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Finite Elements Approaches in the Solution of Field Functions in Multidimensional Space: A Case of Boundary Value Problems

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ABSTRACT

An idealized two dimensional continuum region of GRP composite was used to develop an efficient method for solving continuum problems formulated for space domains. The continuum problem is solved by minimization of a functional formulated through a finite element procedure employing triangular elements and assumption of linear approximation polynomial. The assemblage of elements functional derivatives system of equations through FEM assembly procedure made possible the definition of a unique and parametrically defined model from which the solution of continuum configuration with an arbitrary number of scales is solved. The finite element method(FEM)developed is recommended to be applied in the evaluation of the function of functions in irregular shaped continuum whose boundary conditions are specified such as in the evaluation of displacement in structures and solid mechanics problems, evaluation of temperature distribution in heat conduction problems, evaluation of displacement potential in acoustic fluids evaluation of pressure in potential flows, evaluation of velocity in general flows, evaluation of electric potential in electrostatics, evaluation of magnetic potential in magnetostatics and in the solution of time dependent field problems. A unified computational model with standard error of 0.15 and correlation coefficient of 0.72 was developed to aid analysis and easy prediction of regional function with which the continuum function was successfully modeled and optimized through gradient search and Lagrange multipliers approach. Above all the optimization schemes of gradient search and Lagrangian multiplier confirmed local minimum of function as 0.006-0.00847 to confirm the predictions of FEM and constraint conditions.

Keywords: *finite element, continuum, functional of function, extremum, boundary value*

1. INTRODUCTION

In calculus of variations, instead of attempting to locate points that extremize function of one or more variables that extremize quantities called functional, functions of functions that extremize the functional are found [1]. Also in the finite element process an approximate solution is sought to the problem of minimizing a functional. The concept of the finite element approach to elasticity as a process in which the total potential energy is minimized with respect to nodal displacements can obviously be extended to a variety of physical problems in which an extremum principle exists. The two concepts are combined in this study. Zienkiewicz and Cheung [2] applied similar approach to solve continuum problem expressed in derivative format employing the concept of functional minimization with FEM.

Above all, there are many problems encountered in engineering and physics where the minimization of the integrated quantity usually referred as functional and subject to some boundary conditions results in the exact solution.This functional may represent a physical recognizable variable in some instances, for many purposes it is simply a mathematically defined entity.

The geometry of field quantities or continuum may be a problem to close form solution of field functions encountered in engineering and science that appropriate algorithm becomes necessary to obtain optimum solution, it is then necessary to employ calculus of variation principles and FEM to obtain optimum continuum field functions whose boundary conditions are specified.
The engineering field continuum problems can be basically in form of wave phenomenon, diffusion phenomenon and potential phenomenon usually represented by hyperbolic, parabolic and elliptic differential equations respectively [3].
The objective of this study is therefore to present a methodical approach to solve multiple dimensional field problems using integrated variational and FEM approach to establish relations for all elements functional of continuum where the minimization of the elements functionals system and solution are expected to give the stationary values of the function which extremize the functional.

2. THEORETICAL BACKGROUND

A finite element model of a two dimensional quadratic function is expected to present a methodical approach to employ for solution of multidimensional field functions that may have regular or irregular field regions. Zienkiewicz and Cheung [2] presented Euler theorem to approximate field functions if the integral or functional of the form

.

$$I\,(u) = \iiint f\left(x,y,z,u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right) dxdydz \qquad (1)$$

is to be minimized. The necessary and sufficient condition for this minimum to be reached is that the unknown function u (x, y, z) should satisfy the following differential equation

$$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial(\partial u/\partial x)}\right]+\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial(\partial u/\partial y)}\right]+\frac{\partial}{\partial z}\left[\frac{\partial f}{\partial(\partial u/\partial z)}\right]-\frac{\partial f}{\partial u}=0 \qquad (2)$$

within the same region, provided u satisfies the same boundary conditions in both cases,while the equation governing the behaviour of unknown physical quantity u can generally be expressed as

$$\frac{\partial}{\partial x}\left(k_x\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial u}{\partial z}\right)+Q=0 \qquad (3)$$

where

u = unknown function assumed to be single valued within the region

k_x, k_y, k_z , Q = specified functions of x, y, z

x, y, z = space variables

The equivalent formulation to that of equation (3) is the requirement that the volume integral given below and taken over the whole region, should be

$$\chi = \iiint \left\{\frac{1}{2}\left[k_x\left(\frac{\partial u}{\partial x}\right)^2+k_y\left(\frac{\partial u}{\partial y}\right)^2+k_z\left(\frac{\partial u}{\partial z}\right)^2\right]-Q\,u\right\}dxdydz \qquad (4)$$

subject to u obeying the same boundary conditions.

For two dimensional differential equation representing some physical quantities then

$$\chi = \iint \left\{\frac{1}{2}\left[k_x\left(\frac{\partial u}{\partial x}\right)^2+k_y\left(\frac{\partial u}{\partial y}\right)^2\right]-Q\,u\right\}dxdy \qquad (5)$$

For the case of our interest, the equivalent functional to be minimized for 2-D Laplace model reduces to

$$\chi = \iint \left\{\frac{1}{2}\left[k_x\left(\frac{\partial u}{\partial x}\right)^2+k_y\left(\frac{\partial u}{\partial y}\right)^2\right]\right\}dxdy \qquad (6)$$

The finite element version of an integrated functional is obtained and minimized with respect to degrees of freedoms of the associated elements. The element functional equations are assembled and boundary conditions applied, resulting in a system of equations equal to the number of unconstrained degrees of freedoms of the continuum.

3. FINITE ELEMENT METHOD (FEM)

Euler variational minimum integral theorem was applied with the procedure of [4] on the general equation governing the behavior of field functions presented by [2] to develop a finite element version of elements functions functionals. The elements function functionals are minimized with respect to degrees of freedoms in the finite element method of assembly are applied to obtain the system model that is solved for the field of function . Basic approaches to achieve finite element solouttion of continuum are also available in [5-8}.

3.1 Formulation of Finite Elements Equations

The elements functional of the study are derived for each element and minimized using equation (6). Minimization of element functional entails finding the partial derivatives of the element functional at its nodes. The contributions of each element nodes are established and added for all continuum nodes to obtain the finite element model of the system. The formulation of finite element model starts by choosing the element type and then choosing the approximation polynomial coefficients are determined for establishing the element equations from where the interpolation functions for u are established for all elements. This function u is used then employed in finding the finite element model of the elements functionals from where the sought functions are found.

3.1.1 Discritization and element topology description

The region is discretized into 16 triangular elements with 26 degrees of freedom and assuming displacement in the global system of coordinate (horizontal direction only) only as in Figure 1 elements topologies are described in Table 1 for the establishment of element interpolation functions for the functional equations for the finite element minimization scheme.

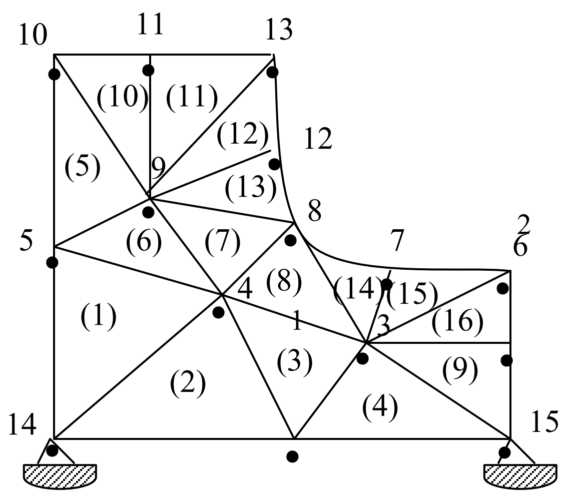


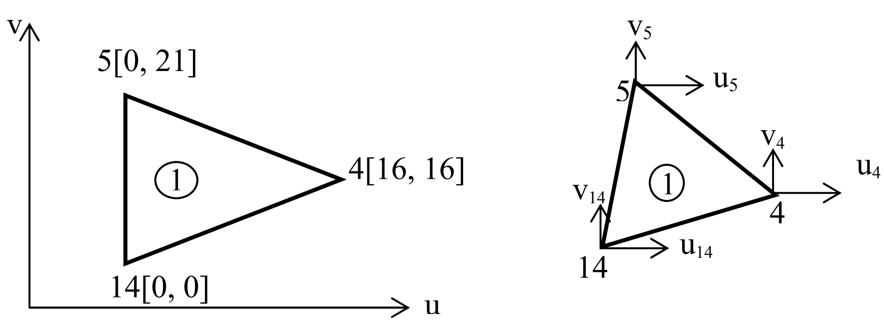
Figure 1: Idealized Finite element Model of two Dimensional Composite Body.

Table 1: Element Topology Description.

Element Number	Active degrees of freedom of elements	Element coordinates	Element nodes
1	$u_5, u_4, u_{14}, v_5, v_4, v_{14}$	(0,0), (0,21), (16,16)	14, 5, 4
2	$u_1, u_4, u_{14}, v_1, v_4, v_{14}$	(0,0), (16,16), (21, 0)	14, 4, 1
3	$u_1, u_4, u_3, v_1, v_4, v_3$	(21, 0), (16, 16), (25, 10)	1, 4, 3
4	$u_1, u_3, u_{15}, v_1, v_3, v_{15}$	(21, 0), (21, 0), (25, 10)	1, 3, 15
5	$u_5, u_{10}, u_9, v_5, v_{10}, v_9$	(0, 21), (0, 37), (10, 25)	5, 10, 9
6	$u_5, u_9, u_4, v_5, v_9, v_4$	(0, 21), (0, 25), (16, 16)	5, 9, 4
7	$u_9, u_4, u_8, v_9, v_4, v_8$	(16, 16), (10, 25), (22, 13)	4, 9, 8
8	$u_8, u_4, u_3, v_8, v_4, v_3$	(16, 16), (22, 23), (25, 10)	4, 8, 3
9	$u_{15}, u_3, u_2, v_{15}, v_3, v_2$	(25, 10), (35, 10), (35, 0)	15, 3, 2
10	$u_9, u_{10}, u_{11}, v_9, v_{10}, v_{11}$	(0, 37), (10, 37), (10, 25)	9, 10, 11
11	$u_9, u_{11}, u_{13}, v_9, v_{11}, v_{13}$	(10, 25), (10, 37), (18, 37)	9, 11, 13
12	$u_9, u_{13}, u_{12}, v_9, v_{13}, v_{12}$	(10, 25), (18, 37), (19, 29)	9, 13, 12
13	$u_9, u_{12}, u_8, v_9, v_{12}, v_8$	(10, 25), (19, 29), (22, 23)	9, 12, 8
14	$u_3, u_8, u_7, v_3, v_8, v_7$	(25, 10), (22, 23), (29, 19)	3, 8, 7
15	$u_3, u_7, u_6, v_3, v_7, v_6$	(25, 11), (29, 19), (35, 18)	3, 7, 6
16	$u_3, u_6, u_{12}, v_3, v_6, v_{12}$	(25, 10), (35, 18), (35, 10)	3, 6, 2

3.2 Determination of FEM Characteristics

3.2.1 Element 1 interpolation and functional equation formulation



By assuming a linear approximation polynomial of the form

$$u(x, y) = a_0 + a_1 x + a_2 y \tag{1}$$

and following the method of Ihueze etal (2009) and Asterly (1992)

Where a_0, a_1, a_2 are called polynomial coefficients or shape constants so that by passing (1) through the nodes of element1the system of unknown function of the element becomes:

$$\begin{aligned} u_{14} &= a_0 + a_1 x_{14} + a_2 y_{14} \\ u_4 &= a_0 + a_1 x_4 + a_2 y_4 \\ u_5 &= a_0 + a_1 x_5 + a_2 y_5 \end{aligned}$$

Putting the above polynomial function in matrix form then

$$\begin{bmatrix} 1 & x_{14} & y_{14} \\ 1 & x_4 & y_4 \\ 1 & x_5 & y_5 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} u_{14} \\ u_4 \\ u_5 \end{bmatrix}$$

By applying Crammers rule

$$\left.\begin{aligned} a_0 &= \tfrac{1}{2A}\{u_{14}(x_4 y_5 - x_5 y_4) + u_4(x_5 y_{14} - x_{14} y_5) + u_5(x_{14} y_4 - x_4 y_{14})\} \\ a_1 &= \tfrac{1}{2A}\{u_{14}(y_4 - y_5) + u_4(y_5 - y_{14}) + u_5(y_{14} - y_4) \\ a_2 &= \tfrac{1}{2A}\{u_{14}(x_5 - x_4) + u_4(x_{14} - x_5) + u_5(x_4 - x_{14}) \end{aligned}\right\} \qquad (2)$$

Substituting (2) in (1) then

$$u_1 = a_0 + a_1 x + a_2 y$$

$$\begin{aligned} u_1 &= \tfrac{1}{2A}\{u_{14}(x_4 y_5 - x_5 y_4) + u_4(x_5 y_{14} - x_{14} y_5) + u_5(x_{14} y_4 - x_4 y_{14})\} + \\ &\tfrac{1}{2A}\{u_{14}(y_4 - y_5) + u_4(y_5 - y_{14}) + u_5(y_{14} - y_4)\}x + \\ &\tfrac{1}{2A}\{u_{14}(x_5 - x_4) + u_4(x_{14} - x_5) + u_5(x_4 - x_{14})\}y \end{aligned} \qquad (3)$$

Recall that the approximation function is given as

$$u = N_{14} u_{14} + N_4 u_4 + N_5 u_5 \qquad (4)$$

Comparing (5) and (6) we evaluate shape and interpolation function thus

$$\left.\begin{aligned}N_{14}(x,y)&=\tfrac{1}{2A}\{(x_4y_5-x_5y_4)+(y_4-y_5)x+(x_5-x_4)\}y\\N_4(x,y)&=\tfrac{1}{2A}\{(x_5y_{14}-x_{14}y_5)+(y_5-y_{14})x+(x_{14}-x_5)\}y\\N_5(x,y)&=\tfrac{1}{2A}\{(x_{14}y_4-x_4y_{14})+(y_{14}-y_4)x+(x_4-x_{14})\}y\end{aligned}\right\} \quad (5)$$

But A = $\frac{1}{2}\{(x_4y_5-x_5y_4)+(x_5y_{14}-x_{14}y_5)+(x_{14}y_4-x_4y_{14})\}$ = 168mm^2 (6)

where A= area of triangular element so that

$$\left.\begin{aligned}N_{14}&=\tfrac{1}{336}(336-5x-16y)\\N_4&=\tfrac{1}{336}(21x)\\N_5&=\tfrac{1}{336}(-16x+16y)\end{aligned}\right\} \quad (7)$$

Substituting (7) in (4)

$$u=\tfrac{1}{336}(336-5x-16y)u_{14}+\tfrac{1}{336}(21x)u_4+\tfrac{1}{336}(-16x+16y)u_5 \quad (8)$$

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=\frac{-5u_{14}}{336}+\frac{21u_4}{336}-\frac{16u_5}{336}\\\frac{\partial u}{\partial y}&=\frac{-16u_{14}}{336}+\frac{16u_5}{336}\end{aligned}\right\} \quad (9)$$

By assuming a two dimensional Laplace function for the continuum function of the form

$$\frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}^2}+\frac{\partial^2 \mathrm{u}}{\partial \mathrm{y}^2}=0 \quad (10)$$

The minimum function integral called functional to be minimized becomes in which case

kx = ky = kz = 1 and Q = 0 (11)

So that (4) reduces to

$$x=\iint\left[\frac{1}{2}\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right\}\right]dxdy \quad (12)$$

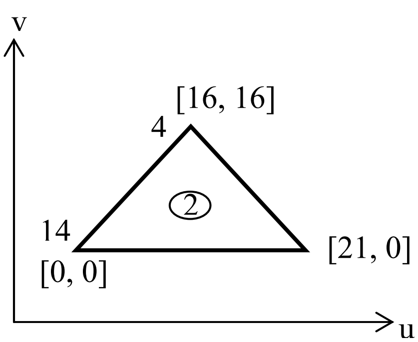
By substitutingg the first partial derivatives of the element 4 interpolation functions in (12) with $dxdy$ = A = 168

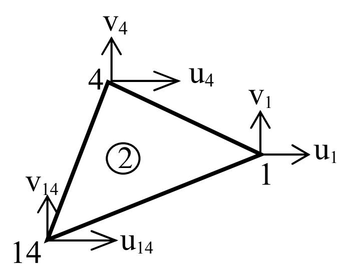
$$x = 0.418 u_{14}^2 - 0.313 u_4 u_{14} - 0.524 u_5 u_{14} - u_4 u_5 \_ 0.656 u_4^2 + 0.762 u_5^2 \tag{13}$$

By differentiating w.r.t.u_{14},u_4,and u_5

$$\left.\begin{aligned}
\frac{\partial x}{\partial u_{14}} &= \left(0.836\, u_{14} - 0.313\, u_4 - 0.524\, u_5\right) * 0.5 \\
\frac{\partial x}{\partial u_4} &= \left(1.312\, u_4 - 0.313\, u_{14} - u_5\right) * 0.5 \\
\frac{\partial x}{\partial u_5} &= \left(1.524\, u_5 - 0.524\, u_{14} - u_4\right) * 0.5
\end{aligned}\right\} \tag{14}$$

3.2.2 Element 2 interpolation and functional equation formulation





By assuming a linear approximation polynomial of the form

$$u(x, y) = a_0 + a_1 x + a_2 y \tag{15}$$

Passing (15) through the nodes then

$$u_{14} = a_0 + a_1 x_{14} + a_2 y_{14}$$
$$u_1 = a_0 + a_1 x_1 + a_2 y_1$$
$$u_4 = a_0 + a_1 x_4 + a_2 y_4$$

Putting the system in matrix form then

$$\begin{bmatrix} 1 & x_{14} & y_{14} \\ 1 & x_1 & y_1 \\ 1 & x_4 & y_4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} u_{14} \\ u_1 \\ u_4 \end{bmatrix}$$

By applying Crammers rule,

$$\left.\begin{aligned} a_0 &= \tfrac{1}{2A}\{u_{14}(x_1y_4 - x_4y_1) + u_{14}(y_1 - y_4) + u_{14}(x_4 - x_1)\} \\ a_1 &= \tfrac{1}{2A}\{u_1(x_4y_{14} - x_{14}y_4) + u_1(y_4 - y_{14}) + u_1(x_{14} - x_4)\} \\ a_2 &= \tfrac{1}{2A}\{u_4(x_{14}y_1 - x_1y_{14}) + u_4(y_{14} - y) + u_4(x_1 - x_{14})\} \end{aligned}\right\} \tag{16}$$

Substituting (16) in (15)

$$u = \tfrac{1}{2A}\Big[\{u_{14}(x_1y_4 - x_4y_1) + u_{14}(y_1 - y_4) + u_{14}(x_4 - x_1)\} + \\ \{u_1(x_4y_{14} - x_{14}y_4) + u_1(y_4 - y_{14}) + u_1(x_{14} - x_4)\} + \\ \{u_4(x_{14}y_1 - x_1y_{14}) + u_4(y_{14} - y_1) + u_4(x_1 - x_{14})\}\Big] \tag{17}$$

Recalling that the approximation function is

$$u = N_{14}u_{14} + N_1u_1 + N_4u_4 \tag{18}$$

Comparing (18) and (17) then

$$\left.\begin{aligned} N_{14} &= \tfrac{1}{2A}\{(x_1y_4 - x_4y_1) + (y_1 - y_4)x + (x_4 - x_1)\}y \\ N_1 &= \tfrac{1}{2A}\{(x_4y_{14} - x_{14}y_4) + (y_4 - y_{14})x + (x_{14} - x_4)\}y \\ N_4 &= \tfrac{1}{2A}\{(x_{14}y_1 - x_1y_{14}) + (y_{14} - y_1)x + (x_1 - x_{14})\}y \end{aligned}\right\} \tag{19}$$

But A = $\tfrac{1}{2}\{(x_1y_4 - x_4y_1) + (x_4y_{14} - x_{14}y_4) + (x_{14}y_1 - x_1y_{14})\}$ =168mm^2 (20)

so that

$$\left.\begin{aligned} N_{14} &= \tfrac{1}{336}\{336 + (0 - 16)x + (16 - 21)\} = \tfrac{1}{336}(336 - 16x - 5y) \\ N_1 &= \tfrac{1}{336}\{0 + (16 - 0)x + (0 - 16)y\} = \tfrac{1}{336}(16x - 16y) \\ N_4 &= \tfrac{1}{336}\{0 + (0) + (21 - 0)y\} = \tfrac{1}{336}(21y) \end{aligned}\right\} \tag{21}$$

Substituting (22) into (18)

$$u = \tfrac{1}{336}(336 - 16x - 5y)u_{14} + \tfrac{1}{336}(16 - 16y)u_1 + \tfrac{1}{336}(21y)u_4 \tag{22}$$

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= \frac{-16u_{14}}{336}+\frac{16u_1}{336}=\frac{-u_{14}}{21}+\frac{u_1}{21}\\ \frac{\partial u}{\partial y} &= \frac{-5u_{14}}{336}-\frac{16u_1}{336}+\frac{21u_{14}}{336}=\frac{-5u_{14}}{336}-\frac{u_1}{21}+\frac{u_4}{16}\end{aligned}\right\} \tag{23}$$

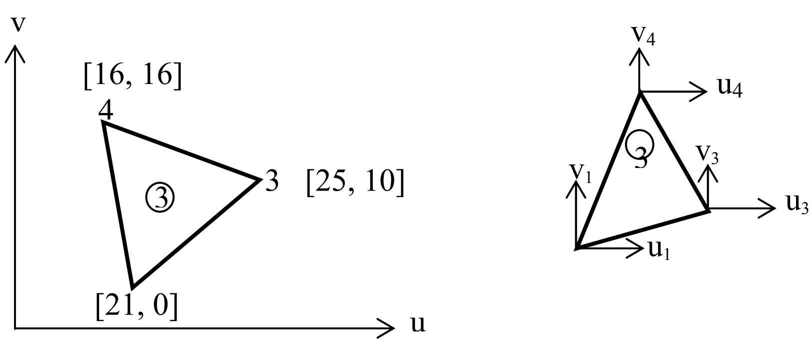
By substitutingg the first partial derivatives of the element 4 interpolarion functions in (12) with dxd*y* = A = 168mm^2

$$x=\left(0.190u_{14}^2-0.3611u_1u_{14}+0.38u_1^2-0.156u_4u_{14}-0.5u_1u_4+0.328u_4^2\right) \tag{24}$$

By differentiating w.r.t.u_{14},u_1,and u_4

$$\left.\begin{aligned}\frac{\partial x}{\partial u_{14}} &= 0.418u_{14}-0.261u_1-0.156u_4\\ \frac{\partial x}{\partial u_1} &= 0.76u_1-0.261u_{14}-0.5u_5\\ \frac{\partial x}{\partial u_4} &= 1.656u_4-0.156u_{14}-0.5u_1\end{aligned}\right\} \tag{25}$$

3.2.3 Element 3 interpolation and functional equation formulation



By assuming a linear approximation polynomial of the form

$$u(x,y)=a_0+a_1x+a_2y \tag{26}$$

and passing (26) through the nodes then

$$u_1 = a_0 + a_1x_1 + a_2y_1$$

$$u_3 = a_0 + a_1x_3 + a_2y_3$$

$$u_4 = a_0 + a_1x_4 + a_2y_4$$

Putting the above equations in matrix form then,

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_3 & y_3 \\ 1 & x_4 & y_4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_3 \\ u_4 \end{bmatrix} \tag{27}$$

By applying Crammers rule,

$$\left.\begin{aligned} a_0 &= \tfrac{1}{2A}\{u_1(x_3y_4 - x_4y_3) + u_1(y_3 - y_4) + u_1(x_4 - x_3)\} \\ a_1 &= \tfrac{1}{2A}\{u_3(x_4y_1 - x_1y_4) + u_3(y_4 - y_1) + u_3(x_1 - x_4)\} \\ a_2 &= \tfrac{1}{2A}\{u_4(x_1y_3 - x_3y_1) + u_4(y_1 - y_3) + u_4(x_3 - x_1)\} \end{aligned}\right\} \tag{28}$$

Substituting (28) into (26)

$$\left.\begin{aligned} u = \tfrac{1}{2A}[&\{u_1(x_3y_4 - x_4y_3) + u_1(y_3 - y_4) + u_1(x_4 - x_3)\} + \\ &\{u_3(x_4y_1 - x_1y_4) + u_3(y_4 - y_1) + u_3(x_1 - x_4)\} + \\ &\{u_4(x_1y_3 - x_3y_1) + u_1(y_1 - y_3) + u_4(x_3 - x_1)\}] \end{aligned}\right\} \tag{29}$$

Recall that the approximation function or interpolation function is expressed as:

$$u = N_1u_1 + N_3u_3 + N_4u_4 \tag{30}$$

Comparing (29) and (30) then,

$$\left.\begin{aligned} N_1 &= \tfrac{1}{2A}\{(x_3y_4 - x_4y_3) + (y_3 - y_4)x + (x_4 - x_3)\}y \\ N_3 &= \tfrac{1}{2A}\{(x_4y_1 - x_1y_4) + (y_4 - y_1)x + (x_1 - x_4)y \\ N_4 &= \tfrac{1}{2A}\{(x_1y_3 - x_3y_1) + (y_1 - y_3)x + (x_3 - x_1)y \end{aligned}\right\} \tag{31}$$

But $A = \frac{1}{2}\{[x_1y_4 - x_4y_3] + [x_4y_1 - x_1y_4] + [x_1y_3 - x_3y_1]\} = 57\text{mm}^2$ (32)

then

$$\left.\begin{aligned} N_1 &= \tfrac{1}{114}(400-160) + (10-16)x + (16-25)y = \tfrac{1}{114}(240 - 6x - 9y) \\ N_3 &= \tfrac{1}{114}(0-336) + (16-0)x + (21-16)y = \tfrac{1}{114}(-336 + 16x - 5y) \\ N_4 &= \tfrac{1}{114}(525-0) + (0-10)x + (25-21)y = \tfrac{1}{114}(525 + 10x - 4y) \end{aligned}\right\} \tag{33}$$

Substituting (30) into (31)

$$\left.\begin{aligned} &u = \tfrac{1}{114}(240 - 6x - 9y)u_1 + \tfrac{1}{114}(-336 + 16x - 5y)u_3 + \tfrac{1}{114}(525 - 10x + 4y)u_4 \\ &\frac{\partial u}{\partial x} = \frac{-6u_1}{114} + \frac{16u_3}{114} - \frac{-10u_{14}}{114} \\ &\frac{\partial \bar{u}}{\partial y} = \frac{-9u_1}{114} - \frac{5u_3}{114} + \frac{4u_4}{114} \end{aligned}\right\} \tag{34}$$

By substitutingg the first partial derivatives of the element 4 interpolarion functions in (12) with dxdy = A = 57

$$x = 0.257u_1^2 - 0.224u_1u_3 + 0.105u_1u_4 + 0.616u_3^2 - 0.789u_3u_4 + 0.254u_4^2 \tag{35}$$

By differentiating w.r.t.u_1,u_3, and u_4

$$\left.\begin{aligned} &\frac{\partial x}{\partial u_1} = 0.514u_1 - 0.224u_3 - 0.105u_4 \\ &\frac{\partial x}{\partial u_3} = 0.232u_3 - 0.224u_1 - 0.789u_4 \\ &\frac{\partial x}{\partial u_4} = 1.508u_4 - 0.105u_1 - 0.789u_3 \end{aligned}\right\} \tag{36}$$

3.2.4 Element 4 interpolation and functional equation formulation

By substitutingg the first partial derivatives of the element 4 interpolarion function in (12)

$$x = 0.357u_1^2 - 0.214u_1u_{15} + 0.207u_{15}^2 - 0.5u_1u_3 - 0.2u_3u_{15} + 0.35u_3^2 \tag{37}$$

By differentiating w.r.t.u_1,u_{15}, and u_3

$$\left.\begin{aligned} &\frac{dx}{du_1} = 0.714u_1 - 0.2144u_{15} - 0.5u_3 \\ &\frac{dx}{du_{15}} = 0.414u_{15} - 0.214u_1 - 0.2u_3 \\ &\frac{dx}{du_3} = 0.7u_3 - 0.5u_1 - 0.2u_{15} \end{aligned}\right\} \tag{38}$$

3.2.5 Element 5 interpolation and functional equation formulation

By substitutingg the first partial derivatives of the element 4 interpolarion functions in (12) with dxdy = A =57mm^2

$$x = 0.35u_5^2 - 0.6u_5u_9 - 0.163u_5u_{10} + 0.4u_9^2 - 0.2u_9u_{10} + 0.181u_{10}^2 \tag{39}$$

By differentiating w.r.t.u_5,u_9,and u_{10}

$$\left.\begin{aligned} \frac{\partial x}{\partial u_5} &= 0.70u_5 - 0.6u_9 - 0.163u_{10} \\ \frac{\partial x}{\partial u_9} &= 0.80u_9 - 0.6u_5 - 0.2u_{10} \\ \frac{\partial x}{\partial u_{10}} &= 0.362u_{10} - 0.163u_5 - 0.2u_9 \end{aligned}\right\} \tag{40}$$

3.2.6 Element 6 interpolation and functional equation formulation

By similar procedures as above,

$$x = 0.254u_4^2 - 0.105u_4u_5 - 0.614u_4u_9 + 0.616u_9^2 - 0.618u_5u_9 + 0.257u_5^2 \tag{41}$$

By differentiating w.r.t.u_4,u_9,and u_5

$$\left.\begin{aligned} \frac{\partial x}{\partial u_4} &= 0.508u_5 - u_9 - u_{10} \\ \frac{\partial x}{\partial u_9} &= 1.232u_9 - 0.614u_5 - 0.618u_5 \\ \frac{\partial x}{\partial u_5} &= 0.514u_5 + 0.105u_4 - 0.618u_9 \end{aligned}\right\} \tag{42}$$

3.2.7 Element 7 interpolation and functional equation formulation

By similar procedures as above,

$$x = 0.385u_4^2 - 0.469u_4u_8 - 0.302u_4u_9 + 0.305u_8^2 - 0.141u_8u_9 + 0.221u_9^2 \tag{43}$$

By differentiating w.r.t.u_4, u_8, and u_9

$$\left.\begin{aligned}\frac{\partial x}{\partial u_4} &= 0.77u_4 - 0.469u_8 - 0.302u_9 \\ \frac{\partial x}{\partial u_8} &= 1.61u_8 - 0.469u_4 - 0.141u_9 \\ \frac{\partial x}{\partial u_9} &= 0.442u_9 - 0.302u_4 - 0.141u_9\end{aligned}\right\} \quad (44)$$

3.2.8 Element 8 interpolation and functional equation formulation

By similar procedures as above,

$$x = 0.215u_3^2 - 0.61u_3u_8 - 0.369u_3u_4 + 0.295u_8^2 - 0.530u_4u_8 + 0.450u_4^2 \quad (45)$$

By differentiating w.r.t.u_3,u_8,and u_4

$$\left.\begin{aligned}\frac{\partial x}{\partial u_3} &= 0.430u_3 - 0.061u_8 - 0.369u_4 \\ \frac{\partial x}{\partial u_8} &= 0.590u_8 - 0.061u_3 - 0.530u_4 \\ \frac{\partial x}{\partial u_4} &= 0.90u_4 - 0.369u_3 - 0.530u_9\end{aligned}\right\} \quad (46)$$

3.2.9 Element 9 interpolation and functional equation formulation

By similar procedures as above,

$$x = 0.5u_2^2 - 0.5u_2u_3 + 0.25u_3^2 + 0.25u_{15}^2 - 0.5u_2u_{15} \quad (47)$$

By differentiating w.r.t.u_2,u_3,and u_{15}

$$\left.\begin{aligned}\frac{\partial x}{\partial u_2} &= u_2 - 0.5u_3 - 0.5u_{15} \\ \frac{\partial x}{\partial u_3} &= 0.5u_3 - 0.5u_2 \\ \frac{\partial x}{\partial u_{15}} &= 0.5u_{15} - 0.5u_2\end{aligned}\right\} \quad (48)$$

3.2.10 Element 10 interpolation and functional equation formulation

By similar procedures as above,

$$x = 0.508u_{11}^2 - 0.61u_{10}u_{11} + 0.3u_{10}^2 + 0.208u_9^2 - 0.417u_9u_{11} \tag{49}$$

By differentiating w.r.t.u_{11},u_{10}, and u_9

$$\left.\begin{aligned} \frac{\partial x}{\partial u_{11}} &= 1.016u_{11} - 0.6u_{10} - 0.417u_9 \\ \frac{\partial x}{\partial u_{10}} &= 0.6u_{10} - 0.6u_{11} \\ \frac{\partial x}{\partial u_9} &= 0.416u_9 - 0.417u_{11} \end{aligned}\right\} \tag{50}$$

3.2.11 Element 11 interpolation and functional equation formulation

By similar procedures as above,

$$x = 0.375u_{13}^2 - 0.75u_{11}u_{13} + 0.542u_{11}^2 + 0.167u_9^2 - 0.333u_9u_{11} \tag{51}$$

By differentiating w.r.t.u_9,u_{13}, and u_1

$$\left.\begin{aligned} \frac{\partial x}{\partial u_9} &= 0.334u_9 - 0.333u_{11} \\ \frac{\partial x}{\partial u_{13}} &= 0.75u_{13} - 0.75u_{11} \\ \frac{\partial x}{\partial u_{11}} &= 1.084u_{11} - 0.75u_{13} - 0.333u_9 \end{aligned}\right\} \tag{52}$$

3.2.12 Element 12 interpolation and functional equation formulation

By similar procedures as above,

$$x = 0.214u_9^2 + 0.68u_9u_{12} + 0.151u_9u_{13} + 0.68u_{12}^2 - 0.158u_{12}u_{13} + 0.319u_{13}^2$$

By differentiating w.r.t.u_9,u_{12}, and u_{13}

$$\left.\begin{aligned} \frac{\partial x}{\partial u_9} &= 0.428u_9 + 0.684u_{12} + 0.151u_{13} \\ \frac{\partial x}{\partial u_{12}} &= 1.368u_{12} + 0.684u_9 - 0.158u_3 \\ \frac{\partial x}{\partial u_{13}} &= 0.638u_{13} - 0.151u_9 - 0.158u_{12} \end{aligned}\right\} \tag{53}$$

3.2.13 Element 13 interpolation and functional equation formulation

By similar procedures as above,

$$x = 0.367u_8^2 - 0.758u_8u_{12} + 0.386u_8u_9 + 0.561u_{12}^2 - 0.182u_9u_{12} - 0.170u_9^2 \tag{54}$$

By differentiating w.r.t.u_8, u_{12}, and u_9

$$\left.\begin{aligned} \frac{\partial x}{\partial u_8} &= 0.734u_8 - 0.758u_{12} + 0.386u_9 \\ \frac{\partial x}{\partial u_{12}} &= 1.122u_{12} + 0.758u_8 - 0.182u_9 \\ \frac{\partial x}{\partial u_9} &= 0.34u_9 + 0.386u_8 - 0.182u_{12} \end{aligned}\right\} \tag{55}$$

3.2.14 Element 14 interpolation and functional equation formulation

By similar procedures as above,

$$x = 0.206u_3^2 - 0.462u_3u_7 + 0.051u_3u_8 + 0.563u_7^2 - 0.665u_7u_8 + 0.307u_8^2 \tag{56}$$

By differentiating w.r.t.u_3,u_7, and u_8

$$\left.\begin{aligned} \frac{\partial x}{\partial u_3} &= 0.412u_3 - 0.462u_7 + 0.051u_8 \\ \frac{\partial x}{\partial u_7} &= 1.126u_7 - 0.462u_3 - 0.665u_8 \\ \frac{\partial x}{\partial u_8} &= 0.614u_8 + 0.51u_3 - 0.665u_7 \end{aligned}\right\} \tag{57}$$

3.2.15 Element 15 interpolation and functional equation formulation

By similar procedures as above,

$$x = 0.178u_3^2 + 0.154u_3u_6 - 0.51u_3u_7 + 0.385u_6^2 - 0.923u_6u_7 + 0.716u_7^2$$

By differentiating w.r.t.u_3, u_6, and u_7

$$\left.\begin{aligned}\frac{\partial x}{\partial u_3} &= 0.356u_3 + 0.154u_6 - 0.51u_7 \\ \frac{\partial x}{\partial u_3} &= 0.356u_3 + 0.154u_6 - 0.51u_7 \\ \frac{\partial x}{\partial u_7} &= 1.432u_7 - 0.51u_3 - 0.923u_6\end{aligned}\right\} \tag{58}$$

3.2.16 Element 16 interpolation and functional equation formulation

By similar procedures as above,

$$x = 0.2u_3^2 - 0.4u_2u_3 + 0.513u_2^2 - 0.625u_2u_6 + 0.313u_6^2$$

By differentiating w.r.t.u_3, u_2 and u_6

$$\left.\begin{aligned}\frac{\partial x}{\partial u_3} &= 0.4u_3 - 0.4u_2 \\ \frac{\partial x}{\partial u_2} &= 0.026u_2 - 0.4u_3 - 0.625u_6 \\ \frac{\partial x}{\partial u_6} &= 0.626u_6 - 0.625u_2\end{aligned}\right\} \tag{59}$$

4. SYSTEM ELEMENTS ASSEMBLY ALGORITHMS

The algorithms for element assembly involves the addition of all elements contributing to minimization $\frac{dX^e}{du}$, this leads to system of equations that equals the degrees of freedoms in the continuum, the derivatives are then added in a special format called assembly. There are 15 effective degrees of freedoms for the assembly of 16 elements

$$\sum\frac{dX^e}{\partial u_i} = 0,\ i = 1, 2, 3,\ldots\ldots, 16$$

For

$$i = 1,\ \sum\frac{\partial X^e}{\partial u_1} = 0 \tag{60}$$

$$i = 2,\ \sum\frac{\partial X^e}{\partial u_2} = 0 \tag{61}$$

$$i = 3, \sum\frac{\partial X^e}{\partial u_3} = 0 \tag{62}$$

$$i = 4, \sum\frac{\partial X^e}{\partial u_4} = 0 \tag{63}$$

$$i = 5, \sum\frac{\partial X^e}{\partial u_5} = 0 \tag{64}$$

$$i = 6, \sum\frac{\partial X^e}{\partial u_6} = 0 \tag{65}$$

$$i = 7, \sum\frac{\partial X^e}{\partial u_7} = 0 \tag{66}$$

$$i = 8, \sum\frac{\partial X^e}{\partial u_8} = 0 \tag{67}$$

$$i = 9, \sum\frac{\partial X^e}{\partial u_9} = 0 \tag{68}$$

$$i = 10, \sum\frac{\partial X^e}{\partial u_{10}} = 0 \tag{69}$$

$$i = 11, \sum\frac{\partial X^e}{\partial u_{11}} = 0 \tag{70}$$

$$i = 12, \sum\frac{\partial X^e}{\partial u_{12}} = 0 \tag{71}$$

$$i = 13, \sum\frac{\partial X^e}{\partial u_{13}} = 0 \tag{72}$$

$$i = 14, \sum\frac{\partial X^e}{\partial u_{14}} = 0 \tag{73}$$

$$i = 15, \sum\frac{\partial X^e}{\partial u_{15}} = 0 \tag{74}$$

5. ELEMENTS EQUATIONS ASSEMBLY

All the partial derivatives resulting from the minimization scheme with respect to the fifteen (15) active degrees of freedom (DOF) are added as follows the superscripts on these equations denote element sources:

$$\frac{\partial X}{\partial u_1} = \sum\frac{\partial X^e}{\partial u_1} = 0 = \frac{\partial X^2}{\partial u_1} + \frac{\partial X^3}{\partial u_1} + \frac{\partial X^4}{\partial u_1}$$
$$= 1.988u_1 - 0.724u_3 - 0.105u_4 - 0.5u_5 - 0.261u_{14} - 0.214u_{15} \tag{75}$$

$$\frac{\partial X}{\partial u_2} = \sum\frac{\partial X^e}{\partial u_2} = 0 = \frac{\partial X^9}{\partial u_2} + \frac{\partial X^{16}}{\partial u_2}$$
$$= 1.026u_2 - 0.9u_3 - 0.625u_6 - 0.5u_{15} \tag{76}$$

$$\frac{\partial X}{\partial u_3} = \sum \frac{\partial X^e}{\partial u_3} = 0 = \frac{\partial X^{16}}{\partial u_3} + \frac{\partial X^{15}}{\partial u_3} + \frac{\partial X^{14}}{\partial u_3} + \frac{\partial X^{9}}{\partial u_3} + \frac{\partial X^{4}}{\partial u_3} + \frac{\partial X^{3}}{\partial u_3} + \frac{\partial X^{8}}{\partial u_3}$$
$$= -\,0.724u_1 - 0.9u_2 + 3.03u_3 - 1.158u_4 + 0.154u_6 - 0.972u_7 - 0.001u_8 - 0.2u_{15} \qquad (77)$$

$$\frac{\partial X}{\partial u_4} = \sum \frac{\partial X^e}{\partial u_4} = 0 = \frac{\partial X^{1}}{\partial u_4} + \frac{\partial X^{2}}{\partial u_4} + \frac{\partial X^{3}}{\partial u_4} + \frac{\partial X^{6}}{\partial u_4} + \frac{\partial X^{7}}{\partial u_4} + \frac{\partial X^{8}}{\partial u_4}$$
$$= -1.158u_3 + 5.49u_4 - 0.395u_1 + 0.008u_5 - 0.469u_8 - 1.832u_9 - u_{10} - 0.001u_{14} \qquad (78)$$

$$\frac{\partial X}{\partial u_5} = \sum \frac{\partial X^e}{\partial u_5} = 0 = \frac{\partial X^{1}}{\partial u_5} + \frac{\partial X^{5}}{\partial u_5} + \frac{\partial X^{6}}{\partial u_5}$$
$$= -\,0.409u_4 + 1.979u_5 - 1.218u_9 - 0.163u_{10} - 0.262u_{14} \qquad (79)$$

$$\frac{\partial X}{\partial u_6} = \sum \frac{\partial X^e}{\partial u_6} = 0 = \frac{\partial X^{15}}{\partial u_6} + \frac{\partial X^{16}}{\partial u_6}$$
$$= 1.396u_6 - 0.625u_2 + 0.154u_3 - 0.923u_7 \qquad (80)$$

$$\frac{\partial X}{\partial u_7} = \sum \frac{\partial X^e}{\partial u_7} = 0 = \frac{\partial X^{15}}{\partial u_7} + \frac{\partial X^{14}}{\partial u_7}$$
$$= 2.549u_7 - 0.972u_3 - 0.923u_6 - 0.665u_9 \qquad (81)$$

$$\frac{\partial X}{\partial u_8} = \sum \frac{\partial X^e}{\partial u_8} = 0 = \frac{\partial X^{7}}{\partial u_8} + \frac{\partial X^{8}}{\partial u_8} + \frac{\partial X^{13}}{\partial u_8} + \frac{\partial X^{14}}{\partial u_8}$$
$$= 0.449u_3 - 0.999u_4 - 0.665u_7 + 3.548u_8 + 0.245u_9 - 0.758u_{12} \qquad (82)$$

$$\frac{\partial X}{\partial u_9} = \sum \frac{\partial X^e}{\partial u_9} = 0 = \frac{\partial X^{13}}{\partial u_9} + \frac{\partial X^{12}}{\partial u_9} + \frac{\partial X^{11}}{\partial u_9} + \frac{\partial X^{10}}{\partial u_9} + \frac{\partial X^{7}}{\partial u_9} + \frac{\partial X^{6}}{\partial u_9} + \frac{\partial X^{5}}{\partial u_9}$$
$$= -\,0.302u_4 - 1.832u_5 + 0.386u_8 + 3.851u_9 - 0.20u_{10} - 0.75u_{11} + 0.502u_{12} + 0.151u_{13} \qquad (83)$$

$$\frac{\partial X}{\partial u_{10}} = \sum \frac{\partial X^e}{\partial u_{10}} = 0 = \frac{\partial X^{5}}{\partial u_{10}} + \frac{\partial X^{10}}{\partial u_{10}}$$
$$= -\,0.163u_5 - 0.2u_9 + 0.962u_{10} - 0.6u_{11} \qquad (84)$$

$$\frac{\partial X}{\partial u_{11}} = \sum \frac{\partial X^e}{\partial u_{11}} = 0 = \frac{\partial X^{11}}{\partial u_{11}} + \frac{\partial X^{10}}{\partial u_{11}}$$
$$= -\,0.75u_9 - 0.6u_{10} + 2.1u_{11} - 0.75u_{13} \qquad (85)$$

$$\frac{\partial X}{\partial u_{12}} = \sum \frac{\partial X^e}{\partial u_{12}} = 0 = \frac{\partial X^{13}}{\partial u_{12}} + \frac{\partial X^{12}}{\partial u_{12}}$$
$$= -\,0.158u_{13} + 0.758u_8 + 0.502u_9 + 2.49u_{12} \qquad (86)$$

$$\frac{\partial X}{\partial u_{13}} = \sum \frac{\partial X^e}{\partial u_{13}} = 0 = \frac{\partial X^{11}}{\partial u_{13}} + \frac{\partial X^{12}}{\partial u_{13}}$$
$$= - 0.151u_9 - 0.75u_{11} - 0.158u_{12} + 1.388u_{13} \quad (87)$$

$$\frac{\partial X}{\partial u_{14}} = \sum \frac{\partial X^e}{\partial u_{14}} = 0 = \frac{\partial X^{2}}{\partial u_{14}} + \frac{\partial X^{1}}{\partial u_{14}}$$
$$= - 0.261u_1 - 0.313u_4 - 0.262u_5 + 0.836u_{14} \quad (88)$$

$$\frac{\partial X}{\partial u_{15}} = \sum \frac{\partial X^e}{\partial u_{15}} = 0 = \frac{\partial X^{4}}{\partial u_{15}} + \frac{\partial X^{9}}{\partial u_{15}}$$
$$= - 0.214u_1 - 0.5u_2 - 0.2u_3 + 0.914u_{15} \quad (89)$$

6. APPLICATION OF BOUNDARY CONDITION

In this work a special case where displacements at the boundaries are limited to 0.5mm for an irregular continuum is considered to predict continuum displacement, strain and stress functions, while the constrained conditions are taken as zero so that by equating $u_{14} = u_{15} = 0$ and $u_2 = u_5 = u_6 = u_8 = u_{10} = u_{13} = 0.50$, (75 - 89) transform to the following:

$$1.988u_1 - 0.724u_3 - 0.105u_4 = 0.25 \quad (90)$$

$$0.900u_3 = 0.201 \quad (91)$$

$$- 0.724u_1 + 3.03u_3 - 0.158u_4 - 0.972u_7 = 0.374 \quad (92)$$

$$- 1.158u_3 + 5.490u_4 - 0.395u_1 = 0.731 \quad (93)$$

$$- 0.409u_4 - 1.218u_9 = - 0.907 \quad (94)$$

$$0.154u_3 - 0.923u_7 = - 0.386 \quad (95)$$

$$2.549u_7 - 0.972u_3 - 0.665u_9 = 0.462 \quad (96)$$

$$0.449u_3 - 0.999u_4 - 0.665u_7 + 0.245u_9 - 0.758u_{12} = - 1.774 \quad (97)$$

$$3.851u_9 - 0.302u_4 - 0.75u_{11} + 0.502u_{12} = 0.748 \quad (98)$$

$$- 0.200u_9 - 0.600u_{11} = - 0.400 \quad (99)$$

$$2.100u_{11} - 0.750u_9 = 0.675 \quad (100)$$

$$- 0.153u_3 + 0.502u_9 + 2.490u_{12} = - 0.379 \quad (101)$$

$$- 0.151u_9 - 0.750u_{11} - 0.158u_{12} = - 0.694 \quad (102)$$

$$- 0.261u_1 - 0.313u_4 = 0.131 \quad (103)$$

$$- 0.214u_1 - 0.200u_3 = 0.250 \quad (104)$$

7. SOLUTION AND POST PROCESSING FOR CONTINUUM FUNCTION

The following nodal displacements in mm are further evaluated by first evaluating $u_3 = 0.222$ from (91) so that other nodal values of the displacement function is as presented in Table 2. The first partial derivatives of the interpolation function evaluated with active degree of freedom in of element with respect to the x axis gives the slope of the function and also gives the value of the strain as presented in Table 2. The computations are achieved with sixteen elements interpolation functions associated with the elements global coordinate axis. The strains so computed may be used with Hooke's law of elasticity to predict the stress distribution function at the respective nodes when the elastic modulus is known from literature.

Table 2: FEM Results.

n(nodes)	u(displacement)	$\frac{\partial u}{\partial x} = \varepsilon$ (strain)
1	0.210	0.02
2	0.500	0.01
3	0.222	0.02
4	0.059	0.015
5	0.500	0.023
6	0.500	0.015
7	0.455	0.082
8	0.500	0.054
9	0.725	0.028
10	0.500	0.01
11	0.424	0.116
12	0.500	0
13	0.500	0.167
14	0.000	0.026
15	0.000	0.011

The stress prediction model of a material within the elastic limit is expressed as

$$\sigma = E\varepsilon \qquad (105)$$

where E = modulus of elasticity

The excel graphics of FEM result using Table 2 of Figure 2 shows a serious indication that the minimum value of the function is between node 14 and 15 hence another extremization method is needed to point at which point of the region is this extremum.

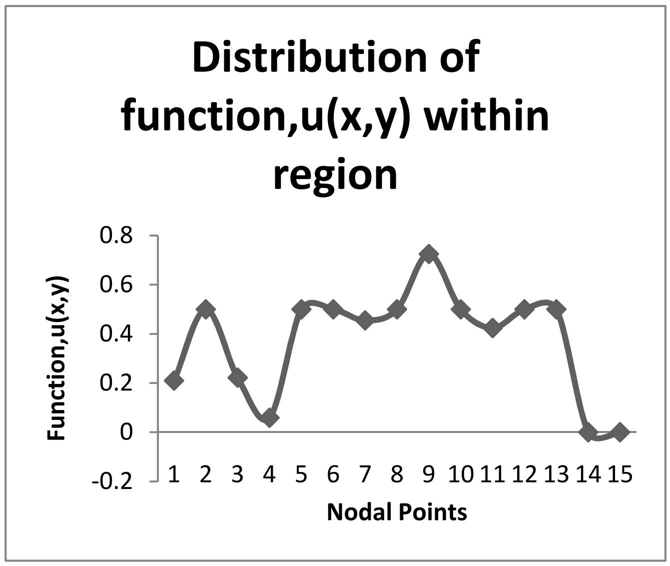


Figure 2: Distribution of Function within the Region.

8. DISCUSSION AND VALIDATION OF RESULTS

Regression analysis was carried out on FEM results to obtain a unified model for elements function interpolation. The regression model so obtained is further used to transform the element functional equation to aid extremization of FEM results.

8.1 Regression Analysis

Multiple linear regression analysis was carried out on finite element results to obtain the following model for the region. By employing the classical multiple linear regression equation of the form

$$u(x, y) = a_o + a_1 x + a_2 y \qquad (106)$$

a regression model for the FEM is obtained with Table 3 and expressed as (107).

$$\mathbf{u(x,y) = 0.065 + 0.0036x + 0.0130y} \quad (107)$$

The goodness of fit of regression was evaluated to obtain: Coefficient of determination, $r^2 = 0.52$, correlation coefficient, $r = 0.72$, standard error, $s_e = 0.1$

where u = field function evaluated through FEM

u^1 = average of FEM function

up = field function predicted with regression model

Table 2 and Figure 2 show the variation of the function within the region. Continuum fluid elements in heat and mass transfer operations associated with pipeline transportation can elegantly be analyzed following the procedure of this work. The FEM developed can be applied in the evaluation of the stress distribution in irregular shaped continuum whose boundary conditions are specified such as in the evaluation of displacement in structures and solid mechanics problems, evaluation of temperature distribution in heat conduction problems, evaluation of displacement potential in acoustic fluids ,evaluation of pressure in potential flows ,evaluation of velocity in general flows, evaluation of electric potential in electrostatics and in evaluation of magnetic potential in magnetostatics.

8.2 Extremization of Functional: Extremization by Lagrange Multipliers Approach

In order to further analyse the FEM results, the functional, χ of any element is transformed to a function of (x, y) using the regression model of (107) to obtain:

$$\chi = f(x,y) = 0.000042x + 0.0017y + 0.000059x^2 + 0.00034y^2 + 0.00015xy + 0.00847 = (108)$$

Figure 4a,b and c show versions of 3D plots of function using Matlab for (108)

The objective function

$$f(x,y) = 0.000042x + 0.0017y + 0.000059x^2 + 0.00034y^2 + 0.00015xy + 0.00847$$

subject to the constraint relations

$$u(x,y) = 0.5 + 0.0225x \qquad = 0.5 \quad (109)$$

$$u(x,y) = -0.0201x + 0.0238y = 0 \quad (110)$$

derived for nodes 14 and 10 of elements 1 and 5 at the boundaries.

Table 3 Computations For Regression and Error Analysis of FEM Results.

N	x	y	u	x^2	y^2	x*y	x*u	y*u	up	$(u-u^1)^2$	$(u-up)^2$
1	21	0.0	0.2100	441	0.0000	0.0000	4.4100	0.0000	0.1406	0.0266	0.004816
2	35	10	0.5000	1225	100	350	17.5000	5.0000	0.321	0.0161	0.032041
3	25	10	0.2220	625	100	250	5.5500	2.2200	0.285	0.0228	0.003969
4	16	16	0.0590	256	256	256	0.944	0.9440	0.3306	0.0986	0.073767
5	0.0	21	0.5000	0.0000	441	0.0000	0.0000	10.5000	0.338	0.0161	0.026244
6	35	18	0.5000	1225	324	630	17.5000	9.0000	0.425	0.0161	0.005625
7	29	19	0.4550	841	361	551	13.195	8.6450	0.4164	0.0067	0.00149
8	22	23	0.5000	484	529	506	11.000	11.5000	0.4432	0.0161	0.003226
9	10	25	0.7250	100	625	250	7.2500	18.1250	0.426	0.1239	0.089401
10	0.0	37	0.5000	0.0000	1369	0.0000	0.0000	18.5000	0.546	0.0161	0.002116
11	10	37	0.424	100	1369	370	4.2400	15.6880	0.582	0.0026	0.024964
12	19	29	0.5000	361	841	551	9.5000	14.5000	0.5104	0.0161	0.000108
13	18	37	0.5000	324	1369	666	9.0000	18.5000	0.6108	0.0161	0.012277
14	0.0	0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.065	0.1391	0.004225
15	35	0.0	0.0000	1225	0.0000	0.0000	0.0000	0.0000	0.191	0.1391	0.036481
sum	**275**	**282**	**5.595**	**7207**	**7684**	**4380**	**100.089**	**133.122**	**5.631**	**0.6721**	**0.32075**

N	x	y	u	x^2	y^2	x*y	x*u	y*u	up	$(u-u^1)^2$	$(u-up)^2$
1	21	0.0	0.2100	441	0.0000	0.0000	4.4100	0.0000	0.1406	0.0266	0.004816
2	35	10	0.5000	1225	100	350	17.5000	5.0000	0.321	0.0161	0.032041
3	25	10	0.2220	625	100	250	5.5500	2.2200	0.285	0.0228	0.003969
4	16	16	0.0590	256	256	256	0.944	0.9440	0.3306	0.0986	0.073767
5	0.0	21	0.5000	0.0000	441	0.0000	0.0000	10.5000	0.338	0.0161	0.026244
6	35	18	0.5000	1225	324	630	17.5000	9.0000	0.425	0.0161	0.005625
7	29	19	0.4550	841	361	551	13.195	8.6450	0.4164	0.0067	0.00149
8	22	23	0.5000	484	529	506	11.000	11.5000	0.4432	0.0161	0.003226
9	10	25	0.7250	100	625	250	7.2500	18.1250	0.426	0.1239	0.089401
10	0.0	37	0.5000	0.0000	1369	0.0000	0.0000	18.5000	0.546	0.0161	0.002116
11	10	37	0.424	100	1369	370	4.2400	15.6880	0.582	0.0026	0.024964
12	19	29	0.5000	361	841	551	9.5000	14.5000	0.5104	0.0161	0.000108
13	18	37	0.5000	324	1369	666	9.0000	18.5000	0.6108	0.0161	0.012277
14	0.0	0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.065	0.1391	0.004225
15	35	0.0	0.0000	1225	0.0000	0.0000	0.0000	0.0000	0.191	0.1391	0.036481
sum	**275**	**282**	**5.595**	**7207**	**7684**	**4380**	**100.089**	**133.122**	**5.631**	**0.6721**	**0.32075**

By taking partial derivatives of Lagrange expression

$$\mathrm{L}\left(\mathrm{x}, \mathrm{y}, \lambda_{1,} \lambda_2\right) = \mathrm{f}(\mathrm{x}, \mathrm{y}) + \lambda_1 \mathrm{g}_1(\mathrm{x}, \mathrm{y}) + \lambda_2 \mathrm{g}_2(\mathrm{x}, \mathrm{y}) \quad (111)$$

$$= 0.000042\mathrm{x} + 0.0017\mathrm{y} + 0.000059\mathrm{x}^2 + 0.00034\mathrm{y}^2 + 0.00015\mathrm{xy} + 0.00847 \\ + \lambda_1(0.0225\mathrm{x}\ \ ) + \lambda_2(-0.0201\mathrm{x} + 0.0238\mathrm{y})$$

to obtain the following relations

$$\frac{\partial \mathrm{L}}{\partial \mathrm{x}} = 0.000042 + 0.0001\mathrm{x} + 0.00015\mathrm{y} + 0.0225\lambda_1 - 0.0201\lambda_2 = 0 \quad (112)$$

$$\frac{\partial \mathrm{L}}{\partial \mathrm{y}} = 0.0017 + 0.0068\mathrm{y} + 0.00015\mathrm{x} + 0.0225\lambda_1 + 0.0238\lambda_2 = 0 \quad (113)$$

$$\frac{\partial \mathrm{L}}{\partial \lambda_1} = 0.0225\mathrm{x} \qquad = 0 \quad (114)$$

$$\frac{\partial \mathrm{L}}{\partial \lambda_2} = 0.0201\mathrm{x} + 0.0238\mathrm{y} \qquad = 0 \quad (115)$$

By solving (107)- (110) from(109)

$$x = y = 0, \lambda_1 = -0.0356, \lambda_2 = -0.0378$$

By substituting the variables in (108) the optimum value of the function is obtained as

u(x,y) = f(x,y) = 0.00847

$$\chi = \mathrm{f}(\mathrm{x}, \mathrm{y}) = 0.000042\mathrm{x} + 0.0017\mathrm{y} + 0.000059\mathrm{x}^2 + 0.00034\mathrm{y}^2 + 0.00015\mathrm{xy} \\ + 0.00847 \quad (108)$$

The prediction of functional, χ with (108) are presented in Table 4 using excel package to draw conclusion with the FEM and multiple linear regression results of Table 3.

Table 4: Prediction of Functional with Equation (108).

N	x	Y	χ
1	21	0	0.035371
2	35	10	0.185715
3	25	10	0.134895
4	16	16	0.176886
5	0	21	0.19411
6	35	18	0.317475
7	29	19	0.296997
8	22	23	0.33281
9	10	25	0.30729
10	0	37	0.53683
11	10	37	0.59865
12	19	29	0.448457
13	18	37	0.656602
14	0	0	0.00847
15	35	0	0.082215

Tables 3 and 4 are compared for u , up and their functional, χ are found approximate.

8.2.1 Extremization by Lagrange gradient search approach

The extremum conditions for continuous and differentiable functions are defined [1] as follows:

$$f_x = 0.000042+0.0001x+0.00015y = 0 \tag{116}$$

$$f_y = 0.0017+0.000068y+0.00015x = 0 \tag{117}$$

$$f_{xx} = 0.0001 \tag{118}$$

$$f_{yy} = 0.00068 \tag{119}$$

Since f_{xx} and $f_{yy} > 0 \Rightarrow$ minimum extremum or local extremum exists.
The extremum at the interior points (x_0, y_0) is evaluated by solving simultaneous equation formed by (100) and (101) to obtain x = 4.9767, y = -3.5978. By substituting this value in equation (108) the function is obtained as 0.006, representing the extrema (minimum) value of the function u(x, y) within the region.

8.2.2 Extremization by Lagrange multipliers approach

By expressing (108) in the form
$\mathrm{f(x, y) = 0.000042x + 0.0017y + 0.000059x^2 + 0.00034y^2 + 0.00015xy + 0.00847}$
Subject to the constraint relations

$$\mathrm{u(x, y) = 0.5 + 0.0225x \qquad = 0.5} \tag{120}$$
$$\mathrm{u(x, y) = \ -0.0201x + 0.0238y = 0} \tag{121}$$
derived for nodes 14 and 10 of elements 1 and 5 at the boundaries.

By taking partial derivatives of Lagrange expression

$$\begin{aligned}\mathrm{L}\big(\mathrm{x, y}, \lambda_{1,}\lambda_2\big) &= \mathrm{(x, y)} + \lambda_1 \mathrm{g}_1\mathrm{(x, y)} + \lambda_2 \mathrm{g}_2\mathrm{(x, y)} \\ &= \mathrm{0.000042x + 0.0017y + 0.000059x^2 + 0.00034y^2 + 0.00015xy + 0.00847} \\ &\qquad + \lambda_1\mathrm{(0.0225x \ )} + \lambda_2\mathrm{(-0.0201x + 0.0238y)}\end{aligned} \tag{122}$$

to obtain the following relations

$$\frac{\partial \mathrm{L}}{\partial \mathrm{x}} = 0.000042 + 0.0001\mathrm{x} + 0.00015\mathrm{y} + 0.0225\lambda_1 - 0.0201\lambda_2 = 0 \tag{123}$$

$$\frac{\partial \mathrm{L}}{\partial \mathrm{y}} = 0.0017 + 0.0068\mathrm{y} + 0.00015\mathrm{x} + 0.0225\lambda_1 + 0.0238\lambda_2 = 0 \tag{124}$$

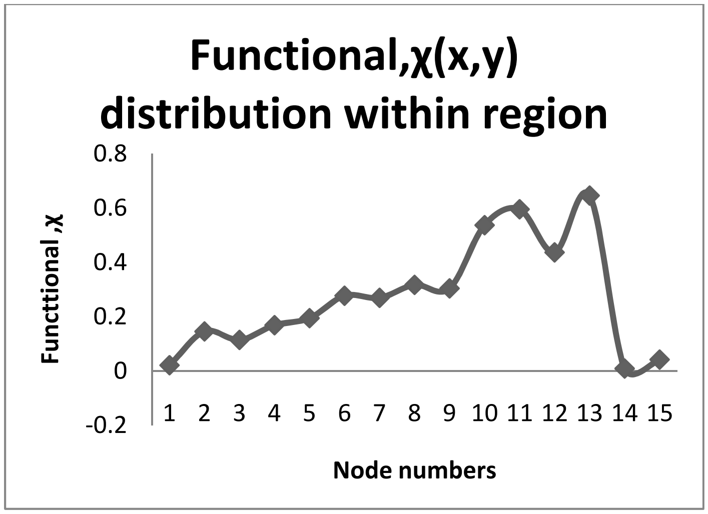
$$\frac{\partial \mathrm{L}}{\partial \lambda_1} = 0.0225\mathrm{x} \qquad\qquad = 0 \tag{125}$$

$$\frac{\partial \mathrm{L}}{\partial \lambda_2} = 0.0201\mathrm{x} + 0.0238\mathrm{y} \qquad = 0 \tag{126}$$

By solving (123)- (126) starting from(125)
$x = y = 0, \lambda_1 = -0.0356, \lambda_2 = \ -0.0378$

By substituting the variables in (108) the optimum value of the function is obtained as u(x,y) = f(x,y) = 0.00847.This value compares favourably with the prediction of 0.006 of gradient search method showing agreement with the graphics of Figure 2 and Figure 3.

a)



b)

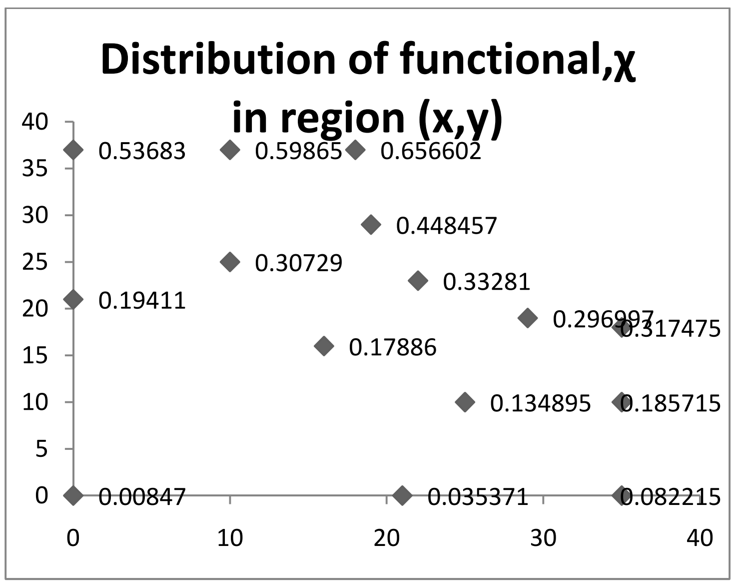


Figure 3a, b Distribution of function within the Region.

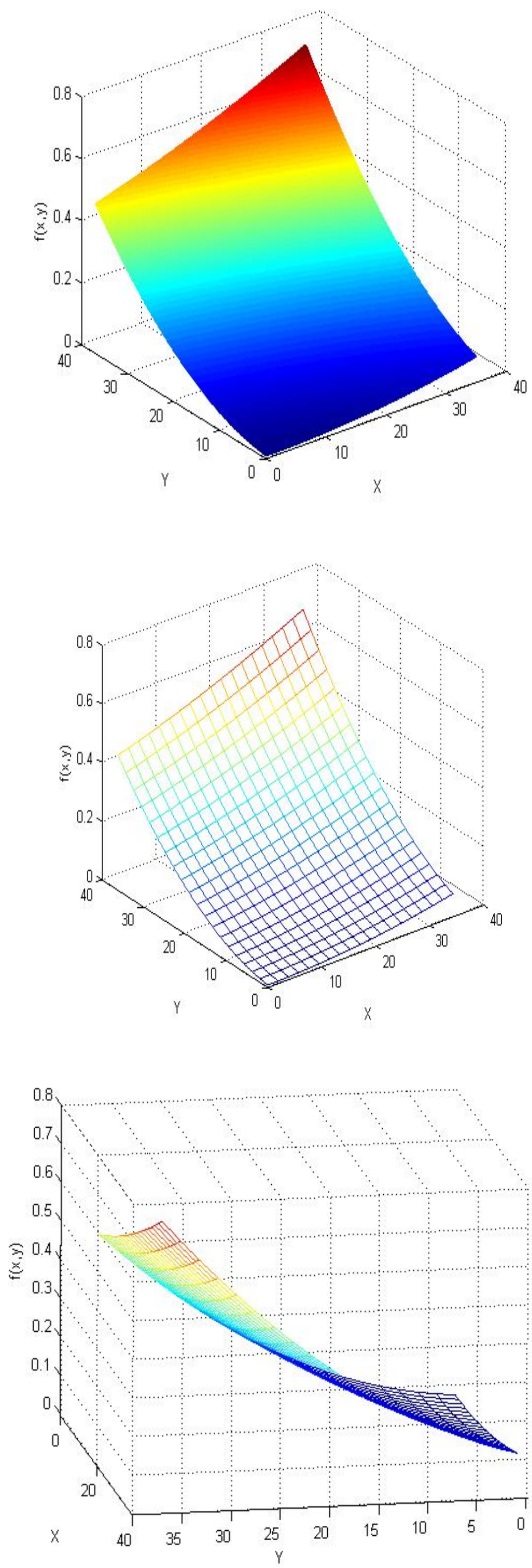


Figure 4a, b and c Versions of 3D Surface Plots of Function.

9. CONCLUSSIONS

The methods of this article apply to:

1. Solution of boundary value engineering phenomena whose function can be expressed as partial differential equation.
2. Solution of of displacement in structures and solid mechanics problems, temperature distribution in heat conduction problems, displacement potential in acoustic fluids , pressure in potential flows , velocity in general flows, electric potential in electrostatics magnetic potential in magnetostatics , torsion of non – homogenous shaft, flow through an anisotropic porous foundation, axi – symmetric heat flow, hydrodynamic pressures on moving surfaces
3. Solution of time dependent field problems such as creep, fracture and fatigue.
4. Equations (97) and (98) are recommended for the prediction of possible values of the displacement function of GRP composites region from where other properties of the region could be evaluated.
5. A unified computational model with standard error of 0.15 and correlation coefficient of 0.72 was developed to aid analysis and easy prediction of regional function with which the continuum function was successfully modeled and optimized through gradient search and Lagrange multipliers approach.
6. The MatLab 3-D graphics of Figure 4 show potential trend of function within the regionwith minimum and maximum at the boundaries.

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