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Spectra of 2 × 2 Upper-Triangular Operator Matrices

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ABSTRACT

In [Perturbation of Spectrums of 2 × 2 Operator Matrices, *Proceedings of the American Mathematical Society*, Vol. 121, 1994], the authors asked whether there was an operator $C_0 \in B(K,H)$ such that $\sigma(M_{C_0}) = \bigcap_{C\in B(K,H)} \sigma(M_C)$ for a given pair (A,B) of operators, where the operator $M_C \in B(K \oplus H)$ was defined by $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. In this note, a partial answer for the question is given.

Keywords: Spectra; Upper-Triangular Operator Matrix; Fredholm Operator

1. Introduction

In the last decades considerable attention has been paid to upper triangular operator matrices, particularly to spectra of operator matrices, see [1-8]. H. Du and J. Pan firstly researched the intersection of the spectra of 2 × 2 upper triangular operator matrices, and also proposed some open problems. In this note, we mainly study these problems.

For the context, we give some notations. Let H and K be Hilbert spaces, $B(H)$, $B(K)$ and $B(K,H)$ denote the sets of all linear bounded operators on H, K and from K into H, respectively. For $A \in B(H)$, $B \in B(K)$, $C \in B(K,H)$, define an operator $M_C \in B(K \oplus H)$ by

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

Let $N(T), R(T), \sigma(T), \sigma_p(T)$, $\sigma_{ap}(T)$, $\rho(T)$, $n(T)$ and $d(T)$ denote the nullspace, the range, the spectrum, the point spectrum, the approximation point spectrum of the resolvent set, the nullity and the deficiency of an operator T, respectively, where

$$n(T) = \dim N(T) \quad \text{and} \quad d(T) = \dim N(T^*)$$

use $F_l(H)$, $F_r(H)$ and $SF(H)$ to denote the sets of left Fredholm operators, right Fredhlom operators and semi-Fredholm operators in $B(H)$, respectively. If T is a semi-Fredholm operator, define the index of T, $indT$, by $indT = n(T) - d(T)$. Note that $indT \in \mathbb{Z} \cup \{\pm\infty\}$ and it is necessary for either $N(T)$ or $N(T^*)$ to be finite dimensional in order for (1) to make sense ([3]).

For $A \in B(H)$, $B \in B(K)$, denote

$$U_k^0(A,B) = \{\lambda \in \sigma(A) \cup \sigma(B) : n(B-\lambda) = d(A-\lambda) = k \text{ and } d(B-\lambda) = n(A-\lambda) = 0\},$$

$$\Lambda_{(A,B)} = (\sigma(A) \cup \sigma(B)) \setminus \bigcap_{C\in B(K,H)} \sigma(M_C).$$

Under the situation that do not cause confusion, we simplify $U_k^0(A,B)$ as U_k^0.

In [2], H. Du and J. Pan have proved that,

$$\bigcap_{C\in B(K,H)} \sigma(M_C) = \sigma_{ap}(A) \cup \sigma_\delta(B) \cup \{\lambda : n(B-\lambda) \neq d(A-\lambda)\}, \tag{1}$$

for given $A \in B(H)$ and $B \in B(K)$, the author asked a question that whether there exists an operator $C_0 \in B(K,H)$ such that

$$\sigma(M_{C_0}) = \bigcap_{C\in B(K,H)} \sigma(M_C)?$$

In this note, when $\Lambda_{(A,B)} = \bigcap_{k=0}^{n} U_k^0$ (n is a natural number), an affirmative answer of the question has been obtained.

2. Main Results and Proofs

To prove the main result, we begin with some lemmas.

Lemma 1. ([2]). Given $A \in B(H)$, $B \in B(K)$, then

$$\Lambda_{(A,B)} = \bigcap_{k=0}^{\infty} U_k^0 .$$

Lemma 2. ([9]). Let G be an open connected subset of $\sigma(A) \setminus \sigma_e(A)$ and suppose $\lambda_0 \in G$ such that $ind(A-\lambda_0) = 0$, then there is a finite-rank operator F such that $A+F-\lambda_0$ is invertible, and also $A+F-\lambda$ is invertible for every $\lambda \in G$.

For any $C' \in B(K,H)$, it is clear that

$$\bigcap_{C \in B(K,H)} \sigma(M_C) \subset \sigma(M_{C'}) \subset \sigma(A) \cup \sigma(B).$$

If there exists a $C_0 \in B(K,H)$ such that

$$\sigma(M_{C_0}) \cap \Lambda_{(A,B)} = \Phi ,$$

then

$$\sigma(M_{C_0}) = \bigcap_{C \in B(K,H)} \sigma(M_C).$$

But how to construct the operator such that

$$\sigma(M_{C_0}) \cap \Lambda_{(A,B)} = \Phi \ ?$$

In the next theorem, we give a necessary condition of the answer of the question.

Theorem 3. For a given pair (A,B) of operators, where $A \in B(H)$, $B \in B(K)$, if $\Lambda_{(A,B)} = \bigcap_{k=0}^{n} U_k^0$ (n is a natural number) and each U_k^0 has finite simple connected open sets, then there exists an operator $C_0 \in B(K,H)$ such that

$$\sigma(M_{C_0}) = \bigcap_{C \in B(K,H)} \sigma(M_C).$$

Proof. For convenience, we divide the proof into two cases.

Case 1. If $n = 0$, that is, $\Lambda_{(A,B)} = U_k^0 = \Phi$, let $C_0 = 0$. It is easy to see that $\sigma(A) \cup \sigma(B) = \sigma(M_C)$ from lemma 1. Thus

$$\sigma(M_0) = \sigma(A) \cup \sigma(B) = \bigcap_{C \in B(K,H)} \sigma(M_C),$$

so the result is obtained.

Case 2. If $n \neq 0$, that is, $\Lambda_{(A,B)} = \bigcap_{k=0}^{n} U_k^0$. Then $\Lambda_{(A,B)}$ has finite simple connected open sets, now reordering and denoting by $U_k (k = 1,2,\cdots,s)$. Thus there exists a natural number m_k such that

$$\begin{aligned} U_k = \{ \lambda \in \sigma(A) \cup \sigma(B) : n(B-\lambda) = d(A-\lambda) \\ = m_k \text{ and } d(B-\lambda) = n(A-\lambda) = 0 \}. \end{aligned}$$

For each U_k choose a $\lambda_k \in U_k$, then λ_k is a finite subset of $\Lambda_{(A,B)}$ and

$$ind(M_{C_0} - \lambda_k) = ind(A-\lambda_k) + ind(B-\lambda_k).$$

Next, the rest of proof is divided into two steps.

Step 1. We construct C_0 as follows:

Let $\{f_j^1\}_{j=1}^{m_1}$ and $\{g_j^1\}_{j=1}^{m_1}$ are orthonormal basis for $N(B-\lambda_1)$ and $R(A-\lambda_1)^{\perp}$, respectively and denote

$$N_0(B-\lambda_1) = N(B-\lambda_1), \ R_0(A-\lambda_1) = R(A-\lambda_1).$$

First define an operator V_1 from $N_0(B-\lambda_1)$ onto $R_0(A-\lambda_1)^{\perp}$ by $V_1 f_j^1 = g_j^1$, $1 \le j \le m_1$. Then define C_1 by

$$\begin{cases} C_1 f = V_1 f, & f \in N_0(B-\lambda_1), \\ C_1 f = 0, & f \perp N_0(B-\lambda_1). \end{cases}$$

It is clear that C_1 is well defined and $C_1 \in B(K,H)$. If $s = 1$, then let $C_0 = C_1$.

If $s \neq 1$, let $\{f_j^1\}_{j=1}^{m_2}$ and $\{g_j^1\}_{j=1}^{m_2}$ be orthonormal basis for $N(B-\lambda_2)$ and $R(A-\lambda_2)^{\perp}$, respectively. It is clear that $\{f_j^1\}_{j=1}^{m_2}$ and $\{f_j^1\}_{j=1}^{m_1}$ are linear independent. then there must be unit vectors

$$f_1'^2 \perp \{f_j^1\}_{j=1}^{m_1}, \ f_2'^2 \perp \{f_j^1\}_{j=1}^{m_1} \cup \{f_1'^2\}, \cdots,$$

$$f_{m_2}'^2 \perp \{f_j^1\}_{j=1}^{m_1} \cup \{f_j'^2\}_{j=1}^{m_2-1}$$

such that

$$f_1^2 = \sum_{j=1}^{m_1} \alpha_{1j} f_j^1 + f_1'^2,$$

$$f_2^2 = \sum_{j=1}^{m_1} \alpha_{2j} f_j^1 + \gamma_{21} f_1'^2 + f_2'^2, \cdots,$$

$$f_{m_2}^2 = \sum_{j=1}^{m_1} \alpha_{mj} f_j^1 + \sum_{j=1}^{m_2-1} \gamma_{2j} f_j'^2 + f_{m_2}'^2.$$

Define an operator V_2 as follows:
Let

$$g_1'^2 = g_1^2 - C_1 f_1^2 \text{ and } V_2 f_1'^2 = g_1'^2,$$

$$g_2'^2 = g_2^2 - C_1 f_2^2 - \gamma_{21} V_2 f_1'^2 \text{ and } V_2 f_2'^2 = g_2'^2, \cdots,$$

$$g_{m_2}'^2 = g_{m_2}^2 - C_1 f_{m_2}^2 - \sum_{j=1}^{m_2-1} \gamma_{2j} V_2 f_j'^2 \text{ and } V_2 f_{m_2}'^2 = g_{m_2}'^2 .$$

Since $\{g_j^1\}_{j=1}^{m_1}$ be and $\{g_j^2\}_{j=1}^{m_2}$ be are linear independent, $\{g_j'^2\}_{j=1}^{m_2}$ is linear independent. Let

$$N_0(B-\lambda_2) = \vee \{f_j'^2\}_{j=1}^{m_2}$$

and

$$R_0(A-\lambda_2)^{\perp} = \vee \{g_j'^2\}_{j=1}^{m_2}.$$

Then $N(B-\lambda_1) \perp N_0(B-\lambda_2)$ and V_2 is an operator from $N_0(B-\lambda_2)$ onto $R_0(A-\lambda_2)^{\perp}$. Define C_2 by

$$\begin{cases} C_2 f = (V_1 \oplus V_2) f, & f \in N_0(B-\lambda_1) \oplus N_0(B-\lambda_1), \\ C_2 f = 0, & f \perp N_0(B-\lambda_1) \oplus N_0(B-\lambda_1). \end{cases}$$

The process can be similarly done continuously.

Let $\{f_j^s\}_{j=1}^{m_s}$ and $\{g_j^s\}_{j=1}^{m_s}$ be orthonormal basis for $N(B-\lambda_s)$ and $R(A-\lambda_s)^{\perp}$, respectively. It is clear that $\left\{\{f_j^k\}_{j=1}^{m_k}\right\}_{k=1}^{s}$ is linear independent. Then there must be unit vectors

$$f'^s_1 \perp \oplus_{k=1}^{s-1} N_0(B-\lambda_k),$$

$$f'^s_2 \perp \oplus_{k=1}^{s-1} N_0(B-\lambda_k) \cup \{f'^s_1\}, \cdots,$$

$$f'^s_{m_s} \perp \oplus_{k=1}^{s-1} N_0(B-\lambda_k) \cup \{f'^s_j\}_{j=1}^{m_s-1}$$

such that

$$f_1^s = \sum_{j=1}^{m_1} \alpha_{1j}^s f_j^1 + \sum_{j=1}^{\sum_k^{s-1} m_k} \gamma_{1j}^s f'^k_j + f'^s_1,$$

$$f_2^s = \sum_{j=1}^{m_1} \alpha_{2j}^s f_j^1 + \sum_{j=1}^{\sum_k^{s-1} m_k} \gamma_{2j}^s f'^k_j + \delta_{s1} f'^s_1 + f'^s_2, \cdots,$$

$$f_{m_s}^s = \sum_{j=1}^{m_1} \alpha_{sj}^s f_j^1 + \sum_{j=1}^{\sum_k^{s-1} m_k} \gamma_{sj}^s f'^k_j + \delta_{sj} f'^s_j + f'^s_{m_2}.$$

Define an operator V_s as follows:
Let

$$g'^s_1 = g_1^s - C_{s-1} f_1^s \text{ and } V_s f'^s_1 = g'^s_1,$$

$$g'^s_2 = g_2^s - C_{s-1} f_2^s - \delta_{21} V_s f'^s_1 \text{ and } V_s f'^s_2 = g'^s_2, \cdots,$$

$$g'^s_{m_s} = g_{m_s}^s - C_{s-1} f_{m_s}^s - \sum_{j=1}^{m_s-1} \delta_{sj} V_s f'^s_j \text{ and } V_s f'^s_{m_s} = g'^s_{m_s}.$$

Since $\left\{\{g_j^1\}_{j=1}^{m_k}\right\}_{k=1}^{s}$ is linear independent, $\{g'^s_j\}_{j=1}^{m_s}$ is linear independent. Denote

$$N_0(B-\lambda_s) = \vee\{f'^s_j\}_{j=1}^{m_s} \text{ and } R_0(A-\lambda_s)^{\perp} = \vee\{g'^s_j\}_{j=1}^{m_s}.$$

Then

$$N_0(B-\lambda_i) \perp N_0(B-\lambda_j),$$

$1 \le i \ne j \le s$ and V_s is an operator from $N_0(B-\lambda_s)$ onto $R_0(A-\lambda_s)^{\perp}$. Define C_s by

$$\begin{cases} C_s f = (\oplus_{k=1}^s V_k) f, & f \in \oplus_{k=1}^s N_0(B-\lambda_k), \\ C_s f = 0, & f \perp \oplus_{k=1}^s N_0(B-\lambda_k). \end{cases}$$

Let $C_0 = C_s$. It is clear that C_0 is well defined and bounded with finite rank. By directly computation, we can get

$$\begin{cases} C_0 f_j^k = g_j^k, & 1 \le k \le s, 1 \le j \le m_k, \\ C_0 f = 0, & f \perp \oplus_{k=1}^s N_0(B-\lambda_k). \end{cases}$$

Step 2. We prove that $C_0 \in B(K,H)$ defined as above such that

$$\sigma(M_{C_0}) = \cap_{C \in B(K,H)} \sigma(M_C).$$

It is sufficient to prove that for any $\lambda \in \Lambda_{(A,B)}$, $M_{C_0} - \lambda$ is invertible. From Lemma 2, it is only to prove for any λ_k, $M_{C_0} - \lambda_k$ is invertible. To finish it, it is to prove that $M_{C_0} - \lambda_k$ is injective and surjective.

If there exists a vector $x_0 = y_0 \oplus z_0$ with

$$(M_{C_0} - \lambda_k) x_0 = 0,$$

where $y_0 \in H$ and $z_0 \in K$, then $z_0 \in N(B-\lambda_k)$ and $C_0 z_0 = -(A-\lambda_k) z_0 \in R(A-\lambda_k)$. By definition of C_0, then $C_0 z_0 \in R(A-\lambda_k)^{\perp}$, thus $C_0 z_0 = 0$. On the other hand, since C_0 is injective on $N(B-\lambda_k)$, then $z_0 = 0$, and so, $(A-\lambda_k) y_0 = 0$. By assumption that $\lambda_k \notin \sigma_{ap}(A)$, hence $y_0 = 0$. Therefore $M_{C_0} - \lambda_k$ is injective.

For any vector $x = y \oplus z$, where $y \in H$ and $z \in K$. Since $\lambda_k \notin \sigma_\delta(B)$ and $\lambda_k \notin \sigma_{ap}(A)$, $R(B-\lambda_k) = K$ and $R(A-\lambda_k)$ is closed. Thus there is a vector $z_1 \in K$ such that $(B-\lambda_k) z_1 = z$. Because $y - C_0 z_1 \in H$, there exist $\xi \in R(A-\lambda_k)$ and $\zeta \in R(A-\lambda_k)^{\perp}$ such that $y - C_0 z_1 = \xi + \zeta$. Hence there exist $z_2 \in N(B-\lambda_k)$ and $y_1 \in H$ such that $C_0 z_2 = \zeta$ and $\xi = (A-\lambda_k) y_1$. The last equality is possible, because C_0 is onto $R(A-\lambda_k)^{\perp}$. Therefore,

$$(M_{C_0} - \lambda_k)\begin{pmatrix} y_1 \\ z_1 + z_2 \end{pmatrix} = \begin{pmatrix} (A-\lambda_k) y_1 + C_0(z_1 + z_2) \\ (B-\lambda_k)(z_1 + z_2) \end{pmatrix} = \begin{pmatrix} \xi + C_0 z_1 + \zeta \\ z \end{pmatrix} = \begin{pmatrix} y \\ z \end{pmatrix}.$$

As x is arbitrary, $M_{C_0} - \lambda_k$ is surjective.

Hence, for any $\lambda \in \Lambda_{(A,B)}$, $M_{C_0} - \lambda$ is invertible, *i.e.*, $\sigma(M_{C_0}) \cap \Lambda_{(A,B)} = \Phi$. So $\sigma(M_{C_0}) = \cap_{C \in B(K,H)} \sigma(M_C)$. The proof is completed.

Example 4. If $A = U$ and $B = U^*$, U is the shift operator on l^1, let

$$C_0(\xi_0, \xi_1, \xi_2, \cdots) = (\xi_0, 0, 0, \cdots),$$

then M_{C_0} is invertible. From directly computation, $\Lambda_{(A,B)} = D$ and $\sigma(A) \cup \sigma(B) = \bar{D}$, where D is the interior of unit disk. For any $\lambda \in \Lambda_{(A,B)}$, $M_{C_0} - \lambda$ is invertible. Thus $\sigma\left(M_{C_0}\right)=\bigcap_{C\in B\left(l^1\right)}\sigma\left(M_C\right)=\bar{D}$.

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