Journal of Applied Mathematics and Physics, **2013, 1, 28-36**
http://dx.doi.org/10.4236/jamp.2013.13006 Published Online August 2013 (http://www.scirp.org/journal/jamp)

Comparison of Solution Methods for some Classical Flow Problems in Rarified Gas Dynamics

Ignatius N. Njoseh[1], Alex Musa[2]
[1]Department of Mathematics and Computer Science, Delta State University, Abraka, Nigeria
[2]Department of Mathematics and Statistics, University of Port-Harcourt, Port-Harcourt, Nigeria
Email: njoseh@delsu.edu.ng, alexmusa66@yahoo.com

Received July 216, 2013; revised August 19, 2013; accepted September 10, 2013

ABSTRACT

A comparison of two methods of solution to classical flow problem in rarefied gas dynamics was presented. The two methods were chosen to examine the effect of the following transport phenomena (pressure gradient and temperature difference) viz Poiseuille and Thermal creep respectively on the flow of rarefied gas. The governing equations were approximated using BGK model. It was shown that while the Discrete Ordinate Method could consider more values of the accommodation coefficients, the Finite Difference Method can only take accommodation coefficient of one. It was also shown that the flow rate has its minimum in both solution methods at $K_n = 0.1$ in the transition regime and that as the channels get wider, the Thermal creep volume flow rates get smaller.

Keywords: Discrete Ordinate; Finite Difference; Pressure Gradient; Temperature Difference; Knudsen Number

1. Introduction

In the recent literature there is a growing interest to solve problems in rarefied gas dynamics. The reader is referred to [1-6], and other references therein for an overview of the recent work in this area. Earlier researches [7-12] solved rarefied gas dynamics problems using different methods. It has been shown that these methods yield good results. The main objective of this work is to do a comparison of two of the most widely used methods in the numerical study of rarefied gas flow problem: the Discrete Ordinate method (DOM) and the Finite Difference Method (FDM). Though the literature concerning our area of study is very intensive, we shall review a few of them.

Barichello, *et al.* [13] studied a version of the discrete-ordinates method to solve in a unified manner some classical flow problems based on the Bhatnagar, Gross and Krook model in the theory of rarefied gas dynamics. In particular, the thermal-creep problem and the viscous-slip (Kramer's) problem are solved for the case of a semi-infinite medium, and the Poiseuille-flow problem, the Couette-flow problem and the thermal-creep problem are all solved for a wide range of the Knudsen number. Also Scherer and Barichello [14] studied an analytical version of the discrete-ordinates method, the ADO method, to solve two problems in the rarefied gas dynamics field, which describe evaporation/condensation between two parallel interfaces and the case of a semi-infinite medium. The modeling of the problems is based on a general expression which may represent four different kinetic models.

In [15], the problem of heat transfer and temperature distribution in a binary mixture of rarefied gases between two parallel plates with different temperatures on the basis of kinetic theory was investigated. Under the assumptions that the gas molecules are hard spheres and undergo diffuse reflection on the plates, the Boltzmann equation was analyzed numerically by means of an accurate finite difference method, in which the complicated nonlinear collision integrals are computed efficiently by the deterministic numerical kernel method. As a result, the overall quantities are obtained accurately for a wide range of the Knudsen number. At the same time, the behavior of the velocity distribution function is clarified with high accuracy.

Muljadi and Yang [16] obtained a direct method for solving rarefied flow of gases of arbitrary particle statistics. The method is based on semi-classical Boltzmann equation with BGK relaxation time approximation. The discrete ordinate method is first applied to render the Boltzmann equation into hyperbolic conservation laws with source terms, and then classes of explicit and implicit time integration schemes are applied to evaluate the discretized distribution function. The method is tested on both transient and steady flow problems of gases of arbitrary statistics at varying relaxation times.

Also worthy of note are the works of [17-22] and other references therein.

2. The Linearized Boltzmann Equation

The non-linearity form of the Boltzmann equation is essential in application if the gas is far from thermal equilibrium. However, if the state of the gas is near thermal equilibrium, a linearized form of the Boltzmann equation will provide a reasonably accurate description of the transport phenomena. This form assumed that the perturbation of the velocity distribution from its equilibrium form is small.

Following the work in [23] a linearized form of the Boltzmann equation was given as

$$\begin{aligned} & c_x\left[\left(c^2-\frac{3}{2}\right)k_x+R_x+2c_xK_0\right]+c_z\left(c^2-\frac{3}{2}\right)K_z \\ & +c_zR_z+\frac{c_x\mathrm{d}h(x,c)}{\mathrm{d}x}+\lambda_0 h(x,c) \\ & =\frac{\lambda_0}{\pi^{\frac{3}{2}}}\int \mathrm{d}c'\exp\left[-c'^2\right]h(x,c') \\ & \quad\times\left[1+2cc'+\frac{2}{3}\left(c^2-\frac{3}{2}\right)\left(c^2-\frac{2}{3}\right)\left(c'^2-\frac{3}{2}\right)\right] \end{aligned} \tag{1}$$

where h is a disturbance caused to the local Maxwellian, R_x is the relative density in the x-direction, K_x is the temperature gradient in the x-direction, $c=v\left(\frac{m}{2KT}\right)^{\frac{1}{2}}$ and $\lambda_0=\lambda\left(\frac{m}{2KT}\right)^{\frac{1}{2}}$.

3. Discrete Ordinate Method

Consider the flow of rarefied gas in z-direction between two parallel plates separated by a distance d. the origin is chosen in the middle of the channel so that the coordinate y varies from $\frac{-d}{2}$ to $\frac{d}{2}$.

Following the linearized Boltzmann Equation (1), we seek the solution to the equation:

$$\xi\frac{\partial z}{\partial x}(x,\xi)+Z(x,\xi)=\pi^{-\frac{1}{2}}\int_{-\infty}^{\infty}\exp\left(-u^2\right)Z(x,u)\mathrm{d}u \tag{2}$$

For $x\in\left(-\frac{d}{2},\frac{d}{2}\right)$ and $\xi\in(-\infty,\infty)$, subject to the boundary conditions:

1) For Couette flow

$$Z(-a,\xi)-(1-\alpha)Z(-a,-\xi)=\alpha \tag{3}$$

$$Z(a,-\xi)-(1-\alpha)Z(a,\xi)=-\alpha \tag{4}$$

2) Poiseuille flow

$$Z(-a,\xi)-(1-\alpha)Z(-a,-\xi)=\alpha\xi^2+a(2-\alpha)\xi \tag{5}$$

$$Z(a,-\xi)-(1-\alpha)Z(a,\xi)=\alpha\xi^2+a(2-\alpha)\xi \tag{6}$$

3) Thermal flow

$$Z(-a,\xi)-(1-\alpha)Z(-a,-\xi)=\frac{1}{2}\alpha\left(\xi^2-\frac{1}{2}\right) \tag{7}$$

$$Z(a,-\xi)-(1-\alpha)Z(a,\xi)=\frac{1}{2}\alpha\left(\xi^2-\frac{1}{2}\right) \tag{8}$$

Rewriting (2) we have

$$\begin{aligned} & \xi\frac{\partial z}{\partial x}Z(x,\xi)+Z(x,\xi) \\ & =\int_{-\infty}^{\infty}\psi(\xi)\left\{Z(x,\xi')+Z(x,-\xi')\mathrm{d}\xi'\right\} \end{aligned}$$

where

$$\psi(\xi)=\pi^{-\frac{1}{2}}\exp\left[-\xi^2\right] \tag{10}$$

for $x\in\left(-\frac{d}{2},\frac{d}{2}\right)$ and $\xi\in(-\infty,\infty)$.

Define W_k = weight and ξ_k = nodes for $k=1,2,\cdots,N$, then the integral term on the right hand side of (9) can be approximated to obtain

$$\begin{aligned} & \xi\frac{\partial z}{\partial x}Z(x,\xi)+Z(x,\xi) \\ & =\sum_{k=1}^{N}W_k\psi(\xi_k)\left[Z(x,\xi_k)+Z(x,-\xi_k)\right] \end{aligned} \tag{11}$$

for $x\in\left(-\frac{d}{2},\frac{d}{2}\right)$ and $\xi\in(-\infty,\infty)$.

To satisfy the requirements of the right hand side of (11) the left hand side was evaluated at the points $\xi=\pm\xi_i$ to obtain a system of differential equations

$$\begin{aligned} & \xi_i\frac{\partial z}{\partial x}Z(x,\xi_i)+Z(x,\xi_i) \\ & =\sum_{k=1}^{N}W_k\psi(\xi_k)\left[Z(x,\xi_k)+Z(x,-\xi_k)\right] \end{aligned} \tag{12}$$

and

$$\begin{aligned} & -\xi_i\frac{\partial z}{\partial x}Z(x,-\xi_i)+Z(x,-\xi_i) \\ & =\sum_{k=1}^{N}W_k\psi(\xi_k)\left[Z(x,\xi_k)+Z(x,-\xi_k)\right] \end{aligned} \tag{13}$$

for $i=1,2,\cdots,N,$ where N is the quadrature points.

Seeking exponential solutions to Equations (12) and (13), set

$$Z\left(x,\pm\xi_i\right)=\Phi\left(v,\xi_i\right)\exp\left(-\frac{x}{v}\right) \quad (14)$$

Substituting Equation (14) into Equations (12) and (13), we have

$$-\frac{\xi_i}{v}\Phi\left(v,-\xi_i\right)+\Phi\left(v,-\xi_i\right) = \sum_{k=1}^{N} W_k\psi\left(\xi_k\right)\left[\Phi\left(v,\xi_i\right)+\Phi\left(v,-\xi_k\right)\right] \quad (15)$$

and

$$\frac{\xi_i}{v}\Phi\left(v,-\xi_i\right)+\Phi\left(v,-\xi_i\right) = \sum_{k=1}^{N} W_k\psi\left(\xi_k\right)\left[\Phi\left(v,\xi_i\right)+\Phi\left(v,-\xi_k\right)\right] \quad (16)$$

For convenience, let

$$\Phi_+=\Phi\left(v,\xi_i\right),\ \Phi_-=\Phi\left(v,-\xi_i\right),$$
$$W_{ij}=W_j\psi\left(\xi_i\right),\ M=\text{diagonals}\left\{\xi_1,\xi_2,\cdots,\xi_N\right\}$$

Then (15) and (16) can be written as

$$\frac{1}{v}M\Phi_+=\left[I-W\right]\Phi_+-W\Phi_- \quad (17)$$

$$-\frac{1}{v}M\Phi_-=\left[I-W\right]\Phi_--W\Phi_+ \quad (18)$$

where I is an $N\times N$ identity matrix

$$\Phi_+=\left[\Phi\left(v,\pm\xi_1\right),\Phi\left(v,\pm\xi_2\right),\cdots,\Phi\left(v,\pm\xi_N\right)\right]^{\mathrm{T}}$$

Now let

$$U=\Phi_++\Phi_- \quad (19)$$

and

$$Y=\Phi_--\Phi_+ \quad (20)$$

Adding (17) and (18) and substituting (19) gives

$$\frac{1}{v}MY=\left(1-2W\right)U \quad (21)$$

Subtracting (18) from (17) and substituting (20) gives

$$\frac{1}{v}MY=Y \quad (22)$$

Eliminating Y from (21) and (22) we have

$$\frac{1}{v^2}MU=\left(D-2M^{-1}WM^{-1}\right)MU \quad (23)$$

where $D=\text{diagonals}\left\{\xi_1^{-2},\xi_2^{-2},\cdots,\xi_N^{-2}\right\}$

Multiplying (23) by a diagonal matrix T with diagonal elements given by

$$T_i=\left[W_i\psi\left(\xi_i\right)\right]^{\frac{1}{2}} \quad (24)$$

we have

$$\left[D-2V\right]X=\frac{1}{v^2}X \quad (25)$$

where

$$V=M'TWT'M' \quad \text{and} \quad X=TMU$$

With the elements $t_1,t_2,\cdots,t_N\in T$, V is made symmetric and hence we can write the eigenvalue in the form

$$\left(D-2ZZ^{\mathrm{T}}\right)X=\lambda X \quad (26)$$

where

$$\lambda=\frac{1}{v^2}\Rightarrow V=\sqrt{\frac{1}{\lambda}}$$

and

$$Z=\left[\frac{\sqrt{W_1\psi\left(\xi_1\right)}}{\xi_1},\frac{\sqrt{W_2\psi\left(\xi_2\right)}}{\xi_2},\cdots,\frac{\sqrt{W_N\psi\left(\xi_N\right)}}{\xi_{N1}}\right]^{\mathrm{T}}$$

Considering that the required eigenvalues has been obtained in (26), a normalization condition is therefore imposed, that is,

$$\sum_{k=1}^{N}W_k\psi\left(\xi_k\right)+\left[\Phi\left(v,+\xi_k\right)+\Phi\left(v,-\xi_k\right)\right]=I \quad (27)$$

Hence the discrete ordinate solution is written as

$$Z\left(x,+\xi_i\right)=\sum_{j=1}^{N}\left[A_j\frac{v_j}{v_j-\xi_i}\mathrm{e}^{-\frac{(a+x)}{v_j}}+B_j\frac{v_j}{v_j+\xi_i}\mathrm{e}^{-\frac{(a+x)}{v_j}}\right] \quad (28)$$

$$Z\left(x,-\xi_i\right)=\sum_{j=1}^{N}\left[A_j\frac{v_j}{v_j+\xi_i}\mathrm{e}^{-\frac{(a+x)}{v_j}}+B_j\frac{v_j}{v_j-\xi_i}\mathrm{e}^{-\frac{(a+x)}{v_j}}\right] \quad (29)$$

where $\left\{A_j\right\}$ and $\left\{B_j\right\}$ are arbitrary constants to be determined from the boundary conditions. v_j is separation constants and is equal to the reciprocal of the positive square root of the eigenvalues as defined by (26), the separation constants $\left(v_j\right)$ will not be allowed to be equal to one of the quadrature points $\left(\xi_j\right)$ and a is the arbitrary scaling constant which we are taking as $2a$ for the full channel width.

The problem based on (2) is "conservative" since

$$\int_{-\infty}^{\infty}\psi\left(\xi\right)\mathrm{d}\xi=I$$

For this reason we expect that one of the eigenvalues defined by Equation (26) will tend to zero as N tends to infinity. Taking this fact into account, v_N, which is the largest of the computed separation constants $\left\{v_j\right\}$ will have to be neglected, hence (28) and (29) are written as

$$Z\left(x,+\xi_i\right)=A+B\left(x-\xi_i\right)+\sum_{j=1}^{N-1}\left[A_j\frac{v_j}{v_j-\xi_i}\mathrm{e}^{-\frac{(a+x)}{v_j}}+B_j\frac{v_j}{v_j+\xi_i}\mathrm{e}^{-\frac{(a+x)}{v_j}}\right] \quad (30)$$

$$Z(x,-\xi_i) = A + B(x-\xi_i) + \sum_{j=1}^{N-1}\left[A_j \frac{v_j}{v_j+\xi_i} e^{-\frac{(a+x)}{v_j}} + B_j \frac{v_j}{v_j-\xi_i} e^{-\frac{(a+x)}{v_j}}\right] \quad (31)$$

The constants $A, B, \{A_j\}$ and $\{B_j\}$ will be determined from the boundary conditions. Equations (30) and (31) represent the discrete ordinate solutions.

To solve the problem of Couette, Poiseuille and Thermal creep, we consider the boundary conditions as defined in (3) to (8) and write

$$Z(-a,\xi) - (1-\alpha) Z(-a,-\xi) = F_1(\xi) \quad (32)$$

and

$$Z(a,-\xi) - (1-\alpha) Z(a,\xi) = F_2(\xi) \quad (33)$$

for $\xi \in (0,\infty)$. From (32) and (33), we can express the boundary conditions as stated in (3) to (8) as

$$F_1(\xi) = \alpha \quad (34)$$

and

$$F_2(\xi) = -\alpha \quad (35)$$

for Couette flow,

$$F_1(\xi) = \alpha\xi^2 + a(2-\alpha)\xi \quad (36)$$

and

$$F_2(\xi) = -\alpha\xi^2 + a(2-\alpha)\xi \quad (37)$$

for Poiseuille flow and

$$F_1(\xi) = \frac{1}{2}\alpha\left[\xi^2 - \frac{1}{2}\right] \quad (38)$$

and

$$F_2(\xi) = -\frac{1}{2}\alpha\left[\xi^2 - \frac{1}{2}\right] \quad (39)$$

for Thermal creep.

Substituting (30) and (31) into the boundary conditions (32) and (33), and evaluate at the quadrature points gives the system of linear algebraic equations

$$\sum_{j=1}^{N-1}\left\{M_{i,j}A_j + N_{i,j}B_j e^{-\frac{2a}{v_j}}\right\} + \alpha A - B\left[\alpha a + \xi_i(2-\alpha)\right] = F_1(\xi_i) \quad (40)$$

and

$$\sum_{j=1}^{N-1}\left\{M_{i,j}A_j + N_{i,j}B_j e^{-\frac{2a}{vj}}\right\} + \alpha A - B\left[\alpha a + \xi_i(2-\alpha)\right] = F_2(\xi_i) \quad (41)$$

for $i = 1, 2, \cdots, N$, and the matrix elements

$$M_{i,j} = v_j\left[\frac{\alpha v_j + \xi_i(2-\alpha)}{v_j^2 - \xi_i^2}\right] \quad (42)$$

and

$$N_{i,j} = v_j\left[\frac{\alpha v_j + \xi_i(2-\alpha)}{v_j^2 + \xi_i^2}\right] \quad (43)$$

Adding (40) and (41) we have

$$2\alpha A + \sum_{j=1}^{N-1}\left[\left(A_j + B_j\right)\left(M_{i,j} - N_{i,j} e^{-\frac{2a}{v_j}}\right)\right] = F_1(\xi) + F_2(\xi) \quad (44)$$

Subtracting (41) from (40) we have

$$\sum_{j=1}^{N-1}\left[\left(A_j - B_j\right)\left(M_{i,j} - N_{i,j} e^{-\frac{2a}{v_j}}\right)\right] - 2B\left[\alpha a + \xi_i(2-\alpha)\right] = F_1(\xi) - F_2(\xi) \quad (45)$$

for $i = 1, 2, \cdots, N$.

Solving (44) and (45) simultaneously to find the values of the constants A, B, $\{A_j\}$ and $\{B_j\}$. Hence we can establish the solutions to the various problems as follows

For Poiseuille flow, we have

1) Velocity profile

$$q_p(\tau) = \frac{1}{2}\left(1 - a^2 + \tau^2\right) - Y_0(\tau) \quad (46)$$

where

$$Y_0(\tau) = A + B\tau + \sum_{J=1}^{N-1}\left[A_j e^{-\frac{(a+\tau)}{v_j}} + B_j e^{-\frac{(a-\tau)}{v_j}}\right] \quad (47)$$

and

2) Poiseuille Volume Flow Rate

$$Q_p = \frac{1}{2a^2}\left[2\alpha A + \sum_{j=1}^{N-1} v_j\left(A_j + B_j\right)\left(1 - e^{-\frac{2a}{v_j}}\right)\right] - \frac{1}{2a}\left(1 - \frac{2}{3}a^2\right) \quad (48)$$

For Couette flow, we compute the stress given by

$$P_{xz} = -\frac{1}{2}\pi^{-\frac{1}{2}}B \quad (49)$$

and for Thermal Creep, we compute the Velocity profile

$$q_T(\tau) = Y_0(\tau_1) \quad (50)$$

and the flow rate

$$Q_T(\tau) = -\frac{1}{2a^2}\left[2\alpha A + \sum_{j=1}^{N-1} v_j\left(A_j + B_j\right)\left(1 - \mathrm{e}^{-\frac{2a}{v_j}}\right)\right] \quad (51)$$

4. Finite Difference Method

Using the linearized two dimensional approach in [7] with the Bhatnagar-Gross-Krook Model (BGK) in [24, 25], the Boltzmann equation to be solved is reduced to

$$\left.\begin{aligned} &\xi_y \frac{\partial\phi}{\partial\gamma} + \xi_z \frac{\partial\phi}{\partial z} = \lambda\left[-\phi + \nu + 2h\xi_z q_z + \tau\left(h\xi^2 - \frac{3}{2}\right)\right] \\ &\nu = \int \phi F_0 \mathrm{d}\xi \\ &\left(\frac{3}{2}h\right)(\nu + \tau) = \int \xi^2 \phi F_0 \mathrm{d}\xi \\ &q = \int \xi\phi F_0 \mathrm{d}\xi \\ &F_0 = \left(\frac{h}{\pi}\right)^{\frac{3}{2}} \exp\left[-h\left(\xi_x^2 + \xi_y^2 + \xi_z^2\right)\right] \\ &h = \frac{m}{2kT} \end{aligned}\right\} \quad (52)$$

where

$\phi =$ relative change in velocity distribution function

$\xi\left(\xi_x, \xi_y, \xi_z\right) =$ the molecular volecity

$q\left(q_x, q_y, q_z\right) =$ the gas velocity

$\nu =$ relative change in the particle density

$\tau =$ relative change in temperature

$\lambda =$ the collision frequency

The perturbation terms ν and τ depend only on z (flow direction) and are related to the pressure and temperature gradient. They are

$$T = k_2\left(\frac{z}{d}\right), \quad \nu + \tau = k_1\left(\frac{z}{d}\right)$$

where k_1 is proportional to pressure gradient and k_2 is proportional to temperature gradient, and both are small compared to unity. The velocity of the reflecting molecules from the wall is specified by the Maxwellian distribution; then the boundary conditions are:

$$\phi^{\pm}\left(-\frac{1}{2}d\,\mathrm{Sgn}\xi_y, z, \xi\right)\left(k_1 - k_2\right)\left(\frac{z}{d}\right) + k_2\left(\frac{z}{d}\right)\left(h\xi^2 - \frac{3}{2}\right) \quad (53)$$

where

$$\mathrm{Sgn}\xi_y = \begin{cases} 1, & \text{if } \xi_y > 0 \\ -1, & \text{if } \xi_y < 0 \end{cases}$$

A solution in the form

$$\phi(\xi, y, z) = \phi_0(\xi)\left(\frac{z}{d}\right) + \phi_1(y, \xi) \quad (54)$$

was sought where

$$\phi_0(\xi) = k_1 + k_2\left(h\xi^2 - \frac{5}{2}\right) \quad (55)$$

Substituting Equation (54) into Equation (52) we have

$$\begin{aligned} &\xi_y \frac{\mathrm{d}\phi}{\mathrm{d}y} + \lambda\phi_1(y, \xi) \\ &= \xi_z\left[\frac{-k_1}{d} - \frac{k_2}{d}\left(h\xi^2 - \frac{5}{2}\right) + 2\lambda h q_z\right] \end{aligned} \quad (56)$$

Multiplying both sides of Equation (56) by

$$\xi_z\left(\frac{h}{\pi}\right)\exp\left[-h\left(\xi_z^2 + \xi_x^2\right)\right]$$

and integrating over full ranges, we have

$$\xi_y \frac{\mathrm{d}F}{\mathrm{d}y} + \lambda F = \frac{1}{2h}\left(2h\lambda q_z - \frac{k_1}{d} + \frac{k_2}{d} + \xi_y^2 \frac{k_2}{d} h\right) \quad (57)$$

where the function F is defined by

$$F(y, \xi) = \frac{h}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \xi_z \exp\left[h\left(\xi_z^2 + \xi_x^2\right)\right]\phi_1\left(y, \xi_y\right)\mathrm{d}\xi_x \mathrm{d}\xi_z \quad (58)$$

Integrating Equation (57) under the boundary conditions

$$\phi_1\left(-\frac{1}{2}d\,\mathrm{Sgn}\xi_y, z, \xi\right) = 0 \quad (59)$$

we have

$$\begin{aligned} &F\left(y, \xi_y\right) \\ &= \left(\xi_y\right)^{-1}\int_{-\frac{d}{2}\mathrm{sgn}\,\xi_y}^{y} (2h)^{-1}\left(2h\lambda q_z - \frac{k_1}{d} - \frac{k_2}{2d} - \xi_y h \frac{2k_2}{d}\right) \\ &\times \exp\left[\frac{\lambda|y-1|}{\left|\xi_y\right|}\right]\mathrm{d}t \end{aligned} \quad (60)$$

When the gas velocity q_z is expressed by

$$\mathrm{sgn}\, q_z(y) = \left(\frac{h}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty} F\exp\left(-h\xi_y^2\right)\mathrm{d}\xi_y \quad (61)$$

Equation (61) now reduces to

$$\begin{aligned} &h^{\frac{1}{2}} q_z(y) \\ &= \pi^{\frac{1}{2}}\int_{-\frac{d}{2}}^{\frac{d}{2}} J_{-1}\left(h^{\frac{1}{2}}\lambda|y-t|\right)\left(h^{\frac{1}{2}} q_z(t) - \frac{k_1}{2dh^{\frac{1}{2}}\lambda} - \frac{k_2}{2dh^{\frac{1}{2}}\lambda}\right)\mathrm{d}t \\ &\quad - \pi^{-\frac{1}{2}}\int_{-\frac{d}{2}}^{\frac{d}{2}} \frac{k_1}{2dh^{\frac{1}{2}}\lambda} J_1\left(h^{\frac{1}{2}}\lambda|y-t|\right)\mathrm{d}t \end{aligned} \quad (62)$$

where J_n is defined by

$$J_n = \int_0^\infty y^n \exp\left(-y^2 - \frac{x}{y}\right) \mathrm{d}y$$

Let

$$\Delta = dh^{\frac{1}{2}}\lambda = \left(\frac{2\delta}{\pi^{\frac{1}{2}}}\right),$$

$$y = h^{\frac{1}{2}}\lambda y, \quad T = dh^{\frac{1}{2}}\lambda t,$$

$$h^{\frac{1}{2}} q_z = \left(2dh^{\frac{1}{2}}\lambda\right)^{-1} \left\{\left[1-\Psi_p(\eta)\right]k_1 - \left[\frac{1}{2} - \Psi_T(\eta)\right]k_2\right\} \quad (63)$$

then Equation (62) will be written as two integral e quations, *i.e.*,

$$\psi_p(\eta) - \pi^{-\frac{1}{2}} \int_{-\frac{d}{2}}^{\frac{d}{2}} \Psi_p(\eta) J_{-1}\left(h^{\frac{1}{2}}\lambda|y-t|\right)\mathrm{d}t = 1 \quad (64)$$

and

$$\begin{aligned}
&\psi_T(\eta) - \pi^{-\frac{1}{2}} \int_{-\frac{d}{2}}^{\frac{d}{2}} \Psi_T(\eta) J_{-1}\left(h^{\frac{1}{2}}\lambda|y-t|\right)\mathrm{d}t \\
&= \frac{1}{2} - \pi^{-\frac{1}{2}} \int_{-\frac{d}{2}}^{\frac{d}{2}} J_{-1}\left(h^{\frac{1}{2}}\lambda|y-t|\right)\mathrm{d}t
\end{aligned} \quad (65)$$

where

δ = the inverse Knudsen number

From Equation (63), we have the velocity of the gas induced by the pressure gradient as,

$$h^{\frac{1}{2}} q_{zP} = \frac{-\pi^{\frac{1}{2}}}{2\delta}\left[1-\Psi_P\right] \quad (66)$$

and that induced by temperature gradient as

$$h^{\frac{1}{2}} q_{zT} = \frac{-\pi^{\frac{1}{2}}}{2\delta}\left[\frac{1}{2} - \Psi_T\right] \quad (67)$$

The volume flow rate is then given by

$$\begin{aligned}
G_P &= \rho \int_{-\frac{d}{2}}^{\frac{d}{2}} q_{zP}(y)\mathrm{d}y \\
&= \left(\frac{\pi^{\frac{1}{2}}}{2\delta} - \frac{\pi}{4\delta^2}\int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \Psi_P(\eta)\mathrm{d}\eta\right) h^{\frac{1}{2}} d^2 \frac{\mathrm{d}p}{\mathrm{d}z}
\end{aligned} \quad (68)$$

$$\begin{aligned}
G_T &= p \int_{-\frac{d}{2}}^{\frac{d}{2}} q_{zT}(y)\mathrm{d}y \\
&= \left(\frac{\pi^{\frac{1}{2}}}{4\delta} - \frac{\pi}{4\delta^2}\int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \Psi_T(\eta)\mathrm{d}\eta\right) h^{\frac{1}{2}} n_0 k d^2 \frac{\mathrm{d}T}{\mathrm{d}z}
\end{aligned} \quad (69)$$

Expressing Equations (68) and (69) in non-dimensional form gives;

$$Q_P = \frac{\pi^{\frac{1}{2}}}{2\delta} - \frac{\pi}{4\delta^2} \int_{-\frac{\delta}{x^{\frac{1}{2}}}}^{\frac{\delta}{x^{\frac{1}{2}}}} \Psi_P(\eta)\mathrm{d}\eta \quad (70)$$

and

$$Q_T = \frac{\pi^{\frac{1}{2}}}{4\delta} - \frac{\pi}{4\delta^2} \int_{-\frac{\delta}{x^{\frac{1}{2}}}}^{\frac{\delta}{x^{\frac{1}{2}}}} \Psi_T(\eta)\mathrm{d}\eta \quad (71)$$

The subscripts P and T imply Poiseuille flow and Thermal creep respectively.

Next, is to solve numerically the unknown functions Ψ_P and Ψ_T in Equations (64) and (65) respectively.

In order to solve Equations (64) and (65), a finite difference method was utilized after discretization as

$$\psi_{Ph} - \pi^{-\frac{1}{2}} \sum_{k=0}^{n-1} \psi_{Pk} \int_{\tau_k}^{\tau_{k+1}} J_{-1}\left[\left|\frac{1}{2}\left(\tau_n + \tau_{n+1}\right) - \tau\right|\right]\mathrm{d}\tau = 1 \quad (72)$$

$$\begin{aligned}
&\psi_{Th} - \pi^{-\frac{1}{2}} \sum_{k=0}^{n-1} \psi_{Tk} \int_{\tau_k}^{\tau_{k+1}} J_{-1}\left[\left|\frac{1}{2}\left(\tau_n + \tau_{n+1}\right) - \tau\right|\right]\mathrm{d}\tau \\
&= \frac{1}{2} - \pi^{-\frac{1}{2}} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} J_1\left[\left|\frac{1}{2}\left(\tau_n + \tau_{n+1}\right) - \tau\right|\right]\mathrm{d}\tau
\end{aligned} \quad (73)$$

where ψ_{Pk} is the stepwise function of ψ_P and ψ_{Tk} the stepwise function of ψ_T

The constant value of the functions ψ_{Pk} and ψ_{Tk} on each interval is interpreted as the value at the midpoint. The transcendental function $T_{-(x)}$ has a singularity when $x \to 0$.

According to the obvious way of differences, Equations (72) and (73) reduce to the matrix

$$\sum_{k=0}^{n-1} A_{hk}\psi_{pk} = 1 \quad \text{for } n = 0,1,2,\cdots,n-1 \quad (74)$$

$$\sum_{k=0}^{n-1} B_{hk}\psi_{Tk} = g_h \quad \text{for } n = 0,1,2,\cdots,n-1 \quad (75)$$

where

$$A_{hk} = \delta_{hk} - \pi^{-\frac{1}{2}} \int_{-\frac{(2k-n-2)\Delta}{2n}}^{\frac{(2k-n-2)\Delta}{2n}} J_{-1}\left(\left|\frac{2h+1-n}{2n}\Delta - \tau\right|\right)\mathrm{d}\tau \quad (76)$$

$$A_{hk} = B_{hk} \quad (77)$$

$$g_h = \frac{1}{2} - \pi^{-\frac{1}{2}} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} J_1\left(\left|\frac{2h+1-n}{2n}\Delta - \tau\right|\right)\mathrm{d}\tau \quad (78)$$

Integrating Equations (76) and (78) using the properties of J_n we have;

If $h \neq k$

$$A_{hk} = \pi^{-\frac{1}{2}} \left\{ J_0 \left[\frac{2\delta}{\pi^{\frac{1}{2}}} \left(\frac{|k-h|}{n} - (2n)^{-1} \right) \right] - J_0 \left[\left[\frac{2\delta}{\pi^{\frac{1}{2}}} \left(\frac{|k-h|}{n} + (2n)^{-1} \right) \right] \right] \right\} \quad (79)$$

If $h = k$

$$Z_{hk} = \left(\frac{2}{\pi^{\frac{1}{2}}} \right) J_0 \left(\frac{\delta}{\pi^{\frac{1}{2}}} n \right) \quad (80)$$

$$g_h = \pi^{-\frac{1}{2}} \left\{ J_2 \left[\frac{\delta}{\pi^{\frac{1}{2}}} \left(1 - \frac{2h+1-n}{n} \right) \right] + J_2 \left[\frac{\delta}{\pi^{\frac{1}{2}}} \left(1 + \frac{2h+1-n}{n} \right) \right] \right\} \quad (81)$$

5. Numerical Results

Using LAPAK and LINPAC solvers, we obtained the following numerical results:

In **Table 1**, we compared the results of Poiseuille flow rate between discrete ordinate and finite difference methods. In the table, the result with accommodation coefficient $\alpha = 1$ was the only one presented. While discrete ordinate method could consider more values of the accommodation coefficients, the finite difference method can only take accommodation coefficient of one. This is due to the fact that the discrete ordinate solution adopted the boundary conditions of diffuse and specular reflections while the finite difference solution adopted the diffuse reflection boundary condition only. A range of inverse Knudsen number from 0.001 to 100 was considered for both solutions, these values accommodated the slip flow, transition flow and the collisionless flow regime.

The results show an agreement of 96.6% within the slip and collisionless regime and 99.9% in the transition regime. The flow rate shows its minimum in both solution methods at $K_n = 1.0$ in the transition regime. This result also agreed with that of Cercignani and Daneri in [9] where it was pointed out that the minimum occurs between 1.0 and 1.2 and the analytical solution as presented in [26] and [27]. It was also observed that as the inverse Knudsen number gets very large, the volume flow rate shoots up drastically; reason was that the mean-free-path becomes larger.

Table 2 was used to compare the Thermal Creep Volume Flow Rate between the Discrete Ordinate and the Finite Difference methods. The same parameters in **Table 1** were used as a basis for this comparison. That is, accommodation coefficient $\alpha = 1$ and an inverse Knudsen number in the range of 0.001 to 100. The result also shows an agreement of 96.6% within the slip and collisionless regime and 99.9% in the transition regime. It was noticed that as the channel gets wider the thermal creep volume flow rates gets smaller.

6. Conclusion

Based on the discussions above, we therefore concluded that: the comparison shows that both schemes give simi

Table 1. Comparison of Poiseuille Flow Rates between Discrete Ordinate Method and Finite Difference Method. Parameter used: Accommodation coefficient α = 1.0000.

Channel width (d_0) or inverse Knudsen number (k_n)	Analytical Solution as in [26] and [27]	Discrete Ordinate Method (DOM) No of Gaussian Points = 60	Finite Difference Method (FDM) No of Elements = 100 No of Gaussian Points = 50
0.0010	-	4.274560	4.194779
0.0100	-	3.049685	3.049363
0.1000	1.9318	2.032716	2.032757
0.5000	1.5607	1.601874	1.601950
1.0000	1.5086	1.538678	1.538786
5.0000	1.9639	1.981093	1.981283
10.000	2.7350	2.768645	2.768504
50.000	-	9.369976	9.263045
100.00	-	17.69330	17.06334

Table 2. Comparison of Thermal Creep Volume Flow Rates between Discrete Ordinate Method and Finite Difference Method. Parameter used: Accommodation coefficient α = 1.0000.

Channel width (d_0) or inverse Knudsen number (k_n)	Analytical Solution as in [26] and [27]	Discrete Ordinate Method (DOM) No of Gaussian Points = 60	Finite Difference Method (FDM) No of Elements = 100 No of Gaussian Points = 50
0.0010	-	1.8541470	1.814151
0.0100	-	1.2358340	1.235673
0.1000	0.7966	0.6949272	0.694946
0.5000	0.5036	0.3984993	0.398527
1.0000	0.3890	0.2949000	0.294933
5.0000	0.1574	0.1107882	0.119890
10.000	0.0898	0.0660763	0.066139
50.000	-	0.0148994	0.015036
100.00	-	0.0075565	0.007810

lar results when computing with the ranges of the inverse Knudsen number; the finite difference method was able to give excellent results on Poiseuille and Thermal creep flows at a relatively much shorter computation and was comparable to the discrete ordinate solutions even up to 99% accuracy. However, the finite difference method could not take accommodation coefficient of order greater than one because of the consideration of only the diffuse boundary condition.

REFERENCES

[1] E. B. Hauser, "Analytical Solution for Three-Dimensional Discrete Ordinates Problems," *Proceedings of* 12*th Pan-American Congress of Applied Mechanic*, Trinidad, 2-6 January 2012, pp. 1-5. http://www.aamech.org/PACAM12/Documents/73.pdf

[2] C. S. Scherer, J. F. Prolo Fihlo and L. B. Barichello, "An Analytical Approach to the Unified Solution of Kinetic Equations in Rarefied Gas Dynamics. I. Flow Problems". *Zeitschrift für Angewandte Mathematik und Physik ZAMP*, Vol. 60, No. 1, 2009, pp. 70-115.

[3] C. S. Scherer, J. F. Prolo Filho and L. B. Barichello, "An Analytical Approach to the Unified Solution of Kinetic Equations in the Rarefied Gas Dynamics. II. Heat Transfer Problems" *Zeitschrift für Angewandte Mathematik und Physik ZAMP*, Vol. 60, No. 4, 2009, pp. 651-687.

[4] S. Taguchi and K. Aoki, "Numerical Analysis of Rarefied Gas Flow induced around a Flat Plate with a Single Heated Side in Rarefied Gas Dynamics," *Proceedings of 27th International Symposium on Rarefied Gas Dynamics*, Pacific Grove, 10-15 July 2010, pp. 790-795.

[5] V. A. Titarev, "Efficient Deterministic Modeling of Three-Dimensional Rarefied Gas Flow," *Communications in Computational Physics*, Vol. 12, No. 1, 2012, pp. 162-192.

[6] J. Y. Yang, B. P. Muljadi, Z. H. Li and H. X. Zhang, "A Direct Solver for Initial Value Problems of Rarefied Gas Flows of Arbitrary Statistics," *Communications in Computational Physics*, Vol. 14, No. 1, 2013, pp. 242-264.

[7] C. Cercignani, "Mathematical Methods in Kinetic Theory," Plenum Press, New York, 1969.

[8] C. Cercignani "Rarefied Gas Dynamics: From Basic Concepts to Actual Calculations," Cambridge University Press, Cambridge, 2000.

[9] C. Cercignani and A. Daneri, "Flow of Rarefied Gas between two Parallel Plates," *Journal of Applied Physics*, Vol. 34 No. 12, 1962. pp. 3509-3513 doi:10.1063/1.1729249

[10] C. E. Siewert and L. B. Barichello, "A Discrete-Ordinate Solution for Poiseuille Flow in Plan Channel," *Zeitschrift für Angewandte Mathematik und Physik ZAMP*, Vol. 50, No. 6, 1999, pp. 972-981. doi:10.1007/s000000050189

[11] S. K. Stefanov, V. M. Roussinov and C. Cerciginani, "Three-Dimensional Rayleigh-Benard Convection of a Rarefied Gas: DSMC and Navier-Stokes Calculations," *AIP Conference Proceedings of* 24*th International Symposium on Rarefied Gas Dynamics*, Vol. 762, 2004, pp. 529-534.

[12] Y. Sone, "Rarefied Gas Flow Induced between Non-Parallel Plane Walls with Different Temperature," In: M. Becker and M. Fiebig, Eds., *Rarefied Gas Dynamics*, DFVLR Press, Porz-Wahn, 1974.

[13] L. B. Barichello, M. Camargo, P. Rodrigues and C. E. Siewert, "Unified Solutions to Classical Flow Problems Based on the BGK Model," *Zeitschrift für Angewandte Mathematik und Physik ZAMP*, Vol. 52, No. 3, 2001, pp. 517-534.

[14] C. S. Scherer and L. B. Barichello, "An Analytical Approach to the Unified Solution of Kinetic Equations in Rarefied Gas Dynamics. III. Evaporation and Condensation Problems," *Zeitschrift für Angewandte Mathematik und Physik ZAMP*, Vol. 61, No. 1, 2012, pp. 95-117.

[15] S. Kosuge, K. Aoki and S. Takata, "Heat Transfer in a Gas Mixture between Two Parallel Plates: Finite-Difference Analysis of the Boltzmann Equation," 22*nd International Symposium on Rarefied Gas Dynamics*, Sydney, 9-14 July 2000, pp. 289-296.

[16] B. P. Muljadi and J. Y. Yang, "Deterministic Solver for Rarefied Flow Problems of Gases of Arbitrary Statistics Based on the Semiclassical Boltzmann-BGK Equation," 7*th International Conference on Computational Fluid Dynamics*, Big Island, 9-13 July 2012, p. 1404.

[17] R. D. M. Garcia and C. E. Siewert, "Particular Solutions of the Linearized Boltzmann Equation for A Binary Mixture of Rigid Spheres," *Zeitschrift für Angewandte Mathematik und Physik ZAMP*, Vol. 59, No. 2, 2008, pp. 281-292.

[18] C. Henri, "Couette Flow for a Gas with a Discrete Velocity Distribution" *Journal of Fluid Mechanics*, Vol. 76, No. 2, 1976, pp. 273-287. doi:10.1017/S0022112076000633

[19] S. L. Lorenzani, A. Gibelli, A. Frezzotti, A. Frangi and C. Cercignani, "Kinetic Approach to Gas Flows in Microchannels," *Nanoscale and Microscale Thermophysical Engineering*, Vol. 11, No. 1-2, 2007, pp. 211-226. doi:10.1080/15567260701333489

[20] S. K. Loyalka, "Motion of a Sphere in a Gas: Numerical Solution of the Linearized Boltzmann Equation," *Physics of Fluids A*: *Fluid Dynamics*, Vol. 4, No. 5, 1992, p. 1049. doi:10.1063/1.858256

[21] S. Kubota and Y. Yoshizawa, "Couette Flow Problems by Thirteen Moment Equations," *Proceedings of Rarefied Gas Dynamics*, DFVLR Press, Porz-Wahn, 1974.

[22] C. E. Siewert, "A Discrete-Ordinate Solution for Heat Transfer in a Plan Channel," *Journal of Computational Physics*, Vol. 152, No. 1, 1999, pp. 251-263. doi:10.1006/jcph.1999.6244

[23] M. M. R. Williams, "A Review of the Rarefied Gas Dynamics Theory Associated with Some Classical Problems in Flow and Heat Transfer," *Zeitschrift für Angewandte Mathematik und Physik ZAMP*, Vol. 52, No. 3, 2001, pp. 500-516.

[24] P. L. Bhatnagar, E. P. Gross and M. Krook, "A Modelfor Collision Processes in Gases. 1. Small Amplitude Processes in Charge and Neutral One-Component Systems,"

Physical Review, Vol. 94, No. 3, 1954, pp. 511-525.

[25] S. Takata, Y. Sone and K. Aoki, " Numerical Analysis of a Uniform Flow of a Rarefied Gas Past a Sphere on the Basis of the Boltzmann Equation for Hard-Sphere Molecules," *Physics of Fluids A*: *Fluid Dynamics*, Vol. 5, No. 3, 1993, p. 716. doi:10.1063/1.858655

[26] S. K. Lokalka and K. A. Hickey, "Kinetic Theory of Thermal Transpiration and Mechanocaloric Effect: Planar Flow of a Rigid Sphere Gas with Arbitrary Accommodation at the Surface," *Journal of Vacuum Science & Technology A*, Vol. 9, No. 1, 1991, p. 158. doi:10.1116/1.577119

[27] D. Valougeorgis, "An Analytic Solution of S-Model Kinetic Equations," *Zeitschrift für Angewandte Mathematik und Physik ZAMP*, Vol. 54, No. 1, 2003, pp. 112-124. doi:10.1007/PL00012627