag{25}$$

Combining the expression (7) with Equation (25) yields

$$w_{\xi_1}^2 = F(w) = w^4 + pw^2 + qw + r, \tag{26}$$

where

$$p = \frac{\sqrt{2}\left(k^2 b_1 - \omega^2\right)}{k^3\sqrt{\delta b_3}},\quad q = \frac{b_2^3 k^2 - 18k^2 b_1 b_2 b_3 + 18\omega^2 b_1 b_2 b_3 + 27\delta k^4 b_3^2}{27\delta k^4 b_3^2}\left(\frac{b_3}{2\delta k^2}\right)^{-\frac{1}{4}},$$

$$\text{And}\quad r = \frac{-b_2^4 k^2 + 6b_2 b_3 k^2 b_1 - 6b_2 b_3 \omega^2 - 18c_1\delta k^4 b_2 b_3^2}{54\delta k^4 b_3^3} + c_0.$$

When $\dfrac{b_3}{\delta} < 0$, we take the following transformation:

$$w = \left(-\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\left(u + \frac{b_2}{3b_3}\right), \xi_1 = \left(-\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\xi. \tag{27}$$

Combining the expression (7) with Equation (27) yields

$$w_{\xi_1}^2 = -F(w) = -\left(w^4 + pw^2 + qw + r\right), \tag{28}$$

where

$$p = -\frac{\sqrt{2}\left(k^2 b_1 - \omega^2\right)}{k^3\sqrt{-\delta b_3}},\quad q = \frac{b_2^3 k^2 - 18k^2 b_1 b_2 b_3 + 18\omega^2 b_1 b_2 b_3 + 27\delta k^4 b_3^2}{-27\delta k^6 b_3^2}\left(-\frac{b_3}{2\delta k^2}\right)^{-\frac{1}{4}},$$

$$\text{and}\quad r = \frac{b_2^4 k^2 - 6b_2 b_3 k^2 b_1 + 6b_2 b_3 \omega^2 + 18c_1\delta k^4 b_2 b_3^2}{54\delta k^4 b_3^3} - c_0.$$

The complete discrimination system for the fourth order polynomial $F(w) = w^4 + pw^2 + qw + r$ as follows:

$$D_1 = 4, D_2 = -p, D_3 = 8rp - 2p^3 - 9q^2,$$
$$D_4 = 4p^4 r - p^3 q^2 + 36prq^2 - 32r^2 p^2 - \frac{27}{4}q^4 + 64r^3, \; E_2 = 9p^2 - 32pr. \tag{29}$$

In order to obtain the solutions to Equation (26) and Equation (28), according to the complete discrimination system for the fourth order polynomial $F(w)$, there are nine cases to be discussed.

Case 2.2.1 $D_2 < 0$, $D_3 = 0$, and $D_4 = 0$, then $F(w) = \left((w-l)^2 + s^2\right)^2$. where $s > 0$. For $\frac{b_3}{\delta} > 0$, the solution of Equation (7) is

$$u = \left(\frac{b_3}{2\delta k^2}\right)^{-\frac{1}{4}} \left[s \tan\left(s\left(\xi_1 - \xi_0\right)\right) + l\right] - \frac{b_2}{3b_3}. \tag{30}$$

Case 2.2.2. $D_2 = 0, D_3 = 0,$ and $D_4 = 0,$ then $F(w) = w^4$. For $\frac{b_3}{\delta} > 0$, the solution of Equation (7) is

$$u = -\left(\frac{b_3}{2\delta k^2}\right)^{-\frac{1}{2}} \left(\xi - \xi_0\right)^{-1} - \frac{b_2}{3b_3}. \tag{31}$$

Case 2.2.3. $D_2 > 0, D_3 = 0,$ $D_4 = 0$ and $E_2 > 0$, $F(w) = (w-\alpha)^2 (w-\beta)^2$, where $\alpha > \beta$. For $\frac{b_3}{\delta} > 0$, when $w > \alpha$ or $w < \beta$, the solution of Equation (7) is

$$u = \left(\frac{b_3}{2\delta k^2}\right)^{-\frac{1}{4}} \left[\frac{\beta - \alpha}{2}\left[\coth \frac{\left(\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}} (\alpha - \beta)(\xi - \xi_0)}{2} - 1\right] + \beta\right] - \frac{b_2}{3b_3}, \tag{32}$$

when $\beta < w < \alpha$,the solution of Equation (7) is

$$u = \left(\frac{b_3}{2\delta k^2}\right)^{-\frac{1}{4}} \left[\frac{\beta - \alpha}{2}\left[\tanh \frac{\left(\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}} (\alpha - \beta)(\xi - \xi_0)}{2} - 1\right] + \beta\right] - \frac{b_2}{3b_3}. \tag{33}$$

Case 2.2.4. $D_2 > 0, D_3 = 0,$ $D_4 = 0$ and $E_2 = 0$, then $F(w) = (w-\alpha)^3 (w-\beta)$, when $\frac{b_3}{\delta} > 0$, $w > \alpha, w > \beta$ or $w < \alpha, w < \beta$, the solution of Equation (7) is

$$u = \left(\frac{b_3}{2\delta k^2}\right)^{-\frac{1}{4}} \left[\frac{4(\alpha - \beta)}{(\beta - \alpha)^2 \left(\frac{b_3}{2\delta k^2}\right)^{\frac{1}{2}} (\xi - \xi_0)^2 - 4} + \alpha\right] - \frac{b_2}{3b_3}, \tag{34}$$

when $\frac{b_3}{\delta} < 0$, $w > \alpha, w < \beta$ or $w < \alpha, w > \beta$, the solution of Equation (7) is

$$u = \left(-\frac{b_3}{2\delta k^2}\right)^{-\frac{1}{4}} \left[\frac{4(\alpha - \beta)}{-(\beta - \alpha)^2 \left(-\frac{b_3}{2\delta k^2}\right)^{\frac{1}{2}} (\xi - \xi_0)^2 - 4} + \alpha\right] - \frac{b_2}{3b_3}. \tag{35}$$

Case 2.2.5 $D_2 > 0, D_3 > 0$, and $D_4 = 0$, then $F(w) = (w-\alpha)^2 (w-\beta)(w-\gamma)$. If $\frac{b_3}{\delta} > 0$, when $\alpha > \beta$ and $w > \beta$ or when $\alpha < \gamma$ and $w < \gamma$, we have

$$\exp\left\{\pm\left(\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}(\alpha-\beta)(\alpha-\gamma)(\xi-\xi_0)\right\} = \frac{\left\{\sqrt{\left(\left(\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\left(u+\frac{b_2}{3b_3}-\beta\right)(\alpha-\gamma)\right)} - \sqrt{\left(\left(\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\left(u+\frac{b_2}{3b_3}\right)-\gamma\right)(\alpha-\beta)}\right\}^2}{\left|\left(\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\left(u+\frac{b_2}{3b_3}\right)-\alpha\right|}, \tag{36}$$

when $\alpha > \beta$ and $w < \gamma$, or when $\alpha < \gamma$, and $w < \beta$, we have

$$\exp\left\{\pm\left(\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}(\alpha-\beta)(\alpha-\gamma)(\xi-\xi_0)\right\} = \frac{\left\{\sqrt{\left(\left(\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\left(u+\frac{b_2}{3b_3}\right)-\beta\right)(\gamma-\alpha)} - \sqrt{\left(\left(\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\left(u+\frac{b_2}{3b_3}\right)-\gamma\right)(\beta-\alpha)}\right\}^2}{\left|\left(\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\left(u+\frac{b_2}{3b_3}\right)-\alpha\right|}, \tag{37}$$

when $\beta > \alpha > \gamma$, we have

$$\pm\sin\left(\left(\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}(\beta-\alpha)(\alpha-\gamma)(\xi-\xi_0)\right) = \frac{\left[\left(\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\left(u+\frac{b_2}{3b_3}\right)-\beta\right](\alpha-\gamma)+\left[\left(\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\left(u+\frac{b_2}{3b_3}\right)-\gamma\right](\alpha-\beta)}{\left|\left[\left(\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\left(u+\frac{b_2}{3b_3}\right)-\alpha\right](\beta-\alpha)\right|}. \tag{38}$$

If $\frac{b_3}{\delta} < 0$, when $\alpha > \beta$ and $w > \beta$ or when $\alpha < \gamma$ and $w < \gamma$, we have

$$\pm\left(\exp\left\{\pm\left(-\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}(\alpha-\beta)(\alpha-\gamma)(\xi-\xi_0)\right\}\right)^{\frac{1}{2}} = \frac{\sqrt{\left[-\left(-\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\left(u+\frac{b_2}{3b_3}\right)-\beta\right](\gamma-\alpha)}}{\sqrt{\left|\left(-\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\left(u+\frac{b_2}{3b_3}\right)-\alpha\right|}} - \frac{\sqrt{\left[\left(-\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\left(u+\frac{b_2}{3b_3}\right)-\gamma\right](\beta-\alpha)}}{\sqrt{\left|\left(-\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\left(u+\frac{b_2}{3b_3}\right)-\alpha\right|}}, \tag{39}$$

when $\alpha > \beta$ and $w < \gamma$, or when $\alpha < \gamma$, and $w < \beta$, we have

$$\pm\left(\exp\left\{\pm\left(-\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}(\alpha-\beta)(\alpha-\gamma)(\xi-\xi_0)\right\}\right)^{\frac{1}{2}}$$

$$=\frac{\sqrt{\left[-\left(-\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\left(u+\frac{b_2}{3b_3}\right)-\beta\right](\alpha-\gamma)}}{\sqrt{\left|\left(-\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\left(u+\frac{b_2}{3b_3}\right)-\alpha\right|}}-\frac{\sqrt{\left[\left(-\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\left(u+\frac{b_2}{3b_3}\right)-\gamma\right](\alpha-\beta)}}{\sqrt{\left|\left(-\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\left(u+\frac{b_2}{3b_3}\right)-\alpha\right|}}, \tag{40}$$

when $\beta > \alpha > \gamma$, we have

$$\pm\sin\left[\left(-\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}(\beta-\alpha)(\alpha-\gamma)(\xi-\xi_0)\right]$$

$$=\frac{\left[-\left(-\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\left(u+\frac{b_2}{3b_3}\right)+\beta\right](\alpha-\gamma)+\left[\left(-\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\left(u+\frac{b_2}{3b_3}\right)-\gamma\right](\alpha-\beta)}{\left|\left[\left(-\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\left(u+\frac{b_2}{3b_3}\right)-\alpha\right](\beta-\alpha)\right|} \tag{41}$$

where $\gamma=\frac{\alpha-2l}{\sqrt{(\alpha-l)^2+s^2}}$, , and $\delta=\sqrt{(\alpha-l)^2+s^2}-\frac{\alpha(\alpha-2l)}{\sqrt{(\alpha-l)^2+s^2}}$.

Case 2.2.6. $D_2 > 0, D_3 > 0$, and $D_4 > 0$, $F(w)=(w-\alpha_1)(w-\alpha_2)(w-\alpha_3)(w-\alpha_4)$, where $\alpha_1 > \alpha_2 > \alpha_3 > \alpha_4$. If $\frac{b_3}{\delta} > 0$, when $w > \alpha_1$, or $w < \alpha_4$, the solution of Equation (7) is

$$u=-\frac{b_2}{3b_3}\left(\frac{b_3}{2\delta k^2}\right)^{-\frac{1}{4}}\left[\frac{\alpha_2(\alpha_1-\alpha_4)\mathrm{sn}^2\left(\left(\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\xi-\xi_0),m\right)-\alpha_1(\alpha_2-\alpha_4)}{(\alpha_1-\alpha_4)\mathrm{sn}^2\left(\left(\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\xi-\xi_0),m\right)-(\alpha_2-\alpha_4)}\right]. \tag{42}$$

when $\alpha_3 < w < \alpha_2$, the solution of Equation (7) is

$$u=-\frac{b_2}{3b_3}\left(\frac{b_3}{2\delta k^2}\right)^{-\frac{1}{4}}\left[\frac{\alpha_4(\alpha_2-\alpha_3)\mathrm{sn}^2\left(\left(\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\xi-\xi_0),m\right)-\alpha_3(\alpha_2-\alpha_4)}{(\alpha_2-\alpha_3)\mathrm{sn}^2\left(\left(\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\xi-\xi_0),m\right)-(\alpha_2-\alpha_4)}\right]. \tag{43}$$

If $\frac{b_3}{2\delta k^2} < 0$, when $\alpha_1 > w > \alpha_2$, the solution of Equation (7) is

$$u=-\frac{b_2}{3b_3}\left(-\frac{b_3}{2\delta k^2}\right)^{-\frac{1}{4}}\left[\frac{\alpha_3(\alpha_1-\alpha_2)\operatorname{sn}^2\left(\left(-\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\xi-\xi_0),m\right)-\alpha_2(\alpha_1-\alpha_3)}{(\alpha_1-\alpha_2)\operatorname{sn}^2\left(\left(-\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\xi-\xi_0),m\right)-(\alpha_1-\alpha_3)}\right], \quad (44)$$

when $\alpha_4 < w < \alpha_3$, the solution of Equation (7) is

$$u=-\frac{b_2}{3b_3}\left(-\frac{b_3}{2\delta k^2}\right)^{-\frac{1}{4}}\left[\frac{\alpha_1(\alpha_1-\alpha_2)\operatorname{sn}^2\left(\left(-\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\xi-\xi_0),m\right)-\alpha_4(\alpha_3-\alpha_1)}{(\alpha_3-\alpha_4)\operatorname{sn}^2\left(\left(-\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\frac{\sqrt{(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)}}{2}(\xi-\xi_0),m\right)-(\alpha_3-\alpha_1)}\right], \quad (45)$$

where $m^2=\frac{(\alpha_1-\alpha_2)(\alpha_3-\alpha_4)}{(\alpha_1-\alpha_3)(\alpha_2-\xi_4)}$.

Case 2.2.7. $D_2D_3\ge 0$, and $D_4<0$, $F(w)=(w-\alpha)(w-\beta)\left((w-l)^2+s^2\right)$. where $\alpha>\beta$ and $s>0$.

The solution of Equation (7) (when $\frac{b_3}{\delta}>0$, we take the positive sign; when $\frac{b_3}{\delta}<0$, we take the negative) is

$$u=\left(\pm\frac{b_3}{2\delta k^2}\right)^{-\frac{1}{4}}\left[\frac{a\operatorname{cn}\left(\left(\pm\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\frac{\sqrt{\mp 2sm_1(\alpha-\beta)}}{2mm_1}(\xi-\xi_0),m\right)+b}{c\operatorname{cn}\left(\left(\pm\frac{b_3}{2\delta k^2}\right)^{\frac{1}{4}}\frac{\sqrt{\mp 2sm_1(\alpha-\beta)}}{2mm_1}(\xi-\xi_0),m\right)+d}\right]-\frac{b_2}{3b_3}, \quad (46)$$

where $a=\frac{1}{2}(\alpha+\beta)c-\frac{1}{2}(\alpha-\beta)d$, $b=\frac{1}{2}(\alpha+\beta)d-\frac{1}{2}(\alpha-\beta)c$, $c=\alpha-l-\frac{s}{m_1}$, $d=\alpha-l-sm_1$,

$E=\frac{s^2+((\alpha-l)\beta-l)}{s(\alpha-\beta)}$, $m_1=E\pm\sqrt{E^2+1}$, $m^2=\frac{1}{1+m_1^2}$.

Case 2.2.8. $D_2D_3\le 0$, and $D_4>0$, then $F(w)=\left((w-l_1)^2+s_1^2\right)\left((w-l_2)^2+s_2^2\right)$, where $s_1\ge s_2>0$. The solution of Equation (7) is

$$u=\left(\pm\frac{b_3}{2\delta k^2}\right)^{-\frac{1}{4}}\left[\frac{a\operatorname{sn}(\eta(\xi_1-\xi_0),m_1)+bcn(\eta(\xi-\xi_0),m_1)}{c\operatorname{sn}(\eta(\xi-\xi_0),m_1)+dcn(\eta(\xi_1-\xi_0),m_1)}\right]-\frac{b_2}{3b_3}, \quad (47)$$

where $\eta=\frac{s_2\sqrt{(c^2+d^2)(m_1^2c^2+d^2)}}{c^2+d^2}$, $a=l_1c+s_1d$, $b=l_1d-s_1c$, $c=-s_1-\frac{s_2}{m_1}$, $d=l_1-l_2$,

$E=\frac{(l_1-l_2)^2+s_1^2+s_2^2}{2s_1s_2}$, $m_1=E+\sqrt{E^2-1}$.

Case 2.2.9. $D_2D_3<0$ and $D_4=0$, then $F(w)=(w-\alpha)^2\left((w-l)^2+s\right)$, where α,l and s are real numbers. If $\frac{b_3}{\delta}>0$, we have

$$u=\left(\frac{b_3}{2\delta k^2}\right)^{-\frac{1}{4}}\left[\frac{\mathrm{e}^{\pm\left(\frac{b_3}{2\delta k^2}\right)^{-\frac{1}{4}}\sqrt{(l-l)^2+s^2}(\xi-\xi_0)}-\gamma+\sqrt{(\alpha-l)^2+s^2}\,(2-\gamma)}{\left[\mathrm{e}^{\left(\pm\frac{b_3}{2\delta k^2}\right)^{-\frac{1}{4}}\sqrt{(l-l)^2+s^2}(\xi-\xi_0)}-\gamma\right]^2-1}\right]-\frac{b_2}{3b_3}, \tag{48}$$

where $\gamma=\dfrac{\alpha-2l}{\sqrt{(\alpha-l)^2+s^2}}$.

4. Conclusion

By the complete discrimination system for polynomial method, we have obtained the classification of single traveling wave solutions to the generalized strong nonlinear Boussinesq without dissipation terms in $p=1$ .These solutions include trigonometric periodic solutions, rational function solution, hyperbolic funtion solutions, Jacobi elliptic function solutions and so on. This method is simple and efficient.

Acknowledgements

The project is supported by Scientific Research Fund of Education Department of Heilongjiang Province of China under Grant No. 12521049.

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