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Is the Space-Time a Superconductor?*

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ABSTRACT

At the fundamental level, the 4-dimensional space-time of our direct experience might not be a continuum and discrete quantum entities might "collectively" rule its dynamics. Henceforth, it seems natural to think that in the "low-energy" regime some of its distinctive quantum attributes could, in principle, manifest themselves even at macroscopically large scales. Indeed, when confronted with Nature, classical gravitational dynamics of spinning astrophysical bodies is known to lead to paradoxes: to untangle them, dark matter or modifications to the classical law of gravity are openly considered. In this article, the hypothesis of a fluctuating space-time acquiring "at large distances" the properties of a Bose-Einstein condensate is pushed forward: firstly, it is shown that a natural outcome of this picture is the production of monopoles, dyons, and vortex lines of "quantized" gravitomagnetic—or gyrogravitational—flux along the transition phase; the minimal supported "charge" (and multiples of it) being directly linked with a nonzero (minimal) vacuum energy. Thus, a world of vibrating, spinning, interacting strings whose only elements in their construction are our topological concepts of space and time is envisioned, and they are proposed as tracers of the superfluid features of the space-time: the archetypal embodiment of these physical processes being set by the "gravitational roton", an analogue of Landau's classic higher-energy excitation used to explain the superfluid properties of helium II. The far and the near field asymptotics of string line solutions are presented and used to deduce their pair-interaction energy. Remarkably, it is found that two stationary, axis-aligned, quantum space-time vortices with the same sense of spin not only exhibit zones of repulsion but also of attraction, depending on their relative geodetic distance.

Keywords: Modified Gravity; Superconductivity; Kinematics; Dynamics; Rotation

1. Introduction

Spiral patterns extending over a large portion of the stelar disk of many galaxies are seen everywhere in the cosmos. Thus, it may seem as if these majestic structures were stable features over a time of many orbital periods. Yet, current theory has a hard time to come up with a convincing explanation of their origin and stability. From the "coffee-cup" theory suggested by von Weizsaecker [1] to the spiral density wave theory of B. Lindblad [2], C.C. Lin, and F. Shu [3,4], it is fair to say, this basic problem of formation and stability of spiral galaxies is still not fully understood. In this article, this very crucial question is reversed, by imagining the sort of features a space-time needs to fulfill in order to explain this apparent stability as a pure gravitational phenomena, without invoking-a priori-the need of cold dark matter. More precisely, V. Rubin's discovery (of an almost constant velocity flow of cool hydrogen clouds outside the bright parts of large spiral galaxies) is pictured here as an indication that the geometry along these special regions is rather uniform, the test orbiting bodies receive the same code of instructions, and the unexplained stiffness in the geometry is primarily due-according to the launched hypothesis-to a second order phase transition where the space-time acquires, at low curvatures, the properties of a superfluid. Basically, Weizsaecker's "coffee-cup" analogy [1] is replaced by a "superfluid-cup" one, where phonons and rotons can flow, see **Figure 1**.

Can the geodesic motion of a radial alignment of test particles resist the winding process when the space-time is a superfluid [5,6]? How, in the first place, do quantum vortices behave if the space-time is a superfluid? In this article a research program is commenced by examining fully this second opening issue.

It should be stressed that the catalog of spiral galaxies is indeed vast: the so called grand design spirals have a well defined two-arm structure, but some others present multiple arms not necessarily symmetrical spaced, while there are others-referred to as flocculent spirals-showing sporadic spiral arm segments [5]; spiral patterns of a very

*This paper is dedicated to the memory of Dr. J. Ize.

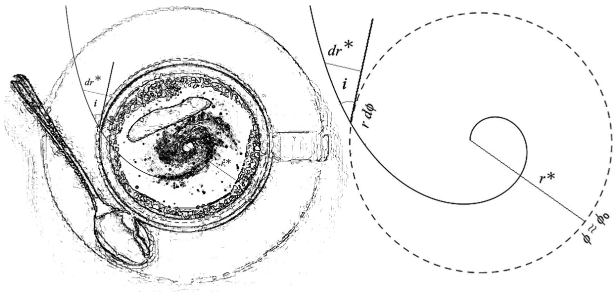

Figure 1. Left: coffee-cup analogy. Right: winding dilemma. Let r^* and r be respectively the radial distance to a central point and the local spherical radius. If a thin disk of matter rotates around such a point with an angular velocity $\Omega\left(r^*\right)\neq 0$, and if $\Omega_{,r^*}\neq 0$; then, an original aligned configuration

$$\phi=\Omega\left(r^*\right)t+\phi_o$$

of physical bodies belonging to the disk will transform, over time, into a spiral arm with a pitching angle i approximately given by

$$\cot i=\left|r\,\mathrm{d}\phi/\mathrm{d}r^*\right|=\left|r\left(\mathrm{d}\Omega/\mathrm{d}r^*\right)\right|t;t\geq 0.$$

And when the local spherical radius r is assumed to change as the radial distance r^*, *i.e.* $r\approx r^*+\text{constant}$, it is concluded that for typical galaxies the spiral arm must be tightly wound, contrary to observation. A rotational velocity

$$\Omega r^*\approx 220\ \mathrm{km\cdot s^{-1}}\ \ (\textit{i.e.}\ \ \Omega\propto 1/r^*\ ),\ \ r\approx r^*=10\ \mathrm{Kpc}\ ,$$

and $t=10^{10}$ yr implies $i\approx 0.25^\circ$ as well as an interarm separation of $0.28\ \mathrm{Kpc}$ [see [5]]. Presumably such alleged relationship between r and r^* breaks down in spiral galaxies [6].

bizarre shape also show up in Nature: for instance, the spiral galaxy NGC4622 not only posses inner spiral arms that are trailing but also has a pair of outer arms that are leading, contrary to most expectations [7]. The oddest thing of all is that according to standard theory, if the material originally making up a spiral arm remains in the arm; then, the differential rotation of the galaxy will wind up the arm in a time short compare with the age of the galaxy. But most spiral arms (often logarithmic in nature) are far from being too tightly wound, with a pitch angle absolute value ranging from 6° to 27° [8-10]. How can this be?

This acute observation creates a fundamental challenge to theories on the origin of the spiral structure and it is referred to as "the winding dilemma" [11]. A description of this winding process, when there is an annular disk of material with a constant pattern speed-thus fulfilling the flat rotation curve criterion-is given in **Figure 1**.

At first sight, these bearings seem no different if one assumes that the orbiting objects are governed by Kepler's laws of planetary motion or if they move with an approximately constant pattern speed; that is why, in the 1960s, an hypothesis was advanced: where the spiral features were assumed not only to be long lasting, but also that they were the result of a quasi-stationary density wave that rotated rigidly, at a slow paced rate, through the galactic disk-meaning in particular that stars should stream in and out of the spiral arms as they orbit the galaxy. This theory, however, has not been satisfactorily confirmed as even the question of longevity of the spiral arms, whether they are short-live transient patters (perhaps breaking apart and reforming periodically) or not has not yet been settled [12,13]. In Binney & Tremaine comprehensive treatise on galaxy formation this peculiar situation is depicted as follows [5]:

"*The common thread of several of these mechanism is that because of the swing amplifier, galactic disk respond with remarkable vigor to a wide variety of perturbations, whether these be tidal forces, gravitational instability of some local pattern of gas or stars, or fresh leading density waves. In some cases there is clear evidence that Lindblad's original conception of the spiral arm as a density wave is correct. However, there is little or no direct evidence for the hypothesis that the spiral pattern is stationary* (*i.e. that it looks the same in* 10^9 *yr or so*)."

Intriguingly, if the density wave theory were correct, a spatial ordering of different stages of star formation would be expected in the arms of galaxies: with very young objects on the leading edges of the arm (where star birth would be triggered by a compression wave) and the oldest ones on the trailing edge. However, research involving computer algorithms to examine twelve nearby spiral galaxies of different variety: such as the 'whirlpool galaxy' *M*51*a*, *M*63, *M*66, *M*74, and *M*95-an interacting, a flocculent, an arm-distorted, a grand design, and a barred spiral respectively-did not find such an ordering, leading to the conclusion that spiral density waves in their simplest form are not an important aspect of explaining spirals in large disk galaxies [14].

The purpose of this article is two fold:

Firstly: to get a deeper understanding of the physics of rotating astrophysical bodies in models where the space-time exhibits non trivial macroscopic quantum effects.

Secondly: to deduce, in some quantitative way, part of the relevant signatures (topological traces) which might help to reveal whether or not such exotic behaviour is present in our universe.

Contents: the plan of the paper is as follows. The Euler-Lagrange equations for a quantum gravitational action are solve for vortexes and monopoles, in Section 4 and 6 respectively, exhibiting in full the superfluid properties of the space-time. The spin interaction of an array of axis-aligned quantum vortices is analysed in Section 5. Next, in Section 6.1, Dirac's quantization condition is applied to quantize the size of the cosmological constant, which in the superconducting theory of gravitation plays a role analogous to an electric charge. Finally, the basic results are discussed and summarised in Section 9, where future directions for research are indicated.

2. Space-Time as a Charged Superfluid

In the late 1930s, W. H. Kessom, P. Kapitza, J. F. Allen and A. D. Misener, initiated a series of low-temperature experiments that led to the discovery of superfluidity [15,16], a quantum many-body effect responsible of very striking properties in a superfluid, such as: an infinity heat conductivity, *i.e.* the boiling abruptly stops, a zero viscosity (superleaking with zero resistance), the fountain and mechanocaloric effects, to cite some appearing below a certain critical temperature (the λ - point for He II) and strictly at speeds under some critical velocity V_c.

Is the space-time at galactic scales acting as a superfluid?

According to the prevailing view, at extragalactic scales the expanding universe is best think of as consisting of two parts: one luminous (obeying Newtonian mechanics in the limit of slowly moving bodies and large distances) and the other dark, or to use perhaps a better word: invisible (which is several times more abundant than the first one, and from which the formation and stability of the large scale structure of the universe presumably rests upon). For this second component, the quality of being invisible (or dark) is bring at front since it is only through its gravitational interaction with other bodies that this hypothetical form of matter (so far) has been accounted for.

In our view, the whole mystery of cold dark matter, and thus, the appearance of a two-fluid like model to describe the universe, where one component is behaving normally, while the other posses very odd properties, is a symptom of a bigger crisis than the one usually cured by just adding a new type of particle:

It is the failure of a proper understanding of how the quanta of mass-energy "*there*" *rules inertia* "*here*". Indeed much is gained by flipping from the dark matter perspective into the realm of quantum gravitational phenomena, since there is now-as D. Hilbert could have put it, "a guide post on the mazy paths of hidden truths" for quantizing the gravitational field. "Quantum gravity is a very tough problem"-warned W. Pauli to B. S. De-Witt [17,18]. How are we going to unify "the strange world" of Max Born's *probability* wave amplitudes Ψ 's with the peculiarities of the Einstein's four-dimensional curved space-time *continuum* ?

Perhaps, as the dark mater conundrum seems to imply, we have various clues already:

There is an electrically neutral, QCD colourless, quasi-substance with local (or non-local) mass that is in a cold, stable (or long-lived) unexcited state far away of any strong field; it flows freely (without resistance) but only at non relativistic speeds-as if there were a limiting velocity that it cannot surpass, it has a negligible nongravitational interaction with ordinary baryonic matter or itself.

What could it be? To cope with the subtleties imposed by the above scenario let us turn to mathematics since as Max Born put it [19]: "*When in conflict, mathematics—as often happens—is cleverer than interpretative thought*."

3. Quantum Mathematical Model

In 1956 W. Pauli remarked [20]:

"*The question of whether Kaluza's formalism has any future in physics is thus leading to the more general unsolved main problem of accomplishing a synthesis between the general theory of relativity and quantum mechanics*."

A deep connection between Einstein's law of gravity (with a nonzero cosmological constant) and quantum physical phenomena better associated to the theory of superconductivity was explored in [6], where the Kaluza-Klein idea of splitting the space-time metric as:

$$\mathrm{d}s^2 = -N^2\left(\mathrm{d}t + A_k \mathrm{d}x^k\right)^2 + |\Psi|^4 \tilde{\lambda}_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{1}$$

and thus:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0k} \\ g_{i0} & g_{ik} \end{pmatrix} = \begin{pmatrix} -N^2 & -N^2 A_k \\ -N^2 A_i & |\Psi|^4 \tilde{\lambda}_{ik} - N^2 A_i A_k \end{pmatrix}, \tag{2}$$

was adapted to offer a phenomenological, Ginzburg-Landau model of a 4-dimensional "quantum space-time". A_k is the gravitomagnetic vector potential, N is a scalar field, and $\tilde{\lambda}_{ij}$ is referred to as the 3-space base metric. The novelty of this approach is that although all the metric components are held real, Ψ is set to be a complex scalar field:

$$\Psi = \rho^{1/2}\exp[ie\varphi], \tag{3}$$

characterising the onset of order of a phase transition affecting the intrinsic features of the space-time itself, which-at galactic scales, it is imagined developing the properties of a highly coherent quantum system in parallelism with superfluids, lasers, and superconductors. Ψ, in other words, is a measure of symmetry violation. φ will play the role of a Goldstone boson field.

Every direct comparison between this and the (traditional) ADM setting should always kept in mind the dual transformation:

$$g_{\mu\nu} = g^{\mu\nu}_{ADM};\ g^{\mu\nu} = g^{ADM}_{\mu\nu}. \tag{4}$$

More comments on this very issue are given in [6].

In this article, Greek and Latin indices are employed to mark 4-dimensional and 3-dimensional tensors respectively ($\mu = 0,1,2,3$; $k = 1,2,3$), as it is done in (2) and

(4).

Key points of this bold proposal are briefly described next, leaving the details to the original article, where the theory was first developed [6]. First pay attention that by virtue of the complex nature of Ψ, the scheme by H. Weyl [21-24] to unite general relativity with electromagnetism can be adapted to treat the gravitomagnetic field A_k, so that in theory, the primeval gauge transformations set by:

$$\Psi \mapsto \Psi' \equiv \exp\left[ie\varphi\right]\Psi, \tag{5}$$

and

$$A_k \mapsto A'_k \equiv A_k + \partial\varphi/\partial x^k\,;\ k = 1,2,3, \tag{6}$$

become a symmetry of the physical gravitating system. Weyl's original view of a 4-dimensional conformally invariant universe (described by a conformally invariant action where only purely real exponents get involved in the gauge transformation laws) was abandoned as a model for the actual state of the universe: for as much as the prediction that physical observables, such as the lengths and times of measuring rods and clocks, would depend of their prehistory, which would in turn introduce spectral blur effects which simply do not show up in reality [20,24]. Yet, gauge invariance (which has been very successful as guidance principle for formulating the electroweak and the strong nuclear interactions) can be incorporated into gravitation in another way [6] which seems more in unison with the principles of quantum mechanics.

As it is argue in pages to come, an utterly natural, Ginzburg-Landau-action principle for gravitation is:

$$\begin{aligned}\mathcal{L}_s = -\chi^{-1}\int\Bigg\{ & 4e^2\left|\Psi\right|^2\tilde{\lambda}^{ij}\left(\varphi_{,i}-A_i\right)\left(\varphi_{,j}-A_j\right)\\ & +4\tilde{\nabla}_k\left|\Psi\right|\tilde{\nabla}^k\left|\Psi\right| - \left(\frac{12e^2}{N^2}\left|\Psi\right|^6 - \frac{1}{2}{}^{(3)}\tilde{R}\left|\Psi\right|^2\right)\\ & +\frac{N^2}{8\left|\Psi\right|^2}\tilde{\lambda}^{ik}\tilde{\lambda}^{jm}F_{ij}F_{km}\Bigg\}N\left(\det\tilde{\lambda}\right)^{1/2}\mathrm{d}V,\end{aligned} \tag{7}$$

where the third term plainly depends on the Ricci scalar of the 3-space, base metric $\tilde{\lambda}_{ij}$. By keeping N fixed, the Ψ^6-term becomes a constant multiplying the physical four-volume. Thus, e^2/N^2 can be identified with a vacuum energy, and e^2 must be proportional to the only constant present in the classical Einstein's field equations which surely, is completely determined by the microphysics of the gravitating system, expressly, Einstein's (1917) cosmological constant. The gravitomagnetic field-stress tensor is given by $F_{ik} = A_{k,i} - A_{i,k}$ and attention should be brought to the curious sign in front of the $F^{ik}F_{ik}$-term (whose origin is traced back to the temporal nature of the fourth dimension).

Equation (7) is expected to be valid in stationary situations, where the temporal variations of the gravitomagnetic vector potential A_i and the base metric $\tilde{\lambda}_{ij}$ can be neglected [6]. It is exactly in this case when the parallelism between gravitation and a metallic super conductor looks more straightforward, after all, stationary space-times with *horizons* follow mechanical rules resembling the laws of thermodynamics [25]. In light of this, it can be stated that the infrared quantum macroscopic effects inherent to gravity seem best fitted by a word introduced by Kamerlingh Onnes in 1911, namely "superconductivity".

Significant aspects of this action are immediately assessed by taking a look to the condensation energy: having both a sixth and inverse-square power terms, and depicted in **Figure 2**. Its shape is dictated by the Einstein-Hilbert action itself. Indeed, by varying $\left|\Psi\right|$ and A_i, the least action principle (7) leads respectively to the energy and momentum constraint equations of Einstein's theory of gravity, as it was developed by A. Lichnerowicz, J. W. York, and Y. Choquet-Bruhat in the 40s and 80s [26,27].

The mere existence of a phase in the ubiquitous complex gravitational potential introduced in (3) and (7) has the most amazing implications [6]:

Firstly: it allows the generation of supercurrents:

$$\tilde{J}_k = \left(\frac{2e}{N}\right)\left|\Psi\right|^2\left(\tilde{\nabla}_k\varphi - A_k\right), \tag{8}$$

transporting vacuum energy while deforming the gravitomagnetic (or gyrogravitational) lines of force. Be aware that closed strings are natural carriers of vacuum energy.

Secondly: second-order phase transitions controlled by

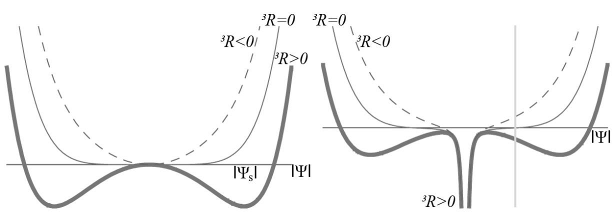

Figure 2. Phenomenological condensation energy affecting the nature of the space-time itself. On the left, the gravitomagnetic field is switched off: in close analogy with the Meissner-Ochsenfeld effect of the theory of super conductivity of metals. It is present, however, on the right side. There is a local minimum at $\left|\Psi\right| = 0$ for ${}^{(3)}\tilde{R} < 0$ and at $\left|\Psi\right| = \left|\Psi_s\right|$ for ${}^{(3)}\tilde{R} > 0$. The negative blow up of the condensation potential exhibited on the right is expected to be cut off by matter or hidden deep inside an absolute event horizon (in the black hole case $\left|\Psi\right|$ vanishes at the singularity). The vertical line indicates the value taken by $\left|\Psi\right|$ at the event horizon, usually this becomes a minimum if the space-time is restricted to lie within an isotropic coordinate chart.

the curving of space can set in, subtlety raising the mass of the gravitomagnetic vector potential A_k, due to its interaction with an all-pervading gravitational degree of freedom [6]: expressly, the modulus of the complex potential Ψ.

Thirdly: when a spinning point-like mass in empty space gets surrounded by supercurrents, the net effect is the generation of space-time superconducting zones, in which the associated rotation curves display non Keplerian features such as the ones exhibited in large spiral galaxies. Such rotations curves can be regarded as arising from the spontaneous breaking of $U(1)$-symmetry induced by the condensation of a Goldstone field coordinate φ to an azimuthal angular value; thus, defining a preferred orbital direction of reference [see [6]].

Finally: at short distances, covering only a sufficiently small open neighbourhood of the space-time, when Λ (and henceforth e) has not too much relevance, the predictions of Einstein's theory of gravity are recovered. The same is truth if e vanishes identically.

How does this work? Well, the string and monopole cases are provided below.

Notation and nomenclature—it is convenient to denote by $|\Psi_s|$, the value taken by the modulus of the complex field Ψ under the peculiar situation when: $F_{ij} \to 0$, at the minimum of the condensation potential. The identity:

$$|\Psi_s|^4 = \frac{1}{2}(N/6e)^2\,{}^{(3)}\tilde{R} \tag{9}$$

is then a direct consequence of this definition, see **Figure 2**. Direct inspection to (7) suggest that ${}^{(3)}\tilde{R}$ is physically related to a measurable mass [6].

Write next

$$\rho_s = \langle\rho\rangle = |\Psi_s|^2, \tag{10}$$

and set

$$\lambda^{-2} = 4^2(e/N)^2\langle\rho\rangle^2, \tag{11}$$

also demanding that

$$\xi = \sqrt{2}\left({}^{(3)}\tilde{R}\right)^{-1/2};\ \left({}^{(3)}\tilde{R} > 0\right). \tag{12}$$

λ is called the "London parameter" (or the penetration depth) and ξ is referred to as the "correlation length".

The physical significance of all these expressions will be worked out with examples later.

Equations (11) and (12) give a dimensionless Ginzburg-Landau (G-L) parameter:

$$\kappa = \lambda/\xi \tag{13}$$

equal to three halves, in line with type II super-conductivity. The (G-L) parameter, however, changes its value if one allows the $(e/N)^2|\Psi|^6$-term to be multiplied by a different coefficient than 12. This arbitrariness is discussed in more detail in [6]. A space-time fulfilling a principle of least action of the form given by (7) will be said to be a charged, space-time superfluid.

Space-time defects (mathematical preliminaries): Let the initial-data hypersurface $\left(\Sigma_t^3, \tilde{\lambda}_{ij}\right)$ be a Riemannian space of constant sectional curvature; that is to say:

$${}^{(3)}\tilde{R}_{ij}{}^{kl} = 2K\left(\delta^k_i\delta^l_j - \delta^k_j\delta^l_i\right), \tag{14}$$

where K is a given constant. Then, according to the theorem of H. Hopf and W. Killing [28,29], locally that space is isometric to one of the following models: a 3-sphere $\left(\mathbb{S}^3\right)$, a 3-Euclidean space $\left(\mathbb{E}^3\right)$, or an hyperbolic 3-space $\left(\mathbb{H}^3\right)$, with the same Ricci-scalar curvature ${}^{(3)}\tilde{R} = 12K$. Setting $|K|$ as l^{-2}, appropriated line elements for the neighbourhood containing a given point $p \in \Sigma_t^3$ as origin are:

$$\mathrm{d}s^2_{\mathbb{S}^3(l)} = l^2\left[\mathrm{d}\chi^2 + \sin^2\chi\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right], \tag{15}$$

$$\mathrm{d}s^2_{\mathbb{E}^3} = \mathrm{d}r^2 + r^2\left(d\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right), \tag{16}$$

and

$$\mathrm{d}s^2_{\mathbb{H}^3(l)} = l^2\left[\mathrm{d}\chi^2 + \sinh^2\chi\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right], \tag{17}$$

in the spherical, Euclidean, and hyperbolic instances respectively. By (17), the associated Laplace-Beltrami operator $\left(-{}^{(3)}\tilde{\Delta} + {}^{(3)}\tilde{R}\right)f$ can be written as:

$$\begin{aligned}\left(-{}^{(3)}\tilde{\Delta}_{\mathbb{H}^3(l)} + {}^{(3)}\tilde{R}\right)f = \frac{-1}{l^2\sinh^2\chi}\Big[\left(\sinh^2\chi f_{,\chi}\right)_{,\chi} \\ + \frac{1}{\sin\theta}\left(\sin\theta f_{,\theta}\right)_{,\theta} + \sin^{-2}\theta f_{,\phi\phi}\Big] - \left(12/l^2\right)f,\end{aligned} \tag{18}$$

Defining $r = l\chi$ gives:

$$\begin{aligned}l^2\sinh^2\chi = r^2\Big(1 + (1/3)(r/l)^2 \\ + (2/45)(r/l)^4 + \mathcal{O}\left[(r/l)^6\right]\Big),\end{aligned}$$

$$l^2\sin^2\chi = r^2\left(1 - (1/3)(r/l)^2 - (1/90)(r/l)^4 + \mathcal{O}\left[(r/l)^6\right]\right).$$

Since

$$\Delta_{\mathbb{R}^3}f = r^{-2}\left[\left(r^2 f_{,r}\right)_{,r} + \frac{1}{\sin\theta}\left(\sin\theta f_{,\theta}\right)_{,\theta} + \frac{1}{\sin^2\theta}f_{,\phi\phi}\right],$$

the following identity must be satisfied:

$$\begin{aligned}\left(-\tilde{\Delta}_{\mathbb{H}^3(l)} + {}^{(3)}\tilde{R}\right)f = \left[-\Delta_{\mathbb{R}^3} - \frac{12}{l^2}\right]f \\ + \frac{1}{3}(r/l)^2 \times \left[\Delta_{\mathbb{R}^3}f - r^{-4}\left(r^4 f_{,r}\right)_{,r}\right] + \mathcal{O}\left[(r/l)^4\right],\end{aligned} \tag{19}$$

likewise,

$$\begin{aligned}\left(-\tilde{\Delta}_{\mathbb{S}^3(l)} + {}^{(3)}\tilde{R}\right)f = \left[-\Delta_{\mathbb{R}^3} + \frac{12}{l^2}\right]f - \frac{1}{3}(r/l)^2 \\ \times\left[\Delta_{\mathbb{R}^3}f - r^{-4}\left(r^4 f_{,r}\right)_{,r}\right] + \mathcal{O}\left[(r/l)^4\right].\end{aligned} \tag{20}$$

The nomenclature introduced in this passage as well as the pair set by (19) and (20), find an immediate application in the analysis of topological space-time defects, coming next.

4. String Solution

The most basic features of the line defects predicted by the quantum rule set down by (7) are determined by the set of equations:

$$r^2 A_{,rr} + rA_{,r} - \left[\frac{\rho^2}{\rho_s^2}\frac{r^2}{\lambda^2} + 1\right] A - r(rA)_{,r}\left(\ln|\rho|\right)_{,r} = 0, \quad (21)$$

$$\begin{aligned} &-8r^{-1}\left(r|\Psi|_{,r}\right)_{,r} + 8e^2A^2|\Psi| - 2^{-1}N^2\tilde{H}^2|\Psi|^{-3} \\ &+\left(2/\xi^2\right)|\Psi|\left[1-\left(|\Psi|/|\Psi_s|\right)^4\right] = 0, \end{aligned} \quad (22)$$

where A is a scalar field; ρ_s, λ, and ξ are given by (9), (10), (11), and (12) respectively; $\tilde{H}$ is referred to as the gravitomagnetic field and it is given by $\tilde{H} = \nabla\times\boldsymbol{A}$.

This can be found as follows. As in the previous paragraph, let Σ_t^3 be a Riemannian space of constant sectional curvature. Let $\chi = r/l \ll 1$, *i.e.* allow the curvature radius of the universe be large enough. Then, by (20), at leading order it must be true that

$$\left(\tilde{\Delta}_{\mathbb{S}^3(l)} + {}^{(3)}\tilde{R}\right)f \approx \left[\Delta_{\mathbb{E}^3} + {}^{(3)}\tilde{R}\right]f,$$

and similarly with other expressions having the divergence, the gradient, and so on-as it is intuitive from (15) and (17). Thus, it is seen that there exists a convenient way to promote various calculations in flat space to curve space. Following this simplifying lead, make the subsequent choices. Employing cylindrical coordinates, write $\tilde{\boldsymbol{H}}$ as $(B_r; B_\theta; B_z)$. Then, in the corresponding orthonormal frame, the gradient of φ and the curl of $\tilde{\boldsymbol{H}}$ (or any other vector field) have respectively the form:

$$\nabla\varphi = \varphi_{,r}\hat{\boldsymbol{e}}_r + r^{-1}\varphi_{,\theta}\hat{\boldsymbol{e}}_\theta + \varphi_{,z}\hat{\boldsymbol{e}}_\phi = \left(\varphi_{,r}; r^{-1}\varphi_{,\theta}; \varphi_{,z}\right), \quad (23)$$

and

$$\begin{aligned} \nabla\times\tilde{\boldsymbol{H}} = \Big(&\left[r^{-1}B_{z,\theta} - B_{\theta,z}\right]; \left[B_{r,z} - B_{z,r}\right]; \\ &r^{-1}\left[\left(rB_\theta\right)_{,r} - B_{r,\theta}\right]\Big). \end{aligned} \quad (24)$$

Assuming that not far away from the axis of symmetry, here symbolized by the z-axis, the “probability-like” current density:

$$\begin{aligned} \tilde{J}_k &= \frac{1}{N}\left[\Psi\left(\frac{1}{i}\tilde{\nabla}_k - eA\right)^* \Psi^* + \Psi^*\left(\frac{1}{i}\tilde{\nabla}_k - eA_k\right)\Psi\right] \\ &= \frac{2e}{N}|\Psi|^2\left(\nabla_k\varphi - A_k\right), \end{aligned} \quad (25)$$

girdles always in the azimuthal direction, and moreover, that its magnitude at any given point $q\in\Sigma_t^3$ proximate to the centre line only depends on its geodesic distance to such line, let $\boldsymbol{A}-\nabla\varphi$ take the form:

$$\begin{aligned} \boldsymbol{A}-\nabla\varphi &= \left(\boldsymbol{A}_{\hat{r}} - \varphi_{,r}; \boldsymbol{A}_{\hat{\theta}} - r^{-1}\varphi_{,\theta}; \boldsymbol{A}_{\hat{z}} - \varphi_{,z}\right) \\ &= \left(0; A(r); 0\right). \end{aligned} \quad (26)$$

These restrictions immediately imply:

$$\tilde{\boldsymbol{H}} = \nabla\times\left(\boldsymbol{A}-\nabla\varphi\right) = \left(0;0;r^{-1}\left(rA\right)_{,r}\right), \quad (27)$$

and henceforth:

$$\nabla\times\tilde{\boldsymbol{H}} = -\left(0; B_{z,r}; 0\right) = -\left(0; A_{,rr} + r^{-1}A_{,r} - r^{-2}A; 0\right). \quad (28)$$

Additionally, if only the radial part of the modulus field $|\Psi|$ is considered [see (3)], Combining (10) and (11) with the Euler-Lagrange equation:

$$N^2\tilde{\nabla}_l\tilde{F}_k^{\ l} = 4^2e^2\rho^2\left(\tilde{\nabla}_k\varphi - A_k\right) + N^2\tilde{F}_k^{\ l}\tilde{\nabla}_l\ln|\rho|, \quad (29)$$

resulting from the variation $\delta\boldsymbol{A}$ in the action principle set by (7), a differential relationship between the two unknown functions: $A(r) = \left(\boldsymbol{A}-\nabla\varphi\right)_{\hat{\theta}}$ and $\rho = |\Psi|^2$, follows immediately:

$$r^2 A_{,rr} + rA_{,r} - \left[\frac{\rho^2}{\rho_s^2}\frac{r^2}{\lambda^2} + 1\right] A - r(rA)_{,r}\left(\ln|\rho|\right)_{,r} = 0. \quad (30)$$

By letting further, the order parameter $|\Psi|$ to be a function of r only, the Lichnerowicz equation, obtained by the $\delta|\Psi|$-variation of the Ginzburg-Landau action set by (7), becomes at leading order [using (12) and (20)]:

$$\begin{aligned} &-8r^{-1}\left(r|\Psi|_{,r}\right)_{,r} + 8e^2A^2|\Psi| - 2^{-1}N^2\tilde{H}^2|\Psi|^{-3} \\ &+\left(2/\xi^2\right)|\Psi|\left[1-\left(|\Psi|/|\Psi_s|\right)^4\right] = 0, \end{aligned} \quad (31)$$

completing the system, where the relation: $\tilde{F}^{km}\tilde{F}_{km} = 2\tilde{H}^2$ entailing the gravitomagnetic field $\tilde{\boldsymbol{H}}$, has been used.

4.1. Asymptotic Analysis near the String Axis

The action principle set by (7) implies the following:

Firstly: the vortex-gravitomagnetic flux is quantized.

Secondly: the minimal flux $\Phi_o = \pi/e$ is achieved by some regular Ψ_{II} -vortex profile. *Thirdly*: the order ρ-parameter for such a vortex of minimal vorticity vanishes in a linear fashion, along cylindrical ring-like structures of nonzero finite radius. *Fourhtly*: near the vortex core, the asymptotic metrical aspects of the quantum, regular vortex of minimal vorticity are determined (say at the initial time $t=0$) by

$$\begin{aligned} \mathrm{d}s^2 = &-N^2\left[c\mathrm{d}t + (rB/2)r\mathrm{d}\theta\right]^2 + (1/3)\left[\left(r/r_{\min}\right)-1\right]^2 \\ &\times N^2\left(rB/2\right)^2\left(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + \mathrm{d}z^2\right), \end{aligned}$$

B is here the intensity of the gravitomagnetic field along the flux tube and it is assumed to be nonzero. *Finally but not least*-by (58), a natural way to express the fitting "charge" e is in the form $e = q/\hbar$, where h is Planck's constant $\approx 6.626 \times 10^{-34}$ joule-seconds, meaning that Λ can be regarded as introducing an h^{-2} factor into the main gravitational equations.

Proof: For the sake of argument, ignore first the *A*-*ρ* coupling in (21). Then, at sufficiently close distance from the axis of symmetry, when

$$r \ll \left(\lambda^2 + \xi^2\right)^{1/2}$$

and $\chi = r/l \ll 1$, (21) implies:

$$A(r) \approx Br/2 + C/er;\; r \to 0^+, \tag{32}$$

where B and C are integration constants. Inserting (32) into (27) shows that the B-constant physically gives the intensity of the gravitomagnetic field $\nabla \times \boldsymbol{A}$ along the z-axis, *i.e.*

$$\left(B_r; B_\theta; B_z\right) = \boldsymbol{B} = \nabla \times \left(\boldsymbol{A} - \nabla\varphi\right) = (0;0;B). \tag{33}$$

Setting

$$\nu = NB/4, \tag{34}$$

(22) reads, in the limit imposed by (32), as

$$r^2 \left|\Psi\right|_{,rr} + r\left|\Psi\right|_{,r} + \left(\nu^2 r^2 \left|\Psi\right|^{-4} - C^2\right)\left|\Psi\right| = 0;\; r \to 0^+. \tag{35}$$

Equation (35) is reminiscent of the Bessel differential equation; however, it contains the non-linear $\left|\Psi\right|^{-4}$-factor, multiplying ν^2 and spoiling an all-encompassing similarity. To solve (35), follow these simple steps:

Firstly: verify the expression below is an exact, regular solution.

$$\left|\Psi(r)\right| = \frac{\left|\nu\right|^{1/2}}{\left(\left|C\right|^2 - 1/4\right)^{1/4}} r^{1/2};\; \left|C\right| > 1/2. \tag{36}$$

Secondly: spot that clearly another regular asymptotic answer is given by:

$$\rho(r) = \left|\Psi(r)\right|^2 \approx a^2_{|C|} r^{|2C|};\; r \approx 0^+;\; \left|C\right| < 1/2: \tag{37}$$

whenever $r \approx 0^+$. What about the distinctive $\left|C\right| = 1/2$ value? Well, several transformations simplify the problem.

Last step: as suggested by (36) and (37), pick

$$C = 1/2 \tag{38}$$

and set $\left|\Psi(r)\right| = r^{1/2} f(r)$. Then, (35) reduces to

$$r^{-2}\left(r^2 f_{,r}\right)_{,r} + \nu^2 r^{-2} f^{-3} = 0, \tag{39}$$

which is symmetric under the specular transformation: $r \mapsto -r$. By the change of variables: $r = \zeta^{-1}$, it transforms into

$$f_{,\zeta\zeta} + \nu^2 f^{-3} \zeta^{-2} = 0, \tag{40}$$

complying with the canonical form of the so-called Emdem-Fowler (E-F) equation:

$$y_{,\zeta\zeta} = k y^m \zeta^n. \tag{41}$$

If $m \neq 0$, the E-F equation has the exact solution:

$$y = \left[(n+2)(n+m+1)/k(m-1)^2\right]^{1/(m-1)} \zeta^{(n+2)/(1-m)} \tag{42}$$

as it is readily verified. Unfortunately $m = -3$ and $n = -2$, thus no formal solution can be extracted from this previous knowledge, as the coefficient multiplying $\zeta^{(n+2)/(1-m)}$ diverges. To advance further, introduce the change of variable $\zeta = \exp[t]$, then, (40) becomes instead:

$$f_{,tt} - f_{,t} + \nu^2 f^{-3} = 0, \tag{43}$$

which has coefficients which do not depend explicitly on the independent t-variable. A standard trick is then to pick $u(t) = f_{,t}$, hence $f_{,tt} = uu_{,f}$, and (43) simplifies to:

$$u_{,f} = 1 - \left(\nu^2/f^3 u\right), \tag{44}$$

which is intended to be solved for $u = u(f)$. Thus, proceeding in the reverse order, $f = f(r)$ is obtained by inverting (if possible)

$$r = \mathrm{e}^{-\int \mathrm{d}f/u(f)}. \tag{45}$$

The general features of the solutions, $u = u(f)$, of (44) are depicted in the phase diagram: u versus f in **Figure 3**. The vertical axis not only gives a measure of the magnitude of $u(t) = f_{,t}$ but also of $-rf_{,r}$, as can be seen by the chain of relations $u = f_{,t} = \zeta f_{,\zeta} = -rf_{,r}.$ Thus, the locus of points of the form $\left(r, f(r)\right)$ where, as a function of r, f is an extremum are mapped into the horizontal axis $\left(f(r), 0\right)$ of **Figure 3**; the points set by $\left(f, u(f)\right)$ where, as a function of f, $u(f)$ is an extremum falls over the dotted curve $u = \nu^2 f^{-3}$ labeled by the a latin symbol in **Figure 3** Use next (44) to obtain

$$u_{,ff} = \nu^2 f^{-6} u^{-3}\left[3f^2u^2 + f^3 u - \nu^2\right]. \tag{46}$$

The superior (and by the same note, inferior) branch of the "inflexion curve":

$$u = 6^{-1} f^{-1}\left(\pm\sqrt{f^4 + 12\nu^2} - f^2\right), \tag{47}$$

drew from (46) by the condition $u_{,ff} = 0$, is labeled b (jointly c) in **Figure 3**. The solutions $u = u(f)$ can be separated into two distinct classes, referred to as type I and type II for definiteness. A representative of each class has been found numerically and depicted in the same figure: type-I solutions do not cross the $(u = 0)$ f-axis,

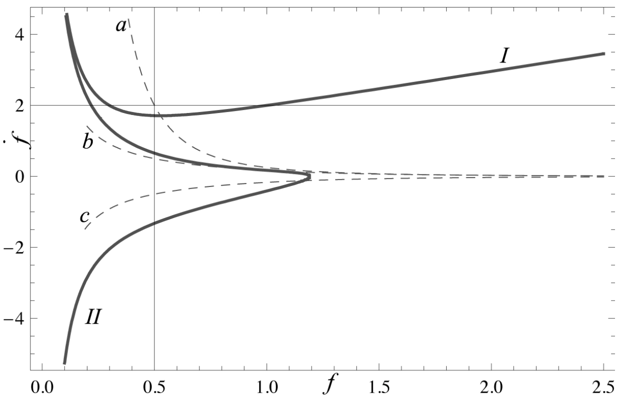

Figure 3. Phase space: $f_{,t}$ versus f (u versus f).

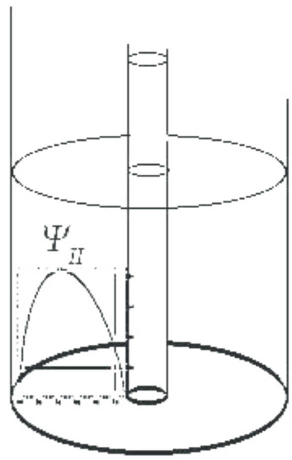

Figure 4. An inner, quantum, regular vortex of minimal flux, upheld by two-coaxial cylinders of different radii, $r_{\max}$ and $r_{\min}$. Stick to the inner cylinder, the core of the vortex, extending even further by a distance controlled by the coherent length ξ, is shielded by an annular cylindrical domain where the space-time becomes superconducting: a quantum effect accurately described by the gravitational potential: $\left|\Psi_{II}(r)\right|$. The space-time settles down to its normal state in the neighbourhood of the outermost cylinder, where $\left|\Psi_{II}(r)\right|$ vanishes as an $\left[1-(r/r_{\max})\right]^{1/2}$ power.

type-II make that cross. Let $u_I(f)$ be a type-I solution, clearly u_I is bounded from below by some positive constant, lets say $u_I(f)\geq k^2$. By (44), as f approach infinity, $u_{,f}$ tends to 1. This means that $u_I(f)$ takes the asymptotic form

$$u_I(f)\approx(f-f_N)+u_N \quad \text{as} \quad f\to\infty,$$

where (f_N,u_N) is a point in $(f,u_I(f))$. Therefore, according to (45) one has

$$r\approx(u_N/f-f_N+u_N)\mathrm{e}^{-\int_o^{f_N}\mathrm{d}f/u_I(f)}, \tag{48}$$

as $f\to+\infty$. Meaning, by inverting the relation, that

$$\left|\Psi_I(r)\right|\approx(f_N-u_N)r^{1/2}+u_N\mathrm{e}^{-\int_o^{f_N}\mathrm{d}f/u_I(f)}r^{-1/2};r\to 0. \tag{49}$$

The important point to make is that this is not a regular solution, since

$$u_N\mathrm{e}^{-\int_o^{f_N}\mathrm{d}f/u_I(f)}$$

is obviously different from zero. Turning now to type-II solutions, consider the situation when one has both: $u\to\pm\infty$ and $f\to+0$.

Applying l'Hôpital's rule to (44), it is established that in this regime:

$$u_{,f}\approx 1+\left(\nu^2u_{,f}/3u^2f^2\right), \tag{50}$$

which gives

$$\begin{aligned}1&\approx u_{,f}\left[1-\left(\nu^2/3u^2f^2\right)\right]\\&\approx\left[1-\left(\nu^2/uf^3\right)\right]\left[1-\left(\nu^2/3u^2f^2\right)\right].\end{aligned} \tag{51}$$

Thus $3f^2u^2+f^3u-\nu^2=0$ and henceforth

$$u_{II}(f)\approx-(f/6)\left[1\pm\sqrt{1+12\nu^2f^{-4}}\right]\approx\pm\left(|\nu|/\sqrt{3}f\right).$$

Inserting such an expression in (45), the following asymptotic formulae for the Ψ-field are obtained, from which some characteristics observed on **Figure 4** are deduced:

$$\rho(r)=\left|\Psi_{II}(r)\right|^2\approx\left(2|\nu|/\sqrt{3}\right)r\left[1-(r/r_{\max})\right]; \quad \text{if } r\to r_{\max}\neq 0;\ |C|=1/2, \tag{52}$$

$$\rho(r)=\left|\Psi_{II}(r)\right|^2\approx\left(2|\nu|/\sqrt{3}\right)r\left[(r/r_{\min})-1\right]; \quad \text{if } r\to r_{\min}\neq 0;|C|=1/2. \tag{53}$$

Let

$$0<r_{\min}\ll r\ll\left(\lambda^2+\xi^2\right)^{1/2}$$

and insert the (53) result into (21), it gives a linear non homogeneous equation whose general solution for $r>0$ is the sum of a particular solution, say $V(r)$, to the solution of the homogeneous problem given by (32) again, choosing a particular solution satisfying the initial conditions: $V=V_{,r}=0$ at $r=r_c$ (where r_c is the value of the coordinate radius at some point of the permitted interval), it is seen then that the non particular solution can only bring quadratic $(r-r_c)^2$-corrections to the previous answer. The C/er term in (32) still dominates the limiting behaviour at small radii. Thus, if

$$A(r)\equiv\mathbf{A}_{\hat{\theta}}-r^{-1}\varphi_{,\theta}\approx Br/2+C/er;\ r\to 0^+, \tag{54}$$

it is consistent to set

$$\boldsymbol{A}_{\hat{\theta}}(r)\approx Br/2;\ r\to 0^+, \tag{55}$$

and also

$$\varphi \approx -(C/e)(\theta-\theta_o)+\varphi_o;\ r\to 0^+. \tag{56}$$

Yet, the Friedmann-Lemaître-Robertson-Walker-like scale factor introduced in the Kaluza-Klein-like metric (1), that is to say, the $\Psi\Psi=\rho\mathrm{e}^{2ie\varphi}$ piece, must be a single-valued function. Not having any restriction on the polar angle θ, it must be true, if $\rho\neq 0$, that

$$|C|=n/2;\ n\in\mathbb{Z}, \tag{57}$$

implying in turn a quantum law over the allowed values for the gravitomagnetic flux, namely

$$\begin{aligned}\Phi_{flux}&=\iint_{\mathcal{A}_\Gamma}\boldsymbol{H}\cdot\mathrm{d}\boldsymbol{S}=\oint_\Gamma\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{s}\\&=\oint_\Gamma\nabla\phi\cdot\mathrm{d}\boldsymbol{s}=\pi n/e;\ n\in\mathbb{Z},\end{aligned} \tag{58}$$

in the understanding that Γ is a planar, smooth, closed curve of winding number one, surrounding the axis of symmetry; each point of Γ, it is assumed also, falls deep inside a large enough zone where the space-time becomes superconducting, and thus where $\boldsymbol{A}-\nabla\varphi$ vanishes identically. This type of flux quantization has exactly the same form than in metallic superconductors, where the carriers of electric current consist of pairs of electrons. A pair of charged quantum fields $\Psi\Psi$, actually a field coupled to itself, appears instead in the line element (1) from which the action principle (7) is based on. Here, the fitting charge, however, is in essence pure vacuum energy.

The law of gravitation (7) outlines the Bose-Einstein condensation of wave-particle pairs and it bring us closer to some of the most fundamental queries posed by Newton about the origin gravity [30,31]:

"*I have not been able to discover the cause of those properties of gravity from phenomena, and I frame no hypotheses; for whatever is not deduced from the phenomena is to be called a hypothesis, and hypotheses, whether metaphysical or physical, whether of occult qualities or mechanical, have no place in experimental philosophy.*"—*Principia* 2*nd edition*.

"*Is not this Æthereal Medium much rarer within the dense Bodies of the Sun, Stars, Planets and Comets, than in empty celestial Spaces between them? And in passing from them to great distances, doth it not grow denser and denser perpetually, and thereby cause gravity of those great Bodies toward one another, and of their parts towards the Bodies; every Body endeavouring to go from the denser parts of the Medium towards the rarer?*" — *Opticks* Query 21.

In what proportion is the intensity of what we call Gravity affected by an increase in mass of the gyromagnetic field which, by a Higgs-like mechanism, gets transformed as we move further and further away from macroscopic dense Bodies like the Sun, Stars, Planets and Comets? Is the local spherical radius on the verge of becoming rather uniform so that orbiting test bodies like Stars at different radii move through paths of almost equal length? And how this rigidity (or uniformity) of the space distorts a beam of light when it departs from a point where gravity is normal, then—as it travels—the gyro-gravitational field becomes massive, to finally end at another point where the spacetime is not superconducting? Is this a step forward towards a consisitent solution to the stabilization problem of spiral galaxies?

4.2. Far Away Asymptotics

A regular, infinite, string line, obeys the asymptotic formulae provided below if the conditions $|\Psi|\to|\Psi_s|$ and $r\gg\lambda$ are met:

$$|\Psi|\approx|\Psi_s|+\sqrt{\frac{2\xi}{\pi r}}\psi_o\cos\left(\frac{r}{\xi}-\frac{\pi}{4}-\delta\right), \tag{59}$$

$$A(r)\approx\frac{\ell}{2\lambda e}K_1(r/\lambda), \tag{60}$$

where $K_1(r/\lambda)$ is the Macdonald function, decaying with distance at leading order as

$$\sqrt{\pi\lambda/2r}\exp[-r/\lambda].$$

To look for the asymptotic distance decay of the gravitational Ψ-potential, turn back to the basic system of cylindrically symmetric equations:

$$r^2A_{,rr}+rA_{,r}-\left[\frac{\rho^2}{\rho_s^2}(r/\lambda)^2+1\right]A-r(rA)_{,r}(\ln|\rho|)_{,r}=0 \tag{61}$$

$$\begin{aligned}&8r^{-1}\left(r|\Psi|_{,r}\right)_{,r}-2\xi^{-2}|\Psi|\left(1-\frac{|\Psi|^4}{|\Psi_s|^4}\right)\\&-8e^2A^2|\Psi|+\frac{1}{2}N^2\tilde{H}^2|\Psi|^{-3}=0,\end{aligned} \tag{62}$$

As the gravitomagnetic vector potential becomes pure gauge, as the space-time becomes superconducting, the $|\Psi|$-field becomes, to a high degree of accuracy, given by an asymptotic expansion of the form:

$$\begin{aligned}&|\Psi|=\left|\Psi^{(0)}\right|+\left|\Psi^{(1)}\right|+\left|\Psi^{(2)}\right|+\cdots\\&\left|\Psi^{(0)}\right|=|\Psi_s|,\ \left|\Psi^{(1)}\right|=\sigma,\ \left|\Psi^{(2)}\right|=\cdots\end{aligned} \tag{63}$$

the system (61) and (62), in the limit $r\to+\infty$, $\rho\to\langle\rho\rangle$, simplifies to:

$$r^2A_{,rr}+rA_{,r}-\left(\frac{r^2}{\lambda^2}+1\right)A=0, \tag{64}$$

and

$$r^2\sigma_{,rr}+r\sigma_{,r}+\left(\frac{r^2}{\xi^2}-0\right)\sigma=0. \tag{65}$$

Put attention that the $\sigma - A$ coupling implied by (61) is relevant only at second order and be alert on the dissimilarity in sign between the coefficient accompanying our correlation length ξ and the one encountered in standard superconductivity: our sign, one may say, is anomalous. Nevertheless, this does not seem to represent a severe problem; on the contrary, it is necessary to display some of the features observed for the shape of the galactic rotation curves, as it is argued in [6]. (64) is just the modified Bessel equation:

$$z^2\,\mathrm{d}^2 w/\mathrm{d}z^2 + z\,\mathrm{d}w/\mathrm{d}z - \left[(z/\lambda)^2 + \nu^2\right]w = 0. \quad (66)$$

and it has as one of its solutions: the Macdonald function $K_\nu(z/\lambda)$, decaying exponentially to zero according to the asymptotic representation [32]:

$$K_\nu(z/\lambda) \approx \sqrt{\frac{\pi\lambda}{2z}}\mathrm{e}^{-z/\lambda} \times \left(1 + \frac{4\nu^2-1}{1!8z}\lambda + \frac{(4\nu^2-1)(4\nu^2-3^2)}{2!(8z)^2}\lambda^2 + \frac{(4\nu^2-1)(4\nu^2-3^2)(4\nu^2-5^2)}{3!(8z)^3}\lambda^3 + \cdots\right) \quad z \gg |\nu+1|\lambda. \quad (67)$$

One discards the other independent solution, the hyperbolic Bessel function of the first kind $I_\nu(z/\lambda)$, since it grows exponentially with z, giving an apposite effect not in line with (63), unless $|\Psi_s|$ be unbounded. In the next section a rough estimate of the vortex-vortex interaction energy is obtained with the help of some identities satisfied by the Macdonald function, which for convenience's sake are listed here: namely, its divergent behaviour at the origin:

$$K_\nu(z/\lambda) \approx \begin{cases} -\ln|z/2\lambda| - \gamma & \text{If } \nu = 0, \\ \dfrac{\Gamma(\nu)}{2}(2\lambda/z)^\nu & \text{If } \nu > 0,\, z \to 0^+, \end{cases}$$

$$\gamma = \lim_{n\to\infty}\sum\nolimits_{k=1}^{n}\left(\frac{1}{k} - \ln|n|\right), \quad (68)$$

where γ is the Euler-Mascaroni constant, approximately given by

$$\gamma \approx 0.577215664901532860606 5121,$$

and the differential identities:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z^{-\nu}K_\nu(z/\lambda)\right) = -\lambda^{-1}z^{-\nu}K_{\nu+1}(z/\lambda), \quad (69)$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z^{\nu}K_\nu(z/\lambda)\right) = -\lambda^{-1}z^{\nu}K_{\nu-1}(z/\lambda). \quad (70)$$

In the same train of thought, (65) is just the Bessel o.d.e:

$$z^2\,\mathrm{d}^2 w/\mathrm{d}z^2 + z\,\mathrm{d}w/\mathrm{d}z + \left[(z/\xi)^2 - \nu^2\right]w = 0, \quad (71)$$

which has as solutions the cylindrical harmonics $J_\nu(z/\xi)$ and $Y_\nu(z/\xi)$. An asymptotic representation of them for large real arguments is given respectively by

$$J_\nu(r/\xi) \approx \sqrt{\frac{2\xi}{\pi r}}\cos\left(\frac{r}{\xi} - \frac{\nu\pi}{2} + \frac{\pi}{4}\right) \quad (72)$$

and

$$Y_\nu(r/\xi) \approx \sqrt{\frac{2\xi}{\pi r}}\sin\left(\frac{r}{\xi} - \frac{\nu\pi}{2} + \frac{\pi}{4}\right) \quad (73)$$

if $r/\xi \gg |\nu^2 - (1/4)|$. Their distance decay is thus oscillatory and modulated, in part by the two-dimensional nature of the problem, by an inverse square root power of the separation distance from the source. Nonetheless at short radii:

$$Y_o(z/\xi) \approx \frac{2}{\pi}\left[\ln|z/2\xi| + \gamma\right] \quad \text{if } z \to 0 \quad (74)$$

holds.

Now, consider the Green's identity:

$$\iint_{\Omega(R,\epsilon)} f\Delta K_o(r/\lambda) - K_o(r/\lambda)\Delta f = \oint_{\Gamma(R,\epsilon)} f\frac{\partial K_o(r/\lambda)}{\partial n} - K_o(r/\lambda)\frac{\partial f}{\partial n}. \quad (75)$$

over the contour Γ depicted in **Figure 5**, letting f be a regular function on $\Omega(R,\epsilon)$—we already know, by (68), that $K_o(r)$ is indeed regular there. Adding and subtracting $\lambda^{-2}fK_o(r/\lambda)$ to the integrand on left hand side of (75), and using (66), we get:

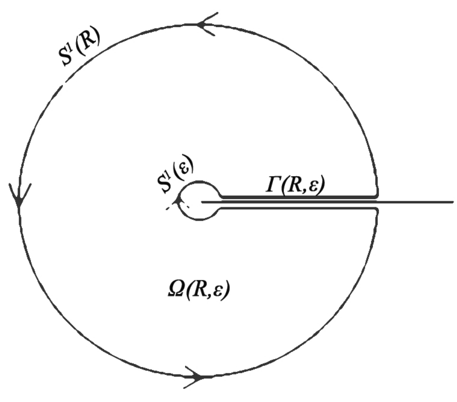

Figure 5. Contour path $\Gamma(R,\epsilon)$ of zero winding number, used to define the Dirac delta distribution $\delta^2(r)$ in a cylindrically symmetric space. The planar curve $\Gamma(R,\epsilon)$ is composed of two concentric circles $S^1(\epsilon)$ and $S^1(R)$ of arbitrarily small and large radii respectively, as well as of two antiparallel segments along the nonnegative x semi-axis, joining both circles.

$$\iint_{\Omega(R,\epsilon)} f\Delta K_o(r/\lambda) - K_o(r/\lambda)\Delta f = -\iint_{\Omega(R,\epsilon)} K_o(r/\lambda)\left(\Delta - \lambda^{-2}\right) f. \quad (76)$$

But the right-hand side of (75) is the sum of two contour integrals: one along the circle of very large radius $S^1(R)$ where the asymptotic representation of (67) applies, and the other along the circle of small radius $S^1(\epsilon)$ where (68) holds asymptotically. Therefore, taking the limit $(R,\epsilon) \to (+\infty, 0^+)$ gives

$$\lim_{(R,\epsilon)\to(+\infty,0^+)} \iint_{\Omega(R,\epsilon)} K_o(r/\lambda)\left(\Delta - \lambda^{-2}\right) f = -2\pi \lim_{\epsilon\to 0^+} f(\epsilon), \quad (77)$$

on the assumption that not only

$$|f| \in C^2\left(\mathbb{R}^2 - \{0\}\right)$$

is bounded from above $(|f| < M)$ outside a disk $B_\rho \subset \mathbb{R}^2$ of sufficiently large radius, but also that

$$\lim_{r\to 0^+} r\frac{\partial f}{\partial r}\ln|r| = 0. \quad (78)$$

Following Laurent Schwartz's theory of distributions, a linear map

$$-\left(\tilde{\Delta} - \lambda^{-2}\right) K_o(r/\lambda) : \mathcal{D} \to \mathbb{R},$$

from a proper space of test functions $\mathcal{D}$ to the reals can then be defined with the help of (77), symbolically written as:

$$-\left(\tilde{\Delta} - \lambda^{-2}\right) K_o(r/\lambda) = 2\pi\delta^2(r), \quad (79)$$

where $\delta^2(r) : \mathcal{D} \to \mathbb{R}$ is Dirac's delta distribution with support at the polar origin. Likewise (73) and (74) leads to

$$-\left(\tilde{\Delta} + \xi^{-2}\right) Y_o(r/\xi) = -4\delta^2(r). \quad (80)$$

Be aware, however, that as the natural space of test functions $\mathcal{D}$ should be in tune with the vanishing hypothesis of boundary terms in the far field regime, in more diverse applications, stronger assumptions than the ones required for making sense of (79) must apply (if needed, a change of measure under which the given integrals are carried out, say by adding proper weighting factors becomes a natural way to follow). Proceeding on such grounds it must be true, by (66) and (67), that

$$A(r) \approx \frac{\ell}{2\lambda e} K_1(r/\lambda); |\Psi| \to |\Psi_s|, \quad (81)$$

which by (27), (69), and (79), immediately gives

$$B_z = \frac{1}{r}(rA)_{,r} = \frac{\ell}{2\lambda e} r^{-1}(rK_1)_{,r} = -\frac{\ell}{2e} r^{-1}\left(rK_{o,r}\right)_{,r} = -\frac{\ell}{2e}\tilde{\Delta}K_o(r/\lambda) = -\frac{\ell}{2\lambda^2 e} K_o(r/\lambda). \quad (82)$$

The minus sign appearing at the end of this expression is expected. Combining (72) and (73), we get

$$|\Psi| \approx \left|\Psi^{(o)}\right| + \left|\Psi^{(1)}\right| = |\Psi_s| + \sqrt{\frac{2\xi}{\pi r}}\psi_o \cos\left(\frac{r}{\xi} - \frac{\pi}{4} - \delta\right), \quad (83)$$

where ψ_o (determining the first order of the perturbation amplitude) and δ (the angular phase) are integration constants. An immediate application of (59) and (60) is the estimation of the vortex-vortex and vortex-antivortex interaction energies.

5. Spin Interaction

Imagine a large pattern of quantum, space-time vortices: each vortex labelled by a unique number i, whose axes are all aligned, as depicted in **Figure 6**. Let the Lagrangian of the system be given by:

$$\mathcal{L}_{\text{system}} = \mathcal{L}_{\text{free}} + \mathcal{L}_{int} = \mathcal{L}^{(0)} + \left[\mathcal{L}_{int}^{(1)} + \mathcal{L}_{int}^{(2)} + \cdots\right]. \quad (84)$$

$\mathcal{L}^{(0)}$ denoting the free Lagrangian: $\mathcal{L}_{\text{free}}$, approximately given by a sum over disjoint regions of space $\Omega(r_i)$ (hereafter referred to as terminals or ends) of compact support centred at each space-time quantum vortex: each vortex treated at leading order as if it were in complete isolation, plus a remainder; that is to say:

$$\mathcal{L}_{\text{free}} = \left(\sum_i \int_{\Omega(r_i)} \mathcal{L}\right) + \text{remainder} \quad (85)$$

where $\mathcal{L}$ is given by (7). To a good degree of accuracy, two stationary, axis-aligned, quantum space-time vortices with the same sense of spin, interact with an interaction energy given by:

$$E_{ij}^{int} \approx \frac{N}{\chi}\left[\frac{\pi N \ell}{\lambda|e|} K_o\left(\left|r_i - r_j\right|/\lambda\right) + 32 Y_o\left(\left|r_i - r_j\right|/\xi\right)\right]. \quad (86)$$

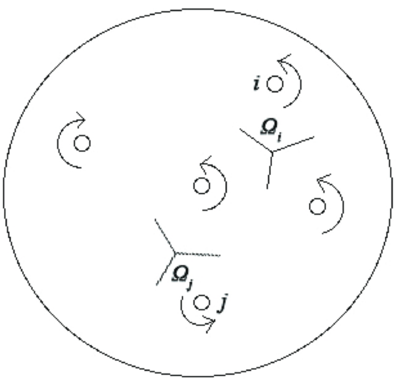

Figure 6. An array of axis-aligned quantum vortices interacting with each other. A vortex-antivortex pair pops out at the left upper corner. Quantized-gravitomagnetic-flux excitations looping back on themselves to form rings might also form on the space-time background, and they could be interpreted as a sort of gravitational roton.

where $\left|r_i - r_j\right|$ is their relative geodetic distance (according to the rest frame of reference attached to the strings themselves) and $Y_o\left(r/\xi\right)$ is a cylindrical harmonic which decays with distance as

$$\sqrt{2\xi/\pi r}\sin\left(r/\xi+\pi/4\right).$$

To first order of approximation, the form of the extra piece in square brackets $\mathcal{L}_{\rm int}$ is dictated by the requirement that the perturbative motion, say of the umpteenth vortex of the pack, roughly described by the triplet

$$\left(A^{(i,0)}+A^{(i,1)};\rho^{(i,0)}+\rho^{(i,1)};\varphi^{(i,0)}+\varphi^{(i,1)}\right)$$

be given by the solution for an isolated vortex

$$\left(A^{(i,0)};\rho^{(i,0)};\varphi^{(i,0)}\right)$$

plus a correction term $\left(A^{(i,1)};\rho^{(i,1)};\varphi^{(i,1)}\right)$ satisfying:

$$\left(\tilde{\Delta}^{(0)}+\xi^{-2}\right)\Psi^{(i,1)}=\sum_{j\neq i}\left(\chi/8\right)\tau^{(j,1)}, \tag{87}$$

$$\left(\tilde{\Delta}^{(0)}-\lambda^{-2}\right)A^{(i,1)}=\sum_{j\neq i}\left(2\chi/N^2\right)J_\theta^{(j,1)}, \tag{88}$$

where

$$\left|\Psi^{(i,1)}\right|^2=\rho^{(i,1)};$$

whereas $\tau^{(j,1)}$ and $J_\theta^{(j,1)}$ are source terms, one for each vortex of the bundle. Compare these equations with (64) and (65). The first entry in the superscript $(j,1)$ of $\tau^{(j,1)}$ and $J_\theta^{(j,1)}$ labels the vortex to which it is referred to, while the second entry (as well as the (0) in $\tilde{\Delta}^{(0)}$) establishes the degree of the perturbation.

In view of this, set

$$\mathcal{L}_{int}=\sum_i\int N\left(\det\tilde{\lambda}\right)^{1/2}\left(\frac{\boldsymbol{J}^{(i)}\cdot\boldsymbol{A}}{\rho}+\tau^{(i)}\left|\Psi\right|\right)\mathrm{d}V, \tag{89}$$

where the sum spans over all the vortex singularities; the equations of motion then become:

$$N^2\tilde{\nabla}_l\tilde{F}_k^{\ l}=4^2e^2\rho^2\left(\tilde{\nabla}_k\varphi-A_k\right)-2\chi\sum_i J_k^{(i)}, \tag{90}$$

and

$$\begin{aligned}0=&-8\tilde{\Delta}\left|\Psi\right|+8e^2\left(\tilde{\nabla}\varphi-\boldsymbol{A}\right)^2\left|\Psi\right|+{}^{(3)}\tilde{R}\left|\Psi\right|-\frac{8(3e)^2}{N^2}\left|\Psi\right|^5\\&-\frac{N^2\tilde{H}^2}{2\left|\Psi\right|^3}+\chi\sum_i\left(\frac{2\boldsymbol{J}^{(i)}\cdot\boldsymbol{A}}{\left|\Psi\right|^3}+\tau^{(i)}\right).\end{aligned} \tag{91}$$

Next, assume that in the neighbourhood $\Omega\left(r_i\right)$ of the i-th vortex of the sample, the asymptotic conditions:

$$\Psi\to\Psi^{(i,0)}+\Psi^{(i,1)};\ r\approx r_i, \tag{92}$$

and

$$\boldsymbol{J}^{(j)}\to J_{\hat{\theta}}^{(i,1)}\hat{\boldsymbol{e}}_\theta^{(j,0)}, \tag{93}$$

hold. Here $\hat{\boldsymbol{e}}_\theta^{(j,0)}$ is the unit vector on the zero-order space-time background, surrounding the j-th vortex and pointing along the associated azimuthal direction. Solve then (90) and (91) perturbatively. At zero order, one gets the system of equations studied previously: (21) and (22). Up to second order terms one recovers the system given by (87) and (88), as requested. To compute the first-order-correction terms of the i-vortex solution, $\Psi^{(i,1)}$ and $A^{(i,1)}$, suppose they are the direct result of an external field obtained by linear superposing each of the zero-order fields, as seen from a long distance, of the remaining $(j\neq i)$ vortices; this can be done by setting:

$$J_\theta^{(j,1)}=\frac{\pi N^2}{2e\chi}\delta^2\left(\left|r-r_j\right|\right)_{,r} \tag{94}$$

and

$$\tau^{(j,1)}=\frac{32}{\chi}\delta^2\left(\left|r-r_j\right|\right), \tag{95}$$

since one must have both

$$\left[K_o\left(r/\lambda\right)\right]_{,r}=-\lambda^{-1}K_1\left(r\right)$$

and (60) holding simultaneously, see (69), whereas the exact coefficients in front of the delta distribution and its distributional derivative are a direct consequence of (79) and (80). Henceforth, in the neighbourhood of the i-vortex, the increment in energy E_{ij} due to the external field produced by an axis-aligned j-vortex moving with the same sense of rotation is given by:

$$E_{ij}=\int_{\Omega(r_i)}\left[\left(\boldsymbol{J}^{(j,1)}\cdot\boldsymbol{A}/\rho\right)+\tau^{(j,1)}\left|\Psi\right|\right]N\left(\det\tilde{\lambda}\right)^{1/2}\mathrm{d}V. \tag{96}$$

To a good degree of approximation E_{ij}, using

$$K_o\left(r/\lambda\right)_{,r}=-\left(1/\lambda\right)K_1\left(r/\lambda\right)$$

and

$$rK_1\left(r/\lambda\right)_{,r}=-\left(r/\lambda\right)K_o\left(r/\lambda\right),$$

reduces to

$$E_{ij}^{int}\approx\frac{N}{\chi}\left[\frac{\pi N\ell}{\lambda|e|}K_o\left(\left|r_i-r_j\right|/\lambda\right)+32Y_o\left(\left|r_i-r_j\right|/\xi\right)\right]. \tag{97}$$

Here r_j gives the location of the j-vortex; the denominator of the first term of (96) is evaluated using $\rho\approx\rho_s$. For small λ the typical shape of this static potential is depicted in **Figure 7**; the first term, by (67), gives a repulsive contribution; the second term, in comparison with the first one, induces in virtue of (73) an attraction at some places, but gives a strong repulsion at others quarters, leading to an spectrum of local stationary minima roughly separated by a distance of order $2\pi\xi$. A variety of interesting solitonic phenomena may arise from (86), perhaps leading to voids where no vortices are

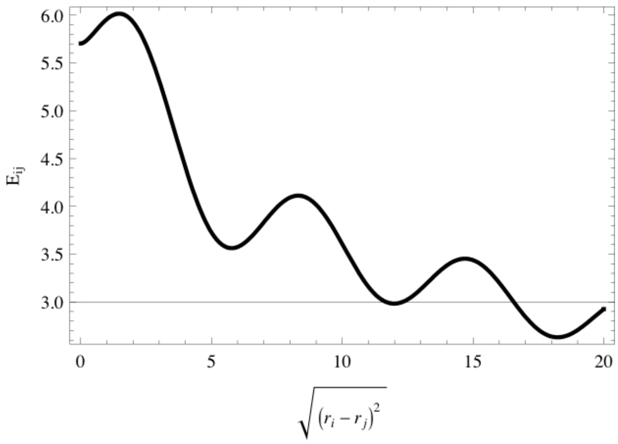

Figure 7. Interaction energy E_{ij} between two-axis-aligned-spatio-temporal vortices moving with the same sense of rotation as a function of their relative distance $|\boldsymbol{r}_i - \boldsymbol{r}_j|$.

expected. Finally, the complete increment in energy, say $E_{ext}^{(i,1)}$ associated with the external fields $\Psi^{(j,1)}$ and $A^{(j,1)}$ acting on the i-vortex is obtained by summing over all the interacting energies, giving:

$$E_{ext}^{(i,1)} = \sum_{j\neq i} E_{ij}^{int}. \qquad (98)$$

In generic situations one may expect to lose some of the finer details resulting from second term of (86), leading to a net repulsive force.

6. Monopole Solution

Ever since its inception in 1861, Maxwell's equations raised a mystery that is still with us. While they seem to emphasize a complete symmetry between the phenomena of electricity and magnetism, only in the vacuum the alleged harmony is known to exist: as particles carrying a single magnetic charge (the magnetic analogue of the electron) so far have not been found [33]. Yet Poincaré, J. J. Thomson, O. Heavyside, and P. Curie contemplated the idea of an exact symmetry at least once [34]. In the years of 1931 and 1948, a consistent quantum theory of magnetic monopoles was put forward by Dirac [35,36], who arrived to a very significant conclusion; namely, that if monopoles (or just one monopole) exist, this would amount of an explanation of why electricity is quantized and given in exact multiples of some smallest charge. Dirac's own initial conviction on the existence of the monopole is succinctly expressed in this famous 1931 statement ([35], p. 71): "Under these circumstance one would be surprised if Nature had made no used of it."

Contrariwise, Bohr was and remained very skeptical of this whole affair [34]. The monopole theory in fact did not seem to generate much interest until 1974, when it was discovered, by Gerard 't Hooft and Alexander M. Polyakov [37,38] independently, that monopoles are an inevitably prediction of certain Grand Unified Theories (GUT's) which rely on the spontaneous breakdown of symmetry. Namely, those in which the electromagnetic group $U(1)$ is taken to be a subgroup of a larger group with a compact covering group, like $SU(5)$, which contains the standard model:

$$SU(5) \supset SU(3) \times SU(2)_L \times U(1). \qquad (99)$$

The GUT monopoles are exceedingly massive, with a mass, say M_{mon}, larger by an inverse square gauge coupling constant than a typical vector boson mass:

$$M_W \sim 10^{15} \text{ - } 10^{16} \text{ GeV}, \quad M_{mon} = 4\pi e^{-2} M_W.$$

They would act as catalysers for the proton decay predicted by grand unified theories [39,40], and would be produced in copious number at the very early stages of the universe. Being highly stable particles, the GUT monopoles would survive as relics to the present epoch. But in order to not enter into conflict with what is observed, it would be necessary that their density be diluted considerable by some unknown mechanism (say inflation) during the cosmic evolution [41]. It is also known that in the so-called Prasad-Sommerfield limit some of these non abelian monopoles can be converted by their mutual interaction into dyons [42,43]: hypothetical particles carrying both electric and magnetic type charges that were first proposed in 1969 by J. Schwinger.

Be that as it may, even if no monopole has been found yet, it brings considerable insight regarding the foundations of physics.

The action principle (7) not only leads to the existence of a gyrogravitational Meissner-Ochsenfeld effect, one of the most fundamental properties expected to arise for a model where the space-time acts like a superconducting body, but also indicates that the cosmological constant, first introduced to gravitation by Einstein in 1917, is not only quantized but also that its square root is given in exact multiples of some smallest value.

To see if this is true, adopt an orthonormal, spherical, coordinate grid of reference outlined by the triplet (r,θ,ϕ), defined by radial, zenith-angular, and azimuthal-angular variables respectively. The gradient of φ, in the corresponding orthonormal frame, is given then by

$$\nabla\varphi = \varphi_{,r}\hat{\mathbf{e}}_r + \frac{1}{r}\varphi_{,\theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\varphi_{,\phi}\hat{\mathbf{e}}_\phi$$
$$= \left(\varphi_{,r}; \frac{1}{r}\varphi_{,\theta}; \frac{1}{r\sin\theta}\varphi_{,\phi}\right),$$

while the curl of any vector field, say $\tilde{\boldsymbol{H}} = (B_r; B_\theta; B_\phi)$ for definiteness, assumes the form:

$$\nabla\times\tilde{\boldsymbol{H}}=\Big(r^{-1}(\sin\theta)^{-1}\Big[\big(\sin\theta B_{\phi}\big)_{,\theta}-B_{\theta,\phi}\Big];\\ r^{-1}\Big[(\sin\theta)^{-1}B_{r,\phi}-\big(rB_{\phi}\big)_{,r}\Big];r^{-1}\Big[\big(rB_{\theta}\big)_{,r}-B_{r,\theta}\Big]\Big). \tag{100}$$

Suppose only one component (the azimuthal one) of the density current (25) is nonzero, and moreover, that it does not depend on the azimuthal φ-variable; the problem reduces then to the evaluation of an scalar potential:

$$A(r,\theta)=(\boldsymbol{A}-\nabla\varphi)_{\hat{\phi}}, \tag{101}$$

taking values on the real line and having poles or singularities, so that the divergence of the gravitomagnetic field be different from zero, as depicted in **Figure 8.** By (100) and the working hypothesis

$$\boldsymbol{A}-\nabla\varphi=\left(\boldsymbol{A}_{\hat{r}}-\varphi_{,r};\boldsymbol{A}_{\hat{\theta}}-\frac{1}{r}\varphi_{,\theta};\boldsymbol{A}_{\hat{\phi}}-\frac{1}{r\sin\theta}\varphi_{,\phi}\right)\\ =\big(0;0;A(r,\theta)\big), \tag{102}$$

it is deduced that:

$$\tilde{\boldsymbol{H}}=\nabla\times(\boldsymbol{A}-\nabla\varphi)=\left(\frac{[A\sin\theta]_{,\theta}}{r\sin\theta};-r^{-1}(rA)_{,r};0\right). \tag{103}$$

Consequently, the curl of the gravitomagnetic field is

$$\nabla\times\tilde{\boldsymbol{H}}=\left(0;0;\frac{1}{r}\Big[\big(rB_{\theta}\big)_{,r}-B_{r,\theta}\Big]\right)\\ =\left(0;0;-\frac{\big(r^2A_{,r}\big)_{,r}}{r^2}-\frac{1}{r^2}\left[\frac{(A\sin\theta)_{,\theta}}{\sin\theta}\right]_{,\theta}\right), \tag{104}$$

and (29) immediately reduces to:

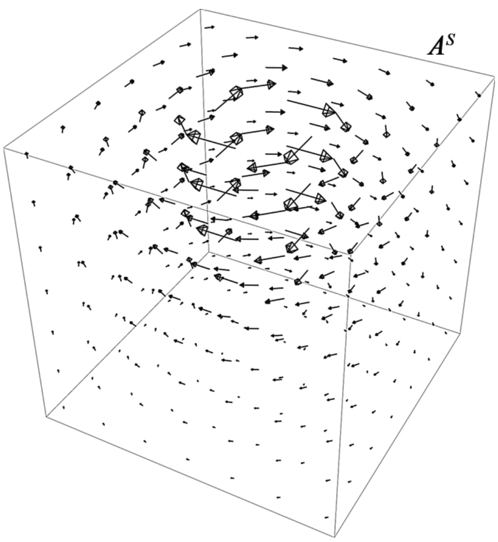

Figure 8. A gravitomagnetic vector potential A^S, well behaved at the south hemisphere (except at the origin) for a gravitomagnetic monopole with a Dirac string running along the nonnegative z semi-axis.

$$\frac{1}{r^2}\big(r^2A_{,r}\big)_{,r}+\frac{1}{r^2}\left[\frac{(A\sin\theta)_{,\theta}}{\sin\theta}\right]_{,\theta}-\frac{\rho^2}{\langle\rho\rangle^2}\frac{A^2}{\lambda^2}\\ -\frac{1}{r}(rA)_{,r}\big(\ln|\rho|\big)_{,r}-\frac{1}{r^2}\frac{(A\sin\theta)_{,\theta}}{\sin\theta}\big(\ln|\rho|\big)_{,\theta}=0. \tag{105}$$

Assuming:

$$\rho=Q(\theta)S(r);\quad A(r,\theta)=\sum_{\ell}g_{\ell}(\theta)R_{\ell}(r); \tag{106}$$

apply the method of separation of variables by letting the A-function to be given in terms of radial and angular eigenfunctions, $R_{\ell}(r)$ and $g_{\ell}(\theta)$, satisfying respectively:

$$\big(r^2R_{\ell,r}\big)_{,r}-\left[\frac{\rho^2}{\langle\rho\rangle^2}\frac{r^2}{\lambda^2}+\ell(\ell-1)\right]R_{\ell}=r\big(rR_{\ell}\big)_{,r}\big(\ln|S|\big)_{,r} \tag{107}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{(g_{\ell}\sin\theta)_{,\theta}}{\sin\theta}\right]+\ell(\ell-1)g_{\ell}=\frac{(g_{\ell}\sin\theta)_{,\theta}}{\sin\theta}\big(\ln|Q|\big)_{,\theta}. \tag{108}$$

For now let ρ be a function of r only. Then, by setting $g_{\ell}(\theta)=P_{\ell}\big(u(\theta)\big)/\sin\theta$ and making the change of variable: $u=\cos\theta$, (108) in turn translates into the form:

$$\big(1-u^2\big)P_{\ell,uu}+\ell(\ell-1)P_{\ell}=0. \tag{109}$$

Thus, if P_{ℓ} is the polynomial series

$$P_{\ell}(u)=\sum_{n=0}^{n=\infty}\alpha_{(\ell,n)}u^n, \tag{110}$$

its coefficients $\alpha_{(\ell,n)}$ must satisfied the recurrence relation:

$$\alpha_{(\ell,n+2)}=\frac{n(n-1)-\ell(\ell-1)}{(n+1)(n+2)}\alpha_{(\ell,n)}; \tag{111}$$

furthermore, since

$$\alpha_{(\ell,n+2)}=0;\ \mathbb{Z}\ni n\geq\ell\in\mathbb{Z}, \tag{112}$$

it is deduced that P_{ℓ} has a finite number of terms whenever ℓ becomes an integer. The P_{ℓ}'s are in fact a limiting case of the so called Jacobi polynomials, denoted by $P_n^{(\alpha,\beta)}(u)$, where one assumes that α and β are bigger than minus one, that is:

$$P_{\ell}(u)\propto\lim_{(\alpha,\beta)\to(-1,-1)}P_n^{(\alpha,\beta)}(u)\equiv P_n^{(-1,-1)}(u). \tag{113}$$

The $P_n^{(-1,-1)}(u)$'s fulfils the Rodrigue's formula:

$$P_n^{(-1,-1)}(u)=\frac{(-1)^n}{2^n\,!}\big(1-u^2\big)\frac{\mathrm{d}^n}{\mathrm{d}u^n}\big(1-u^2\big)^{n-1}. \tag{114}$$

They are connected with the Gegenbauer polynomials also through the relation:

$$P_\ell(u) \propto \lim_{\lambda \to -1/2} C_n^\lambda(u) \equiv C_n^{-1/2}(u). \tag{115}$$

$$P_n^{(-1,-1)}(u) = cte \times C_n^{-1/2}(u). \tag{116}$$

A small set of angular functions g_ℓ, obtained by the repeated application of the recurrence relation (111) or directly through the Rodrigue's formula (114), is given bellow:

$$g_0(\theta) + g_1(\theta) = \frac{1}{\sin\theta}\left(\alpha_{(0,0)} + \alpha_{(1,1)}\cos\theta\right), \tag{117}$$

$$g_2(\theta) = \frac{\alpha_{(2,0)}}{\sin\theta}\left(1-\cos^2\theta\right) = \alpha_{(2,0)}\sin\theta, \tag{118}$$

$$g_3(\theta) = \alpha_{(3,1)}\sin\theta\cos\theta = \frac{1}{2}\alpha_{(3,1)}\sin 2\theta, \tag{119}$$

$$g_4(\theta) = \frac{\alpha_{(4,0)}}{\sin\theta}\left(1-6\cos^2\theta + 5\cos^4\theta\right), \tag{120}$$

$$g_5(\theta) = \frac{\alpha_{(5,1)}\cos\theta}{3\sin\theta}\left(3-10\cos^2\theta + 7\cos^4\theta\right), \tag{121}$$

$$g_6(\theta) = \frac{\alpha_{(6,0)}}{\sin\theta}\left(1-15\cos^2\theta + 35\cos^4\theta - 21\cos^6\theta\right), \tag{122}$$

$$g_7(\theta) = \frac{\alpha_{(7,1)}\cos\theta}{5\sin\theta}\left(5-35\cos^2\theta + 63\cos^4\theta - 33\cos^6\theta\right), \tag{123}$$

and so on. The zero eigenvalue is degenerate and an arbitrary linear combination of the corresponding eigenfunctions is presented in (117). In general, they not lead to (weighted) squared integrable functions, inasmuch as the integral:

$$\int_0^\pi \frac{\left(C_i^{(-1/2)}\right)^2}{\sin\theta}\mathrm{d}\theta = \infty; \; i = 0,1. \tag{124}$$

is divergent for the $i = 0$ and $i = 1$ cases. The rest, however, satisfy the identity

$$\int_0^\pi \frac{C_n^{(-1/2)}(\cos\theta)C_m^{(-1/2)}(\cos\theta)}{\sin\theta}\mathrm{d}\theta = \frac{\delta_{mn}}{n\Gamma\left(n-\frac{1}{2}\right)}; \, n,m \ge 2, \tag{125}$$

having finite limits at the north and south poles. In fact,

$$\lim_{\theta \to 0,\pi} g_\ell(\theta) = 0$$

if $\mathbb{Z} \ni \ell \ge 2$. By an appropriated choice of parameters (117) can be made regular, either at the north or the south pole, but not both! Regularity at the north pole (when $\theta = 0$) implies

$$-g = -\alpha_{(0,0)} = \alpha_{(1,1)}$$

whereas regularity at the south pole (when $\theta = \pi$) requires

$$-g = \alpha_{(0,0)} = \alpha_{(1,1)}.$$

Turn next to the radial part. $R_\ell(r)$ is a solution of the linear differential equation:

$$\begin{aligned} &r^2 R_{\ell,rr} + 2rR_{\ell,r} - \left[\frac{\rho^2}{\langle\rho\rangle^2}\frac{r^2}{\lambda^2} + \ell(\ell-1)\right]R_\ell \\ &= r(rR_\ell)_{,r}(\ln|\rho|)_{,r}. \end{aligned} \tag{126}$$

Two special limits call for inquiry, namely, the behaviour of the R_ℓ-field at *near* and *far* distances from the source. Suppose firstly that, at spatial infinity, ρ tends to $\langle\rho\rangle$. Then, (126) reduces to the modified, spherical Bessel equation, which besides being linear and homogeneous: at infinity has as regular solutions the modified spherical Bessel functions of the third kind, often denoted as $k_n(r/\lambda)$. A few of them are listed below:

$$k_0(r/\lambda) = (r/\lambda)^{-1}\mathrm{e}^{-r/\lambda}, \tag{127}$$

$$k_1(r/\lambda) = (r/\lambda)^{-2}\mathrm{e}^{-r/\lambda}\left[(r/\lambda)+1\right], \tag{128}$$

$$k_2(r/\lambda) = (r/\lambda)^{-3}\mathrm{e}^{-r/\lambda}\left[(r/\lambda)^2 + 3(r/\lambda) + 3\right] \tag{129}$$

$$\begin{aligned} k_3(r/\lambda) = (r/\lambda)^{-4}\mathrm{e}^{-r/\lambda}\Big[&(r/\lambda)^3 \\ &+6(r/\lambda)^2 + 15(r/\lambda) + 15\Big], \end{aligned} \tag{130}$$

$$\begin{aligned} k_4(r/\lambda) = (r/\lambda)^{-4}\mathrm{e}^{-r/\lambda}\Big[&(r/\lambda)^4 + 10(r/\lambda)^3 \\ &+ 45(r/\lambda)^2 + 105(r/\lambda) + 105\Big]. \end{aligned} \tag{131}$$

The differential identity

$$k_{n+1}(r/\lambda) = -\lambda r^n \frac{\mathrm{d}}{\mathrm{d}r}\left(r^{-n}k_n(r/\lambda)\right) \tag{132}$$

generates the rest. The exponential decay with distance shown in Equations (127)-(131) and inferred from (132) is nothing more than the mathematically embodiment of the *gyrogravitational Meissner effect.*

On the other hand, in the vicinity of the source-where the relation $r \ll \lambda$ is expected—the term in (126) depending on such a parameter can be neglected, and moreover, if ρ has a minimum at some spherical core but it does not vanish there, the r.h.s. of (126) can also be ignored up to terms of quadratic order. The radial function, under these assumptions, must take the form:

$$\begin{aligned} &R_\ell(r) \approx B_{\ell-1}r^{\ell-1} + B_{-\ell}r^{-\ell}; \\ &r \approx r_{core} \lll \left(\lambda^2 + \xi^2\right)^{1/2}. \end{aligned} \tag{133}$$

Then, it is seen immediately, by restricting attention to the degenerate case: $R_0(r) = R_1(r) \quad (\ell = 1,0)$, that if

$g \neq 0$, the combination of the radial and angular dependence leads, in the neighbourhood of the centre of symmetry, the following asymptotic behaviour for the vector potential $\boldsymbol{A}(r,\theta)$:

$$A^N_{\hat{\phi}} \approx \frac{g(1-\cos\theta)}{r\sin\theta}, \tag{134}$$

$$A^S_{\hat{\phi}} \approx -\frac{g(1+\cos\theta)}{r\sin\theta}. \tag{135}$$

The first (the second) expression of the pair allows for a gravitomagnetic vector potential that is well defined at north (south) latitudes but not in the semi r-axis:

$$\theta = \pi, r > 0 \left(\theta = 0, r > 0\right),$$

referred to as the "Dirac string". As it has been depicted in **Figure 8**. By direct comparison with (100), it is seen immediately that both $\boldsymbol{A}^N$ and $\boldsymbol{A}^S$ lead to the same Hedgehog-like gravitomagnetic field:

$$\boldsymbol{H} = \nabla\times\boldsymbol{A} = \frac{g}{r^2}\hat{\boldsymbol{r}}, \tag{136}$$

whose exact form implies that g is the gravitational analogue of the "magnetic charge". Thus, the gravitomagnetic field of (136) admits a description in terms of two different gravitomagnetic vector potentials $\boldsymbol{A}^N$ and $\boldsymbol{A}^S$, each of which is not singular (except at the origin) when they are assigned to a chart dividing the north and south hemispheres respectively. Using the former construction, Stokes theorem, and (136), it is readily verified that the total flux around the centre of symmetry $r = 0$ of such a field is given by

$$\Phi = \int_{S^2(\epsilon)} \boldsymbol{H}\cdot\hat{\boldsymbol{n}}\mathrm{d}^2 S = \oint_{\mathcal{C}} \left(\boldsymbol{A}^N - \boldsymbol{A}^S\right)\cdot\hat{\boldsymbol{T}}\mathrm{d}s = 4\pi g. \tag{137}$$

where $S^2(\epsilon)$ is a sphere of radius ϵ with centre at the origin and $\mathcal{C}$ is its equator, as defined by the north and south poles; $\hat{\boldsymbol{n}}$ is a unit normal to $S^2(\epsilon)$ pointing outwards, and $\hat{\boldsymbol{T}}$ is the unit tangent vector to $\mathcal{C}$ obtained by the corkscrew's rule. Moreover, the fulfilment of the mathematical expression:

$$\nabla\cdot\boldsymbol{H} = \nabla\cdot\nabla\left(-g/r\right) = -g\Delta r^{-1} = 4\pi g\delta^3(r), \tag{138}$$

also establishes in differential form the existence of gravitomagnetic monopoles whenever $g \neq 0$.

6.1. Λ and Dirac's Quantization Condition

Gauge invariance, implicit for instance in (136), implies that the gravitomagnetic vector potentials $\boldsymbol{A}^N$ and $\boldsymbol{A}^S$, given by (134) and (135) respectively, must in fact be related by a nonsingular gauge transformation. In effect

$$\boldsymbol{A}^N = \boldsymbol{A}^S + 2g\nabla\phi = \boldsymbol{A}^S - \nabla\varphi, \tag{139}$$

where φ is the Goldstone boson field introduced in (3). Hence

$$\varphi = -2g(\phi - \phi_o), \tag{140}$$

and consequently:

$$\Psi\Psi = \rho\mathrm{e}^{-i(4ge)(\phi-\phi_o)}. \tag{141}$$

Moreover, by (2), which contains the combination $\Psi^*\Psi^*\Psi\Psi$, it is inferred that the gravitational potential amplitude Ψ^2 must be a single valued function, not changing by marching out, going around the origin one or more times, and arriving to the same spacio-temporal point of departure. Thus, if $\rho \neq 0$,

$$2g2e = n;\ n\in\mathbb{Z}. \tag{142}$$

It is seen, therefore, that Dirac's quantization condition, given in [35], means that the existence of just one gravitomagnetic monopole would imply that the cosmological constant is not only quantized but also that its square root is given in exact multiples of some smallest value!

6.2. Macroscopic Ψ for Monopoles

It is time to reflect on the underlying mathematical aspects of the companion, modulus field $\Psi^*\Psi$; technically a sort of symmetry violating dial satisfying a Lichnerowicz's like equation [6]:

$$\begin{aligned} 0 = &-8\Delta|\Psi| + {}^{(3)}\tilde{R}|\Psi| + 8e^2A^2|\Psi| \\ &-8(3e/N)^2|\Psi|^5 - N^2\left(\tilde{H}^2/2\right)|\Psi|^{-3}. \end{aligned} \tag{143}$$

To get the leading order terms, in the hedgehog-like setting, insert the monopolar Equation (134) into (143). In the asymptotic limit $\rho\to\langle\rho\rangle$ when r goes to $+\infty$, the resulting equation becomes separable and a solution of the form:

$$\begin{aligned} |\Psi| = &|\Psi_s| + \sum_{m,l}\sigma_{(m,l)}\Phi_m(\phi)S_l(r)Q^{(m,l)}(-\cos\theta) \\ &+ \text{higher order corrections} \end{aligned} \tag{144}$$

is feasible, where $\sigma_{(m,l)}$ are constant coefficients. The azimuthal, the radial, and the zenith equations that follow straightforwardly from applying the method of separation of variables are:

$$\Phi_{,\phi\phi} + m^2\Phi = 0, \tag{145}$$

$$\left(r^2S_{,r}\right)_{,r} + \left[\frac{r^2}{\xi^2} - l(l+1)\right]S = 0, \tag{146}$$

and

$$\begin{aligned} &\frac{\left(Q_{,\theta}\sin\theta\right)_{,\theta}}{\sin\theta} + l(l+1)Q - \frac{m^2}{\sin^2\theta}Q \\ &-\frac{(eg)^2(1-\cos\theta)^2}{\sin^2\theta}Q = 0 \end{aligned} \tag{147}$$

respectively.

$$\Phi(\phi) = \exp[\pm im\phi]$$

solve evidently (145). If the $l(l+1)$-term is ignored, (146) leads to

$$S_0(r) = (\xi/r)\cos(r/\xi - \delta), \tag{148}$$

compare with (59). To unravel the solutions of (147), set $Q = Q(u(\theta))$ and $u = -\cos(\theta)$, it transforms then into:

$$\begin{aligned} 0 &= \left(\left(1-u^2\right)Q_{,u}\right)_{,u} + l(l+1)Q \\ &\quad - \frac{m^2}{1-u^2}Q - \frac{(eg)^2(1+u)^2}{\left(1-u^2\right)}Q. \end{aligned} \tag{149}$$

For $|u|<1$, a solution to (149) can be found by applying the series expansion method. Putting thus

$$Q^{(l,m)}(u) = \sum_{n=0}^{\infty}\alpha_n u^n \tag{150}$$

into (149) and defining, for notational convenience,

$$\beta_{\pm}(m) = m^2 + (eg)^2 \pm l(l+1), \tag{151}$$

the recurrence relation:

$$\begin{aligned} \alpha_{n+2} &= \frac{1}{(n+2)(n+1)}\Big[\left(\beta_-(m)+2n^2\right)\alpha_n \\ &\quad + \frac{(eg)^2}{2}\alpha_{n-1} + \left(\beta_+(0)-(n-2)(n-5)\right)\alpha_{n-2}\Big], \end{aligned} \tag{152}$$

limited by the rule $\alpha_{-2} = \alpha_{-1} = 0$ is obtained immediately; α_0 and α_1, however, are arbitrary given numbers. D'Alembert's criterion and the inferable property $\lim_{n\to\infty}\alpha_n = 1$ imply that the resulting series converges for $|u|<1$ since

$$\lim_{n\to\infty}\left|\frac{\alpha_{n+1}u^{n+1}}{\alpha_n u^n}\right| = \lim_{n\to\infty}\left|\frac{\alpha_{n+1}}{\alpha_n}\right|\lim_{n\to\infty}\left|\frac{u^{n+1}}{u^n}\right| = |u| < 1. \tag{153}$$

Henceforth,

$$\begin{aligned} Q^{(l,m)}(u) &= \alpha_0\Big\{1+\frac{1}{2}\beta_-(m)u^2 + \frac{g^2e^2}{3}u^3 \\ &\quad + \frac{1}{12}\left[\beta_+(0)+\frac{1}{2}\beta_-(m)(\beta_-(m)+8\right]u^4 \\ &\quad + \frac{g^2e^2}{30}\left(2\beta_-(m)+9\right)u^5 + \mathcal{O}\left(u^6\right)\Big\} \\ &\quad + \alpha_1 u\Big\{1+\frac{1}{6}\left(\beta_-(m)+2\right)u^2 + \frac{g^2e^2}{6}u^3 \\ &\quad + \frac{1}{120}\Big[\left(2+\beta_+(0)\right)+\frac{1}{6}\left(\beta_-(m)+2\right) \\ &\quad \times\left(\beta_-(m)+18\right)\Big]u^4 + \mathcal{O}\left(u^5\right)\Big\}. \end{aligned} \tag{154}$$

In this way, the departure to $|\Psi_s|$ is obtained.

As for the limit of $|\Psi|$ at small radii, the dominant contribution arises from the $\boldsymbol{H}\cdot\boldsymbol{H}$-term. By (136), (143) reduces, in such a limit, to:

$$r^2\left(r^2|\Psi|_{,r}\right)_{,r} = \nu^2|\Psi|,$$

where $\nu = Ng/4$. Using the change of variable: $u = -1/r$, it is easily seen that when the integration constants $\mathcal{E}$ and r_{core} are nonzero, then

$$\rho = |\Psi|^2 = \mathcal{E}^2\left[\left(1/r_{core}\right)-\left(1/r\right)\right]^2 + (\nu/\mathcal{E})^2 \geq (\nu/\mathcal{E})^2 \tag{155}$$

solves the problem, and thus, if $\nu = Ng/4 \neq 0$:

$$\lim_{r\to r_{core}}\left(\ln|\rho|\right)_{,r} = 0. \tag{156}$$

7. A Note on Dyons

From the multipolar expansion (106) of the A-scalar field-which it must not be forgotten, it was obtained under the assumption of an orthonormal, spherically symmetric grid, the gravitomagnetic form $A_\phi \mathrm{d}\phi$ can be extracted. By combining (118) and (129), it is seen immediately that the dipolar $(\ell = 2)$ contribution to the $A_\phi \mathrm{d}\phi$ differential encompass, for instance, the following form:

$$\begin{aligned} A_\phi \mathrm{d}\phi &= (r/\lambda)^{-2}\,\mathrm{e}^{-r/\lambda}\left[(r/\lambda)^2 + 3(r/\lambda)+3\right] \\ &\quad \times 4g\sin^2(\theta/2)\mathrm{d}\phi, \end{aligned} \tag{157}$$

for $r \gg \lambda$ and $\rho \to \rho_s$. However, if $r \to 0^+$, (118) and (133) implies (for $l = 2$) that:

$$A_\phi \mathrm{d}\phi = 4g\sin^2(\theta/2)\mathrm{d}\phi. \tag{158}$$

Both expressions, one when the space-time becomes superconducting and the other valid in the vicinity of the source, contain the

$$A_\phi \mathrm{d}\phi = 4g\sin^2(\theta/2)\mathrm{d}\phi$$

piece of the Taub-NUT space: an exact, spatially homogeneous solution of the vacuum Einstein's equations, first discovered in 1951 by A. H. Taub and extended analytically a little bit later by E. Newman, L. Tamburino, and T. Unti [44,45]. The Taub-NUT solution has topology $\mathbf{R}^1\times\mathbf{S}^3$. It determines the gravitational field produced by a gravitational dyon of mass M and gravitating magnetic mass g [46]. Its metric, in gravitomagnetic units and isotropic coordinates, can be written, if $\bar{r} > M/2$, as:

$$\begin{aligned} \mathrm{d}s^2 &= -|\Psi|^{-4}\left(1-\frac{M^2}{4\bar{r}^2}\right)^2\left[\mathrm{d}t + 4g\sin^2(\theta/2)\mathrm{d}\phi\right]^2 \\ &\quad + |\Psi|^4\left(\mathrm{d}\bar{r}^2 + \bar{r}^2\mathrm{d}\theta^2 + \bar{r}^2\sin^2\theta\mathrm{d}\phi^2\right), \end{aligned} \tag{159}$$

where $\theta \neq \pi$. $|\Psi|$ reduces to:

$$|\Psi|^4 = \left(1+\frac{M}{2\bar{r}}\right)^4 + \frac{g^2}{\bar{r}^2}. \tag{160}$$

Here t, θ, and ϕ are Euler coordinates on $\mathbf{S}^3$, while $\bar{r}$ denotes the isotropic radial coordinate. Common points between (155) and (160) can be established at $\bar{r} = M/2$. The Taub-NUT space-time is better visualized by the product of the Hopf fibering

$$S^1 \rightarrowtail S^3 \xrightarrow{\pi} S^2$$

with the r-axis $\sim \mathbf{R}^1$. $\bar{r} = M/2$ corresponds to an horizon and $\bar{r} = 0$ to an irremovable curvature singularity [46]. In (159) ${}^{(3)}\tilde{R} = 0$; however, any establishment of space-time superconducting requires ${}^{(3)}\tilde{R} > 0 \quad (\Lambda \neq 0)$ as depicted in **Figure 2**. Current observation limits ${}^{(3)}\tilde{R}$ to a rather small value.

8. A Note on Topology Change

When $g = 0$ but B_o does not vanish, (117) and (133) imply:

$$A_{\hat{\phi}}^{N}(r,\theta) = B_0 \frac{1-\cos\theta}{\sin\theta} \tag{161}$$

The r^{-1} factor of (134) has utterly disappeared (similar comments apply for $\boldsymbol{A}^S$). In that case, by (100):

$$\boldsymbol{H} = \nabla \times \boldsymbol{A} = r^{-1} B_o \left(1; -\frac{1-\cos\theta}{\sin\theta}; 0\right), \tag{162}$$

implying at once:

$$\boldsymbol{H}^2 = \left(B_o^2 / r^2 \cos^2(\theta/2)\right). \tag{163}$$

Such gravitomagnetic field is singular along the nonpositive z-semi-axis and it can be interpreted as a tear or cut (labeled a in **Figure 9**) in the very fabric of the space-time. The lines of force create a family of parabolic curves cutting orthogonally the equipotential surfaces: a coaxial set of paraboloids of revolution. In the figure, the gravitomagnetic-field intensity can be read off from the arrow's length. Space-time ripping opens the possibility of topology change in quantum gravity.

9. Remarks and Conclusions

The universe has at least several billions of galaxies. An explanation of their origin and stability remains, however, uncertain. And it rises two of the most fundamental open problems in Physics, like one time ago were the structure and stability problems of the atom, that led to the quantum revolution. In the present state of knowledge, no man of our age, no inflationist, no dark matter theorist, can claim to know for sure the identity of the primeval mechanism giving rise to the rich galactic tapestry observed across the cosmos, or the way that the dismantling of the largest structures of the universe is avoided.

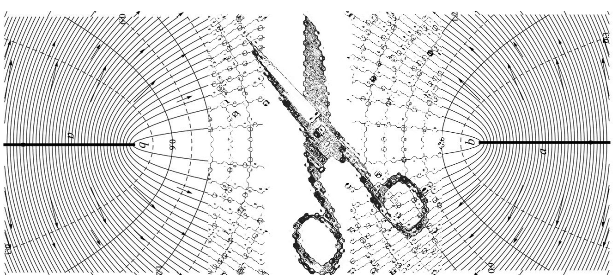

Figure 9. Spatio-temporal ripping of the space-time fabric. The figure shows an artistic representation of the second order phase transition effecting the nature of space and time due to the Bose-Einstein condensation outlined by the law of gravity set by Equation (7).

The lack of a fully satisfactory explanation to such basic issues, and the impression that perhaps inflation and dark matter (although actively pursued) are not enough, because there are still relevant missing pieces needed to achieve complete clarification, like the unification of quantum mechanics with the general theory of relativity, certainly calls for a revision of our most cherished ideas, concerning the nature of space, time, and matter. It is in this spirit, that we have set to explore the likeliness that "even at large scales" the space-time might exhibit some very striking properties of purely quantum origin, which might well have passed unnoticed, in favour of other powerful possibilities that have become very tight to our way of thinking, like the lightest supersymmetric particle idea which depends on the assumption of *R*-parity conserving supersymmetry.

The point of view adopted in this article is that at the core of the dark matter conundrum lies the problem of finding how to develop a consistent law of inertia for a discrete, quantum fluctuating, space-time background, and not necessarily the presence-in some exact proportion-of an entirely new class of particle per se, such as the neutralino or the invisible axion, that may or may not exist in the required amounts.

According to the Einstein-Hilbert action, asymptotic flatness seems like a very natural restriction to follow for the classical geometry due to an isolated, static, point-like source in empty space. However, at the fundamental level, the 4-dimensional space-time of our direct experience might not be a continuum [47] and discrete entities ("space-time atoms") might rule its dynamics [48,49]. This possibility might be enough to radically change the picture provided by classical theory, and to put into question the "relevance" of the asymptotic flatness hypothesis: all the entities that we know about in Nature, at least the ones which are linked (in one away or the other) to indistinguishable particles, follow well defined statistical rules: Bose-Einstein or Fermi-Dirac statistics. Therefore, it seem natural to speculate that perhaps such "space-time atoms" could suffer from a similar identity crisis than the one known to exist in superfluids [6,50], specifically in a situation where phrases like: "low temperature" and "the lowest state of energy" apply. All these reasons, including the analogy of an stable atom surrounded by superconducting currents, but more strongly, the precise mathematical form, piece by piece, of the Einstein-Hilbert action led unavoidably to the exploration of the "space-time as a superconductor" paradigm [6]; an exercise that it is not only useful to provide a fertile arena for contrasting (and estimate) how classical ideas about the nature of the space-time can get altered when quantum affects are taken into account, but also to translate the main difficulties encountered in the dynamical study of galaxies in a completely new language, where different technical tools can be put in practice with the hope of getting better insights about how to handle the fundamental unsolved questions of their dynamics, such as the well known "winding dilemma".

This article focuses primarily on finding out the distinctive, physical consequences of modifying, in precise accord to a gauge principle [see (5), (6), and (7)], the l.h.s. of Einstein's field equations, by the addition of a phase factor to one of the gravitational potentials: Equations (2) and (3). The first thing that comes across is the similarity of such gravitational theory with the theory of superconductivity, which surely cannot be an accident.

It was highlighted that:

- *Firstly*: The cosmological constant (which in terms of Planck units is as small as 10^{-122}) can be linked with the minimal gravitomagnetic (or gyrogravitational) flux supported by a spinning string [see (58) and (142)].
- *Secondly*: the appearance of supercurrents around rotating astrophysical bodies can modify the space-time geometry in such a way that the aforementioned complex potential gets a modulus which remains very close to a characteristic value [e.g. (59), (144) and (148)], while the gravitomagnetic vector potential acquires mass causing an exponentially decay (with distance) of the gravitomagnetic field [e.g. (127)-(131)]; this picture, as discussed in [6], provides an alternative to the dark-matter-halo hypothesis, see **Figure 9**.
- *Thirdly*: vortex and monopole solutions can be found [Section 4 and Section 6] exhibiting in full the superfluid properties of the space-time; the critical point of the quantum phase transition, where the order Ψ-parameter vanishes, takes place at space-time singularities, see **Figure 9**.
- *Fourthly*: the close enough mathematical similitude that can be established with subatomic models where hadrons are viewed as being made up of quarks bound by dual strings [51,52], suggests the application of the scheme just presented to the study of open strings having gravitomagnetic monopoles at their ends, or where, spinning strings (open or closed) break or join when they interact.
- *Finally*: *but not least,* two crucial differences when a comparison is made with the type of superconductivity found in metals like *Nb* is that, firstly, our theory is not renormalizable, and secondly, that two axis-aligned quantum vortices with the same sense of spin not only exhibit zones of repulsion but also of attraction, depending on their relative geodetic distance [see (86)]; this in itself is an invitation to reflect, in this new setting, on the spin-statistic theorem and supersymmetry.

It might seem fitting to recall (as a matter of reflexion or even as an historical panoramic view) Einstein's own remarks, set in the 1920s, regarding the revolutionary impact brought in by the discovery of superconductivity [53]:

"*The theoretical oriented scientist cannot be envied, because Nature, i.e. the experiment, is a relentless and not very friendly judge of his work. In the best case scenario it only says 'maybe' to a theory, but never 'yes' and in most cases 'no'. If an experiment agrees with theory it means 'perhaps' for the later. If it does not agree it means 'no.' Almost any theory will experience a 'no' at one point in time-most theories very soon after they have been developed. In this paper we want to focus on the fate of theories concerning metallic conductivity*."

In summary, the list on the left assembles some of the features-topological traces if you will-that should be present in our universe if the space-time behaves, in some places and times, as a superconductor: obeying (say) an action principle like (7). According to the train of thought pushed forward: when this happens, the $(\kappa = \lambda/\xi = 3/2)$ space-time is quite capable of producing quantum vortices of (minimal) quantized gravitomagnetic flux $\Phi_o = \pi/e$, looping back on themselves to form rings, see **Figure 6**. At a first rough approximation, such entities should obey the Nambu-Goto action to take into account any relativistic Lorentz contraction of their vortex core, or better yet, a sort of Kalb-Ramond effective action to incorporate the topological coupling to the Goldstone boson field; what is more, by virtue of (86), they also self interact. All these features, together with the "hydrodynamical" Magnus effect, are expected to be crucial to correctly obtain their effective equation of motion: these distinctive "hula-hoop" structures not only are capable of reproducing spin-2 effects [52] but also are the natural analogue of the higher energy excitations referred to as "rotons" in helium II [54,55]. Such looped excitations, let's call them gravitational rotons, can form supercurrents: which can persist for very long times, as Cooper pairs do in a superconducting wire, or photons do in a laser, or electrons in an atom, affecting the inertia of test orbiting bodies by a frame-dragging effect and producing places of uniformity in the spatial geometry [see **Figure 4** and Equations (59) and (148)].

Thus, a crucial question arises: Could these theoretical ideas: superfluidity of the space-time, supercurrents of gravitational rotons, quantization of vacuum energy, frame dragging, as well as a Higgs mechanism for gravity-located in the borderline typified by the shifty split between micro-macro, reversible-irreversible, and quatum-classical, bring us closer to identify the nature of cold dark matter?

Is the space-time a superconductor?

We do not know the answer yet, but surely Nature will tell us.

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