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The *H*-Decomposition Problem for Graphs

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ABSTRACT

The concept of *H*-decompositions of graphs was first introduced by Erdös, Goodman and Pósa in 1966, who were motivated by the problem of representing graphs by set intersections. Given graphs *G* and *H*, an *H*-decomposition of *G* is a partition of the edge set of *G* such that each part is either a single edge or forms a graph isomorphic to *H*. Let $\phi(n,H)$ be the smallest number ϕ, such that, any graph of order *n* admits an *H*-decomposition with at most ϕ parts. The exact computation of $\phi(n,H)$ for an arbitrary *H* is still an open problem. Recently, a few papers have been published about this problem. In this survey we will bring together all the results about *H*-decompositions. We will also introduce two new related problems, namely Weighted *H*-Decompositions of graphs and Monochromatic *H*-Decompositions of graphs.

Keywords: Graph Decompositions; Weighted Graph Decompositions; Monochromatic Graph Decompositions; Turán Graph; Ramsey Numbers

1. Introduction

1.1. Terminology and Notations

For notation and terminology not discussed here the reader is referred to [1]. A *graph* is a (finite) set $V = V(G)$, called the *vertices* of *G* together with a set $E = E(G)$ of (unordered) pairs of vertices of *G*, called the *edges*. We do not allow loops and multiple edges. The number of vertices of a graph is its *order* and is denoted by $v(G)$. The number of edges in a graph is its *size* and is denoted by $e(G)$. A vertex v is *incident* with an edge *e* if $v \in e$ and the two vertices incident with an edge are called its *endpoints*. Two vertices x, y of *G* are said to be *adjacent* or *neighbors* if $\{x, y\}$ is an edge of *G*. The *degree of a vertex v* is the number of edges incident with *v* and will be denoted by $\deg_G v$ or simply by $\deg v$ if it is clear which graph is being considered. The complete graph (clique) of order *n* will be denoted by K_n, the complete bipartite graph with parts of size *m* and *n* will be denoted by $K_{m,n}$ and the cycle of length n will be denoted by C_n.

The Turán graph of order *n*, denoted by $T_{r-1}(n)$, is the unique complete $(r-1)$-partite graph on *n* vertices where every partite class has either $\left\lfloor \frac{n}{r-1} \right\rfloor$ or $\left\lceil \frac{n}{r-1} \right\rceil$ vertices. The well-known *Turán*'*s Theorem* [2] states that $T_{r-1}(n)$ is the unique graph on *n* vertices that has the maximum number of edges and contains no complete subgraph of order *r*. We let $t_{r-1}(n)$ denote the number of edges in $T_{r-1}(n)$.

Finally, a *proper colouring* or simply a *colouring* of the vertices of *G* is an assignment of colours to the vertices in such a way that adjacent vertices have distinct colours; $\chi(G)$ is then the minimum number of colours in a (vertex) colouring of *G*. For example, $\chi(K_r) = r$, $\chi(C_{2r}) = 2$ and $\chi(C_{2r+1}) = 3$.

1.2. Motivation and Definitions

Given two graphs *G* and *H*, an *H*-decomposition of *G* is a partition of the edge set of *G* such that each part is either a single edge or forms an *H*-subgraph, *i.e.*, a graph isomorphic to *H*. We allow partitions only, that is, every edge of *G* appears in precisely one part. Let $\phi(G,H)$ be the smallest possible number of parts in an *H*-decomposition of *G*. It is easy to see that $\phi(G,H) = e(G) - p_H(G)(e(H)-1)$, where $p_H(G)$ is the maximum number of pairwise edge-disjoint *H*-subgraphs that can be packed into *G*. Building upon a body of previous research, Dor and Tarsi [3] showed that if *H* has a component with at least 3 edges, then the problem of checking whether an input graph *G* is perfectly decomposable into *H*-subgraphs is NP-complete. Hence, it is NP-hard to compute the function $\phi(G,H)$ for such *H*. Therefore, the aim is to study the function

$$\phi(n,H)=\max\{\phi(G,H)|v(G)=n\},$$

which is the smallest number such that any graph G of order n admits an H-decomposition with at most $\phi(n,H)$ parts.

This function was first studied, in 1966, by Erdös, Goodman and Pósa [4], who were motivated by the problem of representing graphs by set intersections. They proved that $\phi(n,K_3)=t_2(n)$. A decade later, this result was extended by Bollobás [5], who proved that $\phi(n,K_r)=t_{r-1}(n)$, for all $n\ge r\ge 3$.

General graphs H were only considered recently by Pikhurko and Sousa [6]. In Section 2 we will present known results about the exact value of the function $\phi(n,H)$ for some special graphs H and its asymptotic value for arbitrary H. In Sections 3 and 4 two new H-decomposition problems will be introduced, namely the weighted version and the monochromatic version respectively.

2. *H*-Decompositions of Graphs

In 1966, Erdös, Goodman and Pósa [4], who were motivated by the problem of representing graphs by set intersections, proved that $\phi(n,K_3)=t_2(n)$ and a decade later Bollobás [5] proved that $\phi(n,K_r)=t_{r-1}(n)$, for all $n\ge r\ge 3$. Recently, Pikhurko and Sousa [6] studied the function $\phi(n,H)$ for arbitrary graphs H. They proved the following result.

Theorem 2.1. [6] *Let H be any fixed graph with chromatic number* $r\ge 3$. *Then*,

$$\phi(n,H)=t_{r-1}(n)+o(n^2).$$

Let $\mathrm{ex}(n,H)$ denote the maximum number of edges in a graph of order n, that does not contain H as a subgraph. Recall that $\mathrm{ex}(n,K_r)=t_{r-1}(n)$. The same authors also made the following conjecture.

Conjecture 2.2. *For any graph H with chromatic number at least* 3, *there is* $n_0=n_0(H)$ *such that* $\phi(n,H)=\mathrm{ex}(n,H)$ *for all* $n\ge n_0$.

The exact value of the function $\phi(n,H)$ is far from being known, however, this conjecture has been verified for some special graphs. The following results have been proved by Sousa.

Theorem 2.3. [7] *For all* $n\ge 6$ *we have*

$$\phi(n,C_5)=t_2(n)=\lfloor n^2/4\rfloor.$$

Theorem 2.4. [8] *For all* $n\ge 10$ *we have*

$$\phi(n,C_7)=t_2(n)=\lfloor n^2/4\rfloor.$$

For $r\ge 3$, *a clique-extension of order* $r+1$ is a connected graph that consists of a K_r plus another vertex, say x, adjacent to at most $r-1$ vertices of K_r. For $i=1,\cdots,r-1$ the $H_{r,i}$ be the clique-extension of order $r+1$ that has $\deg x=i$.

Theorem 2.5. [9] *For all* $n\ge 4$ *and* $i=1,2$ *we have*

$$\phi(n,H_{3,i})=t_2(n)=\lfloor n^2/4\rfloor.$$

Theorem 2.6. [9] *Let* $r\ge 4$ *and let* H *be any cliqueextension of order* $r+1$. *For all* $n\ge r+1$ *we have*

$$\phi(n,H)=t_{r-1}(n).$$

A graph H is said to be *edge-critical* if there exists an edge $e\in E(H)$ whose deletion decreases the chromatic number, that is, $\chi(H)>\chi(H-e)$. Cliques and odd-cycles are examples of edge-critical graphs. Özkahya and Person [10] were able to prove that Pikhurko and Sousa's conjecture is true for all edge-critical graphs. Their result is the following.

Theorem 2.7. [10] *Let H be any edge-critical graph with chromatic number* $r\ge 3$. *Then, there exists* n_0 *such that* $\phi(n,H)=ex(n,H)$, *for all* $n\ge n_0$. *Moreover, the only graph attaining* $\phi(n,H)$ *is the Turán graph* $T_{r-1}(n)$.

The case when H is a bipartite graph has been less studied. Pikhurko and Sousa [6] determined $\phi(n,H)$ for any fixed bipartite graph with an $O(1)$ additive error. For a non-empty graph H, let $\gcd(H)$ denote the greatest common divisor of the degrees of H. For example, $\gcd(K_{6,4})=2$, while for any tree T with at least 2 vertices we have $\gcd(T)=1$. They proved the following result.

Theorem 2.8. [6] *Let H be a bipartite graph with* m *edges and let* $d=\gcd(H)$. *Then there is* $n_0=n_0H$ *such that for all* $n\ge n_0$ *the following statements hold*.

If $d=1$, then if $\binom{n}{2}\equiv m-1(\mathrm{mod}\, m)$,

$$\phi(n,H)=\phi(n,K_n)=\left\lfloor\frac{n(n-1)}{2m}\right\rfloor+m-1,$$

otherwise,

$$\phi(n,H)=\phi(n,K_n^*)=\left\lfloor\frac{n(n-1)}{2m}\right\rfloor+m-2$$

where K_n^* denotes any graph obtained from K_n after deleting at most $m-1$ edges in order to have $e(K_n^*)\equiv m-1(\mathrm{mod}\, m)$. Furthermore, if G is extremal then G is either K_n or K_n^*.

If $d\ge 2$, then

$$\phi(n,H)=\frac{nd}{2m}\left(\left\lfloor\frac{n}{d}\right\rfloor-1\right)+\frac{1}{2}n(d-1)+O(1).$$

Moreover, there is a procedure with running time polynomial in $\log n$ which determines $\phi(n,H)$ and describes a family $\mathcal{D}$ of n-sequences such that a graph G of order n satisfies $\phi(G,H)=\phi(n,H)$ if and only if the degree sequence of G belongs to $\mathcal{D}$. (It will be the case that $|\mathcal{D}|=O(1)$ and each sequence in $\mathcal{D}$ has $n-O(1)$ equal entries, so $\mathcal{D}$ can be described using $O(\log n)$ bits.)

3. Weighted *H*-Decompositions of Graphs

In 2011, Sousa [11] introduced a weighted version of the *H*-decomposition problem for graphs. More precisely, let *G* and *H* be two graphs and *b* a positive number. A *weighted* (H,b)*-decomposition* of *G* is a partition of the edge set of *G* such that each part is either a single edge or forms an *H*-subgraph, *i.e.*, a graph isomorphic to *H*. We assign a weight of *b* to each *H*-subgraph in the decomposition and a weight of 1 to single edges. The total weight of the decomposition is the sum of the weights of all elements in the decomposition. Let $\phi(G,H,b)$ be the smallest possible weight in an (H,b)-decomposition of *G*.

As before, the goal is to study the function

$$\phi(n,H,b)=\max\{\phi(G,H,b)\mid v(G)=n\},$$

which is the smallest number such that any graph *G* with *n* vertices admits an (H,b)-decomposition with weight at most $\phi(G,H,b)$.

Note that when we take $b=1$ the original *H*-decomposition problem is recovered, hence, it suffices to consider the case when $b\neq 1$. Furthermore, when $b\geq e(H)$ we easily have $\phi(n,H,b)=\binom{n}{2}$. Therefore, one only has to consider the case when $0\leq b\leq e(H)$ and $b\neq 1$. Sousa [11] obtained the asymptotic value of the function $\phi(n,H,b)$ for any fixed bipartite graph *H* when $0\leq b\leq e(H)$ and $b\neq 1$.

Recall that for a non-empty graph *H*, $\gcd(H)$ denotes the greatest common divisor of the degrees of *H*. Sousa proved the following result.

Theorem 3.1. [11] *Let H be a bipartite graph with* m *edges, let* $d=\gcd(H)$ *and* $0<b<m$ *with* $b\neq 1$ *a constant. Then there is* $n_0=n_0(H)$ *such that for all* $n\geq n_0$ *the following statements hold.*

If $d=1$, then

$$\phi(n,H,b)=b\frac{n(n-1)}{2m}+O(1).$$

If $d\geq 2$, let $n-1=qd+r$ where $0\leq r\leq d-1$ is an integer.

If $r\neq 0$ and $d-1\leq\frac{bd}{m}+r$, then

$$\phi(n,H,b)=\frac{b}{m}\binom{n}{2}+\frac{1}{2}n\left(r-\frac{br}{m}\right)+O(1).$$

If $r\neq 0$ and $d-1\geq\frac{bd}{m}+r$, then

$$\phi(n,H,b)=\frac{b}{m}\binom{n}{2}+\frac{1}{2}n\left(d-1-\frac{br-bd}{m}\right)+O(1).$$

If $r=0$ and $\frac{b}{m}<1-\frac{5d^2}{5d^3-2}$, then

$$\phi(n,H,b)=\frac{b}{m}\binom{n}{2}+\frac{1}{2}n\left(d-1-\frac{bd}{m}\right)+O(1).$$

If $r=0$ and $1-\frac{5d^2}{5d^3-2}\leq\frac{b}{m}\leq 1-\frac{1}{d}$, then

$$\frac{b}{m}\binom{n}{2}+\frac{1}{2}n\left(d-1-\frac{bd}{m}\right)-\frac{1}{2}\leq\phi(n,H,b)$$

and

$$\phi(n,H,b)\leq\frac{b}{m}\binom{n}{2}+\frac{m-b}{5md^2}n.$$

If $r=0$ and $\frac{b}{m}\geq 1-\frac{1}{d}$, then

$$\frac{b}{m}\binom{n}{2}\leq\phi(n,H,b)\leq\frac{b}{m}\binom{n}{2}+\frac{m-b}{5md^2}n.$$

The case when *H* is not a bipartite graph is still an open problem.

4. Monochromatic *H*-Decompositions of Graphs

In this section the *H*-decomposition problem is extended to coloured versions of the graph *G* and monochromatic copies of *H*. We define the problem more precisely.

A *k-edge-colouring* of a graph *G* is a function $c:E(G)\to\{1,\cdots,k\}$. We think of *c* as a colouring of the edges of *G*, where each edge is given one of *k* possible colours. Given a fixed graph *H*, a graph *G* of order *n* and a *k*-edge-colouring of the edges of *G*, a *monochromatic H-decomposition* of *G* is a partition of the edge set of *G* such that each part is either a single edge or a monochromatic copy of *H*. Let $\phi_k(G,H)$ be the smallest number such that, for any k-edge-colouring of *G*, there exists a monochromatic *H*-decomposition of *G* with at most $\phi_k(G,H)$ elements. The objective is to study the function

$$\phi_k(n,H)=\max\{\phi_k(G,H)\mid v(G)=n\},$$

which is the smallest number such that, any *k*-edge-coloured graph of order *n* admits a monochromatic *H*-decomposition with at most $\phi_k(G,H)$ elements.

This function was introduced recently by Liu and Sousa [12] and they studied the function $\phi_k(n,K_r)$ for all $k \geq 2$ and $r \geq 3$. Their results involve the Ramsey numbers and the Turán numbers. Recall that for $r \geq 3$ and $k \geq 2$, the *Ramsey number for* K_r, denoted by $R_k(r)$, is the smallest value of *s*, for which every *k*-edge-colouring of K_s contains a monochromatic K_r. The Ramsey numbers are notoriously difficult to calculate, even though, it is known that their values are finite for all $r \geq 3$ and $k \geq 2$. In fact, for the Ramsey numbers $R_k(r)$, only three of them are currently known. In 1955, Greenwood and Gleason [13] were the first to determine $R_2(3) = 6$, $R_2(3) = 17$ and $R_2(4) = 18$. Liu and Sousa [12] proved the following results about monochromatic K_r-decompositions.

Theorem 4.1. [12] *Let* $k = 2,3$. *There is an* n_0 *such that, for all* $n \geq n_0$, *we have*

$$\phi_k(n, K_3) = t_{R_k(3)-1}(n).$$

That is, $\phi_2(n, K_3) = t_5(n)$ and $\phi_3(n, K_3) = t_{16}(n)$. Moreover, the only *k*-edge-coloured graph *G* attaining $\phi_k(n, K_3)$ is the Turán graph $t_{R_k(3)-1}(n)$.

Theorem 4.2. [12] *For all* $k \geq 4$ *we have*

$$\phi_k(n, K_3) = t_{R_k(3)-1}(n) + o(n^2).$$

The same authors also made the following conjecture.

Conjecture 4.3. *Let* $k \geq 4$. *Then* $\phi_k(n, K_3) = t_{R_k(3)-1}(n)$ *for* $n \geq R_k(3)$.

Larger cliques were also studied by Liu and Sousa and they obtained the exact value of the function $\phi_k(n, K_r)$ for all $k \geq 2$ and $r \geq 4$. Recall that the Ramsey number $R_2(4) = 18$ is also well-known.

Theorem 4.4. [12] *Let* $r \geq 4$, $k \geq 2$. *There is an* $n_0 = n_0(r, k)$ *such that, for all* $n \geq n_0$, *we have*

$$\phi_k(n, K_r) = t_{R_k(r)-1}(n).$$

In particular, $\phi_2(n, K_4) = t_{17}(n)$. Moreover, the only graph attaining $\phi_k(n, K_r)$ is the Turán graph $T_{R_k(r)-1}(n)$.

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