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One Sound and Complete *R*-Calculus with Pseudo-Subtheory Minimal Change Property*

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ABSTRACT

The AGM axiom system is for the belief revision (revision by a single belief), and the DP axiom system is for the iterated revision (revision by a finite sequence of beliefs). Li [1] gave an **R**-calculus for **R**-configurations $\Delta|\Gamma$, where Δ is a set of atomic formulas or the negations of atomic formulas, and Γ is a finite set of formulas. In propositional logic programs, one *R*-calculus **N** will be given in this paper, such that **N** is sound and complete with respect to operator $s(\Delta,t)$, where $s(\Delta,t)$ is a pseudo-theory minimal change of t by Δ.

Keywords: Belief Revision; ***R***-Calculus; Soundness and Completeness of a Calculus; Pseudo-Subtheory

1. Introduction

The AGM axiom system is for the belief revision (revision by a single belief) [2-5], and the DP axiom system is for the iterated revision (revision by a finite sequence of beliefs) [6,7]. These postulates list some basic requirements a revision operator $\Gamma\circ\Phi$ (a result of theory Γ revised by Φ) should satisfy.

The R-calculus ([1]) gave a Gentzen-type deduction system to deduce a consistent one $\Gamma'\cup\Delta$ from an inconsistent theory $\Gamma\cup\Delta$, where $\Gamma'\cup\Delta$ should be a maximal consistent subtheory of $\Gamma\cup\Delta$ which includes Δ as a subset, where $\Delta|\Gamma$ is an **R**-configuration, Γ is a consistent set of formulas, and Δ is a consistent sets of atomic formulas or the negation of atomic formulas. It was proved that if $\Delta|\Gamma\Rightarrow\Delta|\Gamma'$ is deducible and $\Delta|\Gamma'$ is an **R**-termination, *i.e.*, there is no **R**-rule to reduce $\Delta|\Gamma'$ to another **R**-configuration $\Delta|\Gamma''$, then $\Delta\cup\Gamma'$ is a contraction of Γ by Δ.

The R-calculus is set-inclusion, that is, Γ,Δ are taken as belief bases, not as belief sets [8-11]. In the following we shall take Δ,Γ as belief bases, not belief sets.

We shall define an operator $s(\Delta,t)$, where Δ is a set of theories and t is a theory in propositional logic programs, such that

- $\Delta,s(\Delta,t)$ is consistent;
- $s(\Delta,t)$ is a pseudo-subtheory of t;
- $s(\Delta,t)$ is maximal such that $\Delta,s(\Delta,t)$ is consistent, and for any pseudo-subtheory η of t, if $s(\Delta,t)$ is a pseudo-subtheory of η and η is not a pseudo-subtheory of η then either $\Delta,\eta\vdash s(\Delta,t)$ and $\Delta,s(\Delta,t)\vdash\eta$, or Δ,η is inconsistent.

Then, we give one R-calculus **N** such that **N** is sound and complete with respect to operator $s(\Delta,t)$, where

- **N** is sound with respect to operator $s(\Delta,t)$, if $\Delta|t\Rightarrow\Delta,s$ being provable implies $s=s(\Delta,t)$, and
- **N** is complete with respect to operator $s(\Delta,t)$, if $\Delta|t\Rightarrow\Delta,s(\Delta,t)$ is provable.

Let $\sqsubseteq$ be the pseudo-subtheory relation, $P(t)$ be the set of all the pseudo-subtheories of t, and $\equiv_\Delta$ be an equivalence relation on $P(t)$ such that for any $\eta_1,\eta_2\in P(t),\eta_1\equiv_\Delta\eta_2$ iff $\Delta,\eta_1\vdash\dashv\Delta,\eta_2$. Given a pseudo-subtheory $\eta\sqsubseteq t$, let $[\eta]$ be the equivalence class of r with respect to $\equiv_\Delta$.

About the minimal change, we prove that $[s(\Delta,t)]$ is $\sqsubseteq$-maximal in $P(t)/\equiv_\Delta$ such that $\Delta,s(\Delta,t)$ is consistent, that is,

- ♦ $\Delta,s(\Delta,t)$ is consistent; and
- ♦ for any η such that $[s(\Delta,t)]\sqsubseteq[\eta]\sqsubseteq[t]$, either $[\eta]\sqsubseteq[s(\Delta,t)]$ or Δ,η is inconsistent.

$[s(\Delta,t)]$ being $\sqsubseteq$-maximal implies that $s(\Delta,t)$ is a minimal change of t by Δ in the syntactical sense, not in the set-theoretic sense, *i.e.*, $s(\Delta,t)$ is a minimal change of t by Δ in the theoretic form such that $s(\Delta,t)$

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is consistent with Δ.

The paper is organized as follows: the next section gives the basic elements of the **R**-calculus and the definition of subtheories and pseudo-subtheories; the third section defines the *R*-calculus **N**; the fourth section proves that **N** is sound and complete with respect to the operator $s(\Delta,t)$; the fifth section discusses the logical properties of t and $s(\Delta,t)$, and the last section concludes the whole paper.

2. The *R*-Calculus

The *R*-calculus ([1]) is defined on a first-order logical language. Let L' be a logical language of the first-order logic; φ,ψ formulas and Γ,Δ sets of formulas (theories), where Δ is a set of atomic formulas or the negations of atomic formulas.

Given two theories Γ and Δ, let $\Delta \mid \Gamma$ be an **R**-configuration.

The **R**-calculus consists of the following axiom and inference rules:

$$(\mathbf{A}^{\neg})\quad \Delta,\neg\varphi \mid \varphi,\Gamma \Rightarrow \Delta,\neg\varphi \mid \Gamma$$

$$(\mathbf{R}^{\mathrm{cut}})\quad \frac{\Gamma_1,\varphi \vdash \psi \quad \varphi \mapsto_T \psi \quad \Gamma_2,\psi \vdash \chi \quad \Delta \mid \chi,\Gamma_2 \Rightarrow \Delta \mid \Gamma_2}{\Delta \mid \varphi,\Gamma_1,\Gamma_2 \Rightarrow \Delta \mid \Gamma_1,\Gamma_2}$$

$$(\mathbf{R}^{\wedge})\quad \frac{\Delta \mid \varphi,\Gamma \Rightarrow \Delta \mid \Gamma}{\Delta \mid \varphi \wedge \psi,\Gamma \Rightarrow \Delta \mid \Gamma}$$

$$(\mathbf{R}^{\vee})\quad \frac{\Delta \mid \varphi,\Gamma \Rightarrow \Delta \mid \Gamma \quad \Delta \mid \psi,\Gamma \Rightarrow \Delta \mid \Gamma}{\Delta \mid \varphi \vee \psi,\Gamma \Rightarrow \Delta \mid \Gamma}$$

$$(\mathbf{R}^{\to})\quad \frac{\Delta \mid \neg\varphi,\Gamma \Rightarrow \Delta \mid \Gamma \quad \Delta \mid \psi,\Gamma \Rightarrow \Delta \mid \Gamma}{\Delta \mid \varphi \to \psi,\Gamma \Rightarrow \Delta \mid \Gamma}$$

$$(\mathbf{R}^{\forall})\quad \frac{\Delta \mid \varphi[t/x],\Gamma \Rightarrow \Delta \mid \Gamma}{\Delta \mid \forall x\varphi,\Gamma \Rightarrow \Delta \mid \Gamma}$$

where in $\mathbf{R}^{\mathrm{cut}}, \varphi \mapsto_T \psi$ means that φ occurs in the proof tree T of ψ from Γ_1 and φ; and in $R^{\forall}, t$ is a term, and is free in φ for x.

Definition 2.1. $\Delta \mid \Gamma \Rightarrow \Delta' \mid \Gamma'$ is an **R**-theorem, denoted by $\vdash^R \Delta \mid \Gamma \Rightarrow \Delta' \mid \Gamma'$, if there is a sequence $\{(\Delta_i,\Gamma_i,\Delta_{i'},\Gamma_{i'}) : i \le n\}$ such that

(i) $\Delta \mid \Gamma \Rightarrow \Delta' \mid \Gamma' = \Delta_n \mid \Gamma_n \Rightarrow \Delta_{n'} \mid \Gamma_{n'}$,

(ii) for each $1 \le i \le n$, either $\Delta_i \mid \Gamma_i \Rightarrow \Delta'_i \mid \Gamma'_i$ is an axiom, or $\Delta_i \mid \Gamma_i \Rightarrow \Delta'_i \mid \Gamma'_i$ is deduced by some **R**-rule of form $\dfrac{\Delta_{i-1} \mid \Gamma_{i-1} \Rightarrow \Delta'_{i-1} \mid \Gamma'_{i-1}}{\Delta_i \mid \Gamma_i \Rightarrow \Delta'_i \mid \Gamma'_i}$.

Definition 2.2. $\Delta \mid \Gamma \Rightarrow \Delta \mid \Gamma'$ is valid, denoted by $\vDash \Delta \mid \Gamma \Rightarrow \Delta \mid \Gamma'$, if for any contraction Θ of Γ' by Δ,Θ is a contraction of Γ by Δ.

Theorem 2.3(The soundness and completeness theorem of the **R**-calculus). For any theories Γ,Γ' and Δ,

$$\vdash \Delta \mid \Gamma \Rightarrow \Delta \mid \Gamma'$$

if and only if

$$\vDash \Delta \mid \Gamma \Rightarrow \Delta \mid \Gamma'.$$

Theorem 2.4. The **R**-rules preserve the strong validity.

Let L be the logical language of the propositional logic. A literal l is an atomic formula or the negation of an atomic formula; a clause c is the disjunction of finitely many literals, and a theory t is the conjunction of finitely many clauses.

Definition 2.5. Given a theory t, a theory s is a sub-theory of t, denoted by $s \preceq t$, if either $t = s$, or

(i) if $t = \neg t_1$ then $s \preceq t_1$;

(ii) if $t = t_1 \wedge t_2$ then either $s \preceq t_1$ or $s \preceq t_2$; and

(iii) if $t = c_1 \vee c_2$ then either $s \preceq c_1$ or $s \preceq c_2$.

Let $t = (p \vee q) \wedge (p' \vee q')$. Then,

$$p \vee q, p' \vee q' \preceq t;$$

and

$$p \wedge p', q \wedge p', p \wedge (p' \vee q') \npreceq t.$$

Definition 2.6. Given a theory $t[s_1,\dots,s_n]$, where s_1 is an occurrence of s_1 in t, a theory $s = t[\lambda,\dots,\lambda] = t[s_1/\lambda,\dots,s_n/\lambda]$, where the occurrence s_i is replaced by the empty theory λ, is called a pseudo-subtheory of t, denoted by $s \sqsubseteq t$.

Let $t = (p \vee q) \wedge (p' \vee q')$. Then,

$$p \vee q, p' \vee q', p \wedge p', q \wedge p', p \wedge (p' \vee q') \sqsubseteq t.$$

Proposition 2.7. For any theories t_1, t_2, s_1 and s_2,

(i) $s_1 \preceq t_1$ implies $s_1 \preceq t_1 \vee t_2$ and $s_1 \preceq t_1 \wedge t_2$;

(ii) $s_1 \sqsubseteq t_1$ and $s_2 \sqsubseteq t_2$ imply $\neg s_1 \sqsubseteq \neg t_1, s_1 \vee s_2 \sqsubseteq t_1 \vee t_2$ and $s_1 \wedge s_2 \sqsubseteq t_1 \wedge t_2$.

Proposition 2.8. For any theories t and s, if $s \preceq t$ then $s \sqsubseteq t$.

Proof. By the induction on the structure of t.

Proposition 2.9. $\preceq$ and $\sqsubseteq$ are partial orderings on the set of all the theories.

Given a theory t, let $P(t)$ be the set of all the pseudo-subtheories of t. Each $s \in P(t)$ is determined by a set $\tau(s) = \{[p_1],\dots,[p_n]\}$, where each $[p_i]$ is an occurrence of p_i in t, such that

$$s = t([p_1]/\lambda,\dots,[p_n]/\lambda).$$

Given any $s_1, s_2 \in P(t)$, define

$$s_1 \sqcap s_2 = \max\{s : s \sqsubseteq s_1, s \sqsubseteq s_2\};$$
$$s_1 \sqcap s_2 = \min\{s : s \sqsupseteq s_1, s \sqsupseteq s_2\}.$$

Proposition 2.10. For any pseudo-subtheories $s_1, s_2 \in P(t), s_1 \sqcap s_2$ and $s_1 \sqcup s_2$ exist.

Let $P(t) = (P(t), \sqcup, \sqcap, t, \lambda)$ be the lattice with the greatest element t and the least element λ.

Proposition 2.11. For any pseudo-subtheories $s_1, s_2 \in P(t), s_1 \sqsubseteq s_2$ if and only if $\tau(s_1) \supseteq \tau(s_2)$. Moreover,

$$\tau(s_1 \sqcap s_2) = \tau(s_1) \cup \tau(s_2);$$
$$\tau(s_1 \sqcup s_2) = \tau(s_1) \cap \tau(s_2).$$

3. The *R*-Calculus N

The deduction system **N**:

$$(N_1^a)\ \frac{\Delta \nvdash \neg l}{\Delta \mid l \Rightarrow \Delta, l} \qquad (N_2^a)\ \frac{\Delta \vdash \neg l}{\Delta \mid l \Rightarrow \Delta, \lambda}$$

$$(N^{\wedge})\ \frac{\Delta \mid t_1 \Rightarrow \Delta, s_1}{\Delta \mid t_1 \wedge t_2 \Rightarrow \Delta, s_1 \mid t_2}$$

$$(N^{\vee})\ \frac{\Delta \mid c_1 \Rightarrow \Delta, d_1 \ \ \Delta \mid c_2 \Rightarrow \Delta, d_2}{\Delta \mid c_1 \vee c_2 \Rightarrow \Delta, d_1 \vee d_2}$$

where Δ, t denotes a theory $\Delta \cup \{t\}$; λ is the empty string, and if s is consistent then

$$\lambda \vee s \equiv s \vee \lambda \equiv s$$
$$\lambda \wedge s \equiv s \wedge \lambda \equiv s$$
$$\Delta, \lambda \equiv \Delta$$

and if s is inconsistent then

$$\lambda \vee s \equiv s \vee \lambda \equiv \lambda$$
$$\lambda \wedge s \equiv s \wedge \lambda \equiv \lambda$$

Definition 3.1. $\Delta \mid t \Rightarrow \Delta, s$ is **N**-provable if there is a statement sequence $\{\Delta_i \mid t_i \Rightarrow \Delta_i, s_i : 1 \leq i \leq n\}$ such that

$$\Delta \mid t \Rightarrow \Delta, s = \Delta_n \mid t_n \Rightarrow \Delta_n, s_n,$$

and for each $i \leq n$, $\Delta_i \mid t_i \Rightarrow \Delta_i, s_i$ is either by an N^a-rule or by an $N^{\wedge}$-,or $N^{\vee}$-rule.

An example is the following deduction for $\neg l_1 \mid l_1 \vee l_2, l_1 \vee \neg l_2$:

$$\begin{array}{lll}
\neg l_1 \mid l_1 & \Rightarrow & \neg l_1 \\
\neg l_1 \mid l_2 & \Rightarrow & \neg l_1, l_2 \\
\neg l_1 \mid l_1 \vee l_2 & \Rightarrow & \neg l_1, \lambda \vee l_2 \equiv \neg l_1, l_2 \\
\neg l_1, l_2 \mid l_1 & \Rightarrow & \neg l_1, l_2 \\
\neg l_1, l_2 \mid \neg l_2 & \Rightarrow & \neg l_1, l_2 \\
\neg l_1, l_2 \mid l_1 \vee \neg l_2 & \Rightarrow & \neg l_1, l_2 \\
\neg l_1 \mid l_1 \vee l_2, l_1 \vee \neg l_2 & \Rightarrow & \neg l_1, l_2 \mid l_1 \vee \neg l_2 \\
 & \Rightarrow & \neg l_1, l_2.
\end{array}$$

Notice that $\neg l_1 \mid l_1 \Rightarrow \neg l_1$ and $l_1 \equiv (l_1 \vee l_2) \wedge (l_1 \vee \neg l_2)$.

Theorem 3.2. For any theory set Δ and theory t, there is a theory s such that $\Delta \mid t \Rightarrow \Delta, s$ is **N**-provable.

Proof. We prove the theorem by the induction on the structure of t.

If $t = l$ is a literal then either $\Delta \vdash \neg l$ or $\Delta \nvdash \neg l$. If $\Delta \vdash \neg l$ then $\Delta \mid l \Rightarrow \Delta, \lambda$ and $s = \lambda$; if $\Delta \nvdash \neg l$ then $\Delta \mid l \Rightarrow \Delta, l$ and $s = l$;

If $t = t_1 \wedge t_2$ then by the induction assumption, there are theories s_1, s_2 such that $\Delta \mid t_1 \Rightarrow \Delta, s_1$ and $\Delta, s_1 \mid t_2 \Rightarrow \Delta, s_1, s_2$. Therefore, $\Delta \mid t_1 \wedge t_2 \Rightarrow \Delta, s_1, s_2$ and $s = s_1 \wedge s_2$.

If $t = c_1 \vee c_2$ then by the induction assumption, there are theories s_1, s_2 such that $\Delta \mid c_1 \Rightarrow \Delta, s_1$ and $\Delta \mid c_2 \Rightarrow \Delta, s_2$. Therefore, $\Delta \mid c_1 \vee c_2 \Rightarrow \Delta, s_1 \vee s_2$ and $s = s_1 \vee s_2$.

Proposition 3.3. If $\Delta \mid t \Rightarrow \Delta, s$ is **N**-provable then $s \sqsubseteq t$.

Proof. We prove the proposition by the induction on the length of the proof of $\Delta \mid t \Rightarrow \Delta, s$.

If the last rule used is (N_1^a) then $t = l$, and $s = l \sqsubseteq t = l$;

If the last rule used is (N_2^a) then $t = l$, and $s = \lambda \sqsubseteq t = l$;

If the last rule used is $(N^{\wedge})$ then $\Delta \mid t_1 \Rightarrow \Delta, s_1$ and $\Delta, s_1 \mid t_2 \Rightarrow \Delta, s_1, s_2$. By the induction assumption, $s_1 \sqsubseteq t_1$ and $s_2 \sqsubseteq t_2$. Hence, $s_1 \wedge s_2 \sqsubseteq t_1 \wedge t_2 = t$;

If the last rule used is $(N^{\wedge})$ then $\Delta \mid c_1 \Rightarrow \Delta, s_1$ and $\Delta \mid c_2 \Rightarrow \Delta, s_2$. By the induction assumption, $s_1 \sqsubseteq t_1$ and $s_2 \sqsubseteq t_2$. Hence, $s_1 \vee s_2 \sqsubseteq c_1 \vee c_2 = t$.

Proposition 3.4. If $\Delta \mid t \Rightarrow \Delta, s$ is **N**-provable then $\Delta \cup \{s\}$ is consistent.

Proof. We prove the proposition by the induction on the length of the proof of $\Delta \mid t \Rightarrow \Delta, s$.

If the last rule used is (N_1^a) then $\Delta \nvdash \neg l$, and $\Delta \mid l \Rightarrow \Delta, l$. Then, $\Delta \cup \{l\}$ is consistent;

If the last rule used is (N_2^a) then $\Delta \vdash \neg l$, and $\Delta \mid l \Rightarrow \Delta, \lambda$. Then, $\Delta \cup \{\lambda\}$ is consistent;

If the last rule used is $(N^{\wedge})$ then $\Delta \mid t_1 \Rightarrow \Delta, s_1$ and $\Delta, s_1 \mid t_2 \Rightarrow \Delta, s_1, s_2$. By the induction assumption, $\Delta \cup \{s_1\}$ and $\Delta \cup \{s_1, s_2\}$ is consistent, and so is $\Delta \cup \{s_1 \wedge s_2\} = \Delta \cup \{s\}$;

If the last rule used is $(N^{\vee})$ then $\Delta \mid c_1 \Rightarrow \Delta, s_1$ and $\Delta \mid c_2 \Rightarrow \Delta, s_2$. By the induction assumption, $\Delta \cup \{s_1\}$ and $\Delta \cup \{s_2\}$ is consistent, and so is $\Delta \cup \{s_1 \vee s_2\} = \Delta \cup \{s\}$.

4. The Completeness of the *R*-Calculus N

For any theory t, define $s(\Delta, t)$ as follows:

$$s(\Delta, t) = \begin{cases} \lambda & \text{if } t = l \text{ and } \Delta \vdash \neg l \\ l & \text{if } t = l \text{ and } \Delta \nvdash \neg l \\ s(\Delta, t_1) \wedge s(\Delta \cup \{s(\Delta, t_1)\}, t_2) & \\ & \text{if } t = t_1 \wedge t_2 \\ s(\Delta, t_1) \vee s(\Delta, t_2) & \text{if } t = t_1 \vee t_2 \end{cases}$$

About the inconsistence, we have the following facts:

- if $\Delta \vdash \neg l$ then $\Delta \cup \{l\}$ is inconsistent;
- $\Delta \cup \{t_1 \wedge t_2\}$ is inconsistent if and only if either $\Delta \cup \{t_1\}$ is inconsistent or $\Delta \cup \{t_1, t_2\}$ is in- consistent;
- $\Delta \cup \{c_1 \vee c_2\}$ is inconsistent if and only if both $\Delta \cup \{c_1\}$ and $\Delta \cup \{c_2\}$ are inconsistent.

Proposition 4.1. For any consistent theory set Δ and a theory $t, \Delta\cup\{s(\Delta,t)\}$ is consistent.

Proof. We prove the proposition by the induction on the structure of t.

If $t=l$ and l is consistent with Δ then $s(\Delta,l)=l$ is consistent with Δ; if $t=l$ and l is inconsistent with Δ then $s(\Delta,l)=\lambda$ is consistent with Δ;

If $t=t_1\wedge t_2$ then by the induction assumption, $\Delta\cup\{s(\Delta,t_1)\}$ and $\Delta\cup\{s(\Delta,t_1), s(\Delta\cup\{s(\Delta,t_1)\},t_2)\}$ are consistent, so $\Delta\cup\{s(\Delta,t_1\wedge t_2)\}$ is consistent;

If $t=c_1\vee c_2$ then by the induction assumption, $\Delta\cup\{s(\Delta,c_1)\}$ and $\Delta\cup\{s(\Delta,c_2)\}$ are consistent, so $\Delta\cup\{s(\Delta,c_1\vee c_2)\}=\Delta\cup\{s(\Delta,c_1)\vee s(\Delta,c_2)\}$ is consistent.

About the consistence, we have the following facts:

- if $\Delta\nvdash\neg l$ then $\Delta\cup\{l\}$ is consistent;
- $\Delta\cup\{t_1\wedge t_2\}$ is consistent if and only if $\Delta\cup\{t_1\}$ is consistent and $\Delta\cup\{t_1,t_2\}$ is consistent;
- $\Delta\cup\{c_1\vee c_2\}$ is consistent if and only if either $\Delta\cup\{c_1\}$ or $\Delta\cup\{c_2\}$ is consistent.

Theorem 4.2. If $\Delta\cup\{t\}$ is consistent then $\Delta,t\vdash s(\Delta,t)$ and $\Delta,s(\Delta,t)\vdash t$.

Proof. We prove the theorem by the induction on the structure of t.

If $t=l$ and l is consistent with Δ then $s(\Delta,l)=l$, and the theorem holds for l;

If $t=t_1\wedge t_2$ then $\Delta\cup\{t_1\}$ and $\Delta\cup\{t_1,t_2\}$ is consistent, and by the induction assumption,

$$\begin{array}{lll}\Delta,t_1 & \vdash & s(\Delta,t_1)\\ \Delta,s(\Delta,t_1) & \vdash & t_1;\\ \Delta,s_1,t_2 & \vdash & s(\Delta\cup\{s_1\},t_2)\\ \Delta,s(\Delta\cup\{s_1\},t_2) & \vdash & t_2,\end{array}$$

where $s_1=s(\Delta,t_1)$. Hence,

$$\begin{array}{lll}\Delta,t_1\wedge t_2 & \vdash & s(\Delta,t_1)\wedge s(\Delta\cup\{s_1\},t_2)\\ \Delta,s(\Delta,t_1)\wedge s(\Delta\cup\{s_1\},t_2) & \vdash & t_1\wedge t_2.\end{array}$$

If $t=c_1\vee c_2$ then either $\Delta\cup\{c_1\}$ or $\Delta\cup\{c_1,c_2\}$ is consistent, and by the induction assumption, either

$$\begin{array}{lll}\Delta,c_1 & \vdash & s(\Delta,c_1)\\ \Delta,s(\Delta,c_1) & \vdash & c_1;\end{array}$$

or

$$\begin{array}{lll}\Delta,c_2 & \vdash & s(\Delta,c_2)\\ \Delta,s(\Delta,c_2) & \vdash & c_2.\end{array}$$

Hence, we have

$$\begin{array}{lll}\Delta,c_1\vee c_2 & \vdash & s(\Delta,c_1)\vee s(\Delta,c_2)\\ \Delta,s(\Delta,c_1)\vee s(\Delta,c_2) & \vdash & c_1\vee c_2.\end{array}$$

Theorem 4.3. $\Delta\,|\,t\Rightarrow\Delta,s$ is **N**-provable if and only if $s=s(\Delta,t)$.

Proof. $(\Rightarrow)$ Assume that $\Delta\,|\,t\Rightarrow\Delta,s$ is **N**-provable. We assume that for any $i<n$, the claim holds.

If $t=l$ and the last rule is (N_1^a) then $\Delta\nvdash\neg l$ and $\Delta\,|\,l\Rightarrow\Delta,l$. It is clear that $s=l=s(\Delta,l)$;

If $t=l$ and the last rule is (N_2^a) then $\Delta\vdash\neg l$ and $\Delta\,|\,l\Rightarrow\Delta,\lambda$. It is clear that $s=\lambda=s(\Delta,l)$;

If $t=t_1\wedge t_2$ and the last rule is $(N^\wedge)$ then $\Delta\,|\,t_1\Rightarrow\Delta,s_1$ and $\Delta\,|\,t_1\wedge t_2\Rightarrow\Delta,s_1\,|\,t_2\Rightarrow\Delta,s_1,s_2$. By the induction assumption, $s(\Delta,t_1)=s_1$ and $s(\Delta\cup\{s_1\},t_2)=s_2$. Then, $s=s_1\wedge s_2=s(\Delta,t_1)\wedge s(\Delta\cup\{s_1\},t_2)=s(\Delta,t_1\wedge t_2)$;

If $t=c_1\vee c_2$ and the last rule is $(N^\vee)$ then $\Delta\,|\,c_1\Rightarrow\Delta,s_1$ and $\Delta\,|\,c_2\Rightarrow\Delta,s_2$. By the induction assumption, $s_1=s(\Delta,c_1), s_2=s(\Delta,c_2)$, and $s=s_1\vee s_2=s(\Delta,c_1)\vee s(\Delta,c_2)=s(\Delta,c_1\vee c_2)$.

$(\Leftarrow)$ Let $s=s(\Delta,t)$. We prove that $\Delta\,|\,t\Rightarrow\Delta,s$ is **N**-provable by the induction on the structure of t.

If $t=l$ and $\Delta\vdash\neg l$ then $s(\Delta,l)=\lambda$, and $\Delta\,|\,l\Rightarrow\Delta,\lambda$, *i.e.*, $\Delta\,|\,l\Rightarrow\Delta,s$;

If $t=l$ and $\Delta\nvdash\neg l$ then $s(\Delta,l)=l$, and $\Delta\,|\,l\Rightarrow\Delta,l$, *i.e.*, $\Delta\,|\,l\Rightarrow\Delta,s$;

If $t=t_1\wedge t_2$ then $s(\Delta,t_1\wedge t_2)=s(\Delta,t_1)\wedge s(\Delta\cup\{s(\Delta,t_1)\},t_2)$. By the induction assumption, $\Delta\,|\,t_1\Rightarrow\Delta,s(\Delta,t_1)$ and $\Delta,s_1\,|\,t_2\Rightarrow\Delta,s_1,s(\Delta\cup\{s(\Delta,t_1)\},t_2)$. Therefore, $\Delta\,|\,t_1\wedge t_2\Rightarrow\Delta,s_1,s(\Delta\cup\{s(\Delta,t_1)\},t_2)$;

If $t=c_1\vee c_2$ then $s(\Delta,c_1\vee c_2)=s(\Delta,c_1)\vee s(\Delta\cup\{s(\Delta,c_1)\},c_2)$. By the induction assumption, $\Delta\,|\,c_1\Rightarrow\Delta,s(\Delta,c_1)$ and $\Delta\,|\,c_2\Rightarrow\Delta,s(\Delta,c_2)$. Therefore, $\Delta\,|\,c_1\vee c_2\Rightarrow\Delta,s(\Delta,c_1)\vee s(\Delta,c_2)$.

5. The Logical Properties of t and $s(\Delta,t)$

It is clear that we have the following

Proposition 5.1. For any theory set Δ and theory t,

$$\xi(\Delta,t)\sqsubseteq s(\Delta,t).$$

Theorem 5.2. For any theory set Δ and theory t,

$$\begin{array}{lll}\Delta,\xi(\Delta,t) & \vdash & s(\Delta,t);\\ \Delta,s(\Delta,t) & \vdash & \xi(\Delta,t).\end{array}$$

Proof. By the definitions of $s(\Delta,\xi),\xi(\Delta,t)$ and the induction on the structure of t.

Proposition 5.3. (i) If $\Delta,s(\Delta,t)\nvdash t$ then Δ,t is inconsistent;

(ii) If $\Delta,s(\Delta,t)\vdash t$ then Δ,t is consistent.

Define

$$\begin{array}{l}C_t^\Delta=\{s\in P(t):\Delta\cup\{s\}\text{ is consistent}\};\\ I_t^\Delta=\{s\in P(t):\Delta\cup\{s\}\text{ is inconsistent}\}.\end{array}$$

Then, $C_t^\Delta\cup I_t^\Delta=P(t)$ and $C_t^\Delta\cap I_t^\Delta=\varnothing$.

Define an equivalence relation $\equiv_\Delta$ on $\mathbf{P}(t)$ such that for any $s_1,s_2\in P(t)$,

$$s_1\equiv_\Delta s_2\ \text{ iff }\ \Delta,s_1\vdash\dashv\Delta,s_2.$$

Given a pseudo-subtheory $s \in P(t)$, let $[r]$ be the equivalence class of s. Then, we have that

$$[s(\Delta,t)],[\xi(\Delta,t)] \subseteq C_t^{\Delta}.$$

Proposition 5.4. $[s(\Delta,t)] = [\xi(\Delta,t)]$.

Define a relation $\simeq$ on $P(t)$ such that for any s_1 and $s_2 \in P(t), s_1 \simeq s_2$ iff

$$\begin{cases} l_1 = l_2 & \text{if } s_1 = l_1 \textit{ and } s_2 = l_2 \\ c_{11} = c_{22} \,\&\, c_{12} = c_{21} \text{ o } c_{11}\text{r} = c_{21} \,\&\, c_{12} = c_{22} & \\ & \text{if } s_1 = c_{11} \vee c_{12} \text{ and } s_2 = c_{21} \vee c_{22} \\ s_{11} = s_{22} \,\&\, s_{12} = s_{21} \text{ o } s_{11}\text{r} = s_{21} \,\&\, s_{12} = s_{22} & \\ & \text{if } s_1 = s_{11} \wedge s_{12} \text{ and } s_2 = s_{21} \wedge s_{22} \end{cases}$$

Proposition 5.5. $\simeq$ is an equivalence relation on $P(t)$, and for any $s_1, s_2 \in P(t)$, if $s_1 \simeq s_2$ then $s_1 \vdash\dashv s_2$.

Theorem 5.6. If $\Delta \mid t \Rightarrow \Delta, s$ is provable then for any η with $s \sqsubseteq \eta \sqsubseteq t, \Delta \mid \eta \Rightarrow \Delta, s$ is provable.

Proof. We prove the theorem by the induction on the structure of t.

If $t = l$ and $\Delta \vdash \neg l$ then $s = \lambda$, and for any η with $s \sqsubseteq \eta \sqsubseteq t, \eta = \lambda$, and $\Delta \mid \eta \Rightarrow \Delta, \lambda$ is provable;

If $t = l$ and $\Delta \nvdash \neg l$ then $s = l$, and for any η with $s \sqsubseteq \eta \sqsubseteq t, \eta = l$, and $\Delta \mid \eta \Rightarrow \Delta, s$ is provable;

If $t = t_1 \wedge t_2$ and the theorem holds for both t_1 and t_2 then $s = s_1 \wedge s_2$, and for any η with $s \sqsubseteq \eta \sqsubseteq t$, there are η_1 and η_2 such that $s_1 \sqsubseteq \eta_1 \sqsubseteq t_1$ and $s_2 \sqsubseteq \eta_2 \sqsubseteq t_2$. By the induction assumption, $\Delta \mid \eta_1 \Rightarrow \Delta, s_1, \Delta, s_1 \mid \eta_2 \Rightarrow \Delta, s_1, s_2$, and by $(N^{\wedge})$, $\Delta \mid \eta_1 \wedge \eta_2 \Rightarrow \Delta, s_1, s_2 \equiv \Delta, s_1 \wedge s_2$;

If $t = c_1 \vee c_2$ and the theorem holds for both c_1 and c_2 then $s = s_1 \vee s_2$, and for any η with $s \sqsubseteq \eta \sqsubseteq t$, there are η_1 and η_2 such that $s_1 \sqsubseteq \eta_1 \sqsubseteq c_1$ and $s_2 \sqsubseteq \eta_2 \sqsubseteq c_2$. By the induction assumption, $\Delta \mid \eta_1 \Rightarrow \Delta, s_1; \Delta \mid \eta_2 \Rightarrow \Delta, s_2$, and by $(N^{\vee})$, $\Delta \mid \eta_1 \vee \eta_2 \Rightarrow \Delta, s_1 \vee s_2$.

Theorem 5.7. For any η with $s \sqsubseteq \eta \sqsubseteq t$, if Δ, η is consistent then $\Delta, \eta \vdash\dashv \Delta, s$, and hence, $[\eta] = [s]$; and if Δ, η is inconsistent then $\Delta, \eta \vdash\dashv \Delta, t$, and hence, $[\eta] = [t]$.

Proof. If Δ, η is consistent then by Theorem 6.6, $\Delta \mid \eta \Rightarrow \Delta, s$, and we prove by the induction on the structure of t that $\Delta, t \vdash\dashv \Delta, s$.

If $t = l$ and $\Delta \nvdash \neg l$ then $s = l$, and $\Delta, t \vdash\dashv \Delta, s$;

If $t = t_1 \wedge t_2$ and the claim holds for both t_1 and t_2 then $s = s_1 \wedge s_2, \Delta, t_1 \vdash\dashv \Delta, s_1$ and $\Delta, t_2 \vdash\dashv \Delta, s_2$. Therefore, $\Delta, t_1 \wedge t_2 \vdash\dashv \Delta, s_1 \wedge s_2$.

If $t = c_1 \vee c_2$ and the theorem holds for both c_1 and c_2 then $d = d_1 \vee d_2$, and there are three cases:

Case 1. Δ, c_1 and Δ, c_2 are consistent. By the induction assumption, we have that $\Delta, c_1 \vdash\dashv \Delta, d_1, \Delta, c_2 \vdash\dashv \Delta, d_2$, and hence, $\Delta, c_1 \vee c_2 \vdash\dashv \Delta, d_1 \vee d_2$;

Case 2. Δ, c_1 is consistent and Δ, c_2 is inconsistent. By the induction assumption, we have that $\Delta, c_1 \vdash\dashv \Delta, d_1$, and $\Delta \mid c_2 \Rightarrow \Delta$. Then,

$$\begin{aligned} \Delta, d_1 &\equiv \Delta, d_1 \vee d_2 \\ &\vdash c_1 \\ &\vdash c_1 \vee c_2; \end{aligned}$$

and

$$\begin{aligned} \Delta, c_1 \vee c_2 &\equiv (\Delta \wedge c_1) \vee (\Delta \wedge c_2) \\ &\equiv \Delta \wedge c_1 \\ &\equiv \Delta, c_1 \vdash d_1 \vdash d_1 \vee d_2, \end{aligned}$$

where $d_2 = \lambda$.

Case 3. Similar to Case 2.

Corollary 5.8. For any η with $s \sqsubseteq \eta \sqsubseteq t$, either $[\eta] = [s]$ or $[\eta] = [t]$. Therefore, $[s]$ is $\sqsubseteq$-maximal such that Δ, s is consistent.

6. Conclusions and Further Works

We defined an *R*-calculus **N** in propositional logic programs such that **N** is sound and complete with respect to the operator $s(\Delta, t)$.

The following axiom is one of the AGM postulates:

$$\text{Extensionality: if } p \vdash\dashv q \text{ then } K \circ p = K \circ q$$

It is satisfied, because we have the following

Proposition 7.1. If $t_1 \vdash\dashv t_2; t_1 \mid s \Rightarrow t_1, s_1$ and $t_2 \mid s \Rightarrow t_2, s_2$ then $s_1 \vdash\dashv s_2$.

It is not true in **N** that

(*) if $s_1 \vdash\dashv s_2; t \mid s_1 \Rightarrow t, s_1'$ and $t \mid s_2 \Rightarrow t, s_2'$ then $s_1' \vdash\dashv s_2'$.

A further work is to give an *R*-calculus having the property $(*)$.

A simplified form of $(*)$ is

(**) if $s_1 \simeq s_2; t \mid s_1 \Rightarrow t, s_1'$ and $t \mid s_2 \Rightarrow t, s_2'$ then $s_1' \vdash\dashv s_2'$, which is not true in **N** either.

Another further work is to give an *R*-calculus having the property $(**)$ and having not the property $(*)$.

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