Scientific Research Publishing

International Journal of Astronomy and Astrophysics, 2022, 12, 147-165
https://www.scirp.org/journal/ijaa
ISSN Online: 2161-4725
ISSN Print: 2161-4717

Gravitational Deflection of Particles of Light by the Earth and by the Sun: A Reconstruction of the Calculations Done by Soldner in 1801

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How to cite this paper: van Kampen, F.F. (2022) Gravitational Deflection of Particles of Light by the Earth and by the Sun: A Reconstruction of the Calculations Done by Soldner in 1801. *International Journal of Astronomy and Astrophysics*, **12**, 147-165.
https://doi.org/10.4236/ijaa.2022.121009

Received: January 11, 2022
Accepted: March 22, 2022
Published: March 25, 2022

Abstract

In 1801, the year of the discovery of Ceres, Johann Georg von Soldner calculated with classical means the gravitational deflection of a lightray grazing the surface of the Sun as 0.84”. According to General Relativity (GR) and using present-day data the deflection amounts to 1.75”. The formula for the deflection is derived with a classical method, with GR and as done by Soldner. The GR formula gives twice as large a deflection as the classical formula. It is shown that the formula of Soldner is equivalent to the classical one. Soldner’s numerical calculation of the classical deflection by the Earth comes out a factor 6.9 larger than using present-day data. This discrepancy is for a factor 6.25 due to a mistaken value for the velocity of the grazing lightray. This factor 6.25 can numerically be accounted for by assuming Soldner made a conceptual mistake related to the Axial Tilt of the Earth. The remaining discrepancy is due to the use of data less accurate than the present-day data. Soldner’s numerical calculation of the deflection by the Sun comes out correctly to the data of those days. In case of the Sun he did not give any further information regarding the data he used. A reconstruction reveals that for the surface gravity of the Sun he used a value close to the present-day value.

Keywords

Johann von Soldner, Gravitational Deflection of Light, History of Science

1. Introduction

In Soldner (1801) [1] Soldner derived with classical means a formula for the gravitational deflection of a lightray grazing the surface of a celestial body.

With that formula Soldner calculated the gravitational deflection by the Earth for a lightray from a star just rising from the Earth's horizon. By the gravitational deflection he meant the angular displacement on the sky as observed by an astronomer on Earth as a consequence of the gravitational field of the Earth. He concluded that for practical astronomy the gravitational deflection could be ignored.

Soldner also paid attention to the different observational situation in which a lightray on its way to the Earth is first deflected by the Moon or by the Sun. He numerically calculated the deflection by the Sun. He concluded the deflection could not be ignored for stars observed very close to the rim of the Sun. Because stars were in practice not observed that close to the Sun he found the deflection by the Sun was for practical astronomy not relevant.

We give short derivations of the classical formula, the GR formula and the formula of Soldner. In the derivations we take the Earth as an example of a celestial body, first of all because Soldner did so, but also because here we (used to) do our experiments and astronomical observations.

The formula Solder derived is shown to be equivalent to the classical formula. The formula only looks different because he used a different concept to describe the strength of the gravitational field than we do today.

The GR deflection comes out to be twice the classical deflection. In **Appendix 1** and **Appendix 2** this is further illustrated.

In the reconstruction of the numerical calculations of Soldner we took the data he mentioned as facts. For data Soldner did not mention but which were needed for the reconstruction we made plausible assumptions.

We compared his results with the results using present-day data. We found Soldner calculated for the deflection by the Earth a value that is a factor 6.9 larger than the present-day classical value. This large discrepancy is for a factor 6.25 due to an incorrect value he used for the velocity of the grazing lightray. The remaining part of the discrepancy is to be attributed to less accurate data.

Trying to explain the discrepancy we suspect Soldner made a conceptual mistake related to the Axial Tilt of the Earth, leading very precise to the factor 6.25.

In the calculation for the Sun Soldner used the correct value for the velocity of the grazing lightray. He calculated the deflection close to the present-day classical value. From a reconstruction of the calculation we found that Soldner used a value of 257 m/s^2 for the surface gravity of the Sun, close to the present-day value of 274 m/s^2.

Two centuries ago the term "particle of light" did not have the same connotation as the modern term "photon", just as for Democritus the term "atom" did not have the same connotation as the modern term. In the classical context we therefore will speak of a particle (of light) and in the GR context we will speak of a photon. Lightrays appear in both theoretical contexts.

2. Derivations of the Formulas

2.1. The Classical Derivation

We use the hyperbolic method as in Soares (2009) [2]. We have a particle coming in from a star at (near) infinity. At P (Perigee) that particle grazes the surface of the Earth.

In the classical context material particles can have any velocity v. We assume the particle at infinity has a velocity $c = 3.00 \times 10^8$ m/s. Coming in from infinity the particle will accelerate. The velocity of the particle at P we call c_p.

The hyperbolic path is described by $r(\phi) = \dfrac{R_\oplus (e+1)}{1 + e\cos(\phi)}$.

The angular coordinate ϕ goes clockwise from P.

The eccentricity e of the path is geometrically defined as $e = \dfrac{a + R_\oplus}{a}$.

The semi major axis a is defined as in **Figure 1**. The radius of the Earth is represented by $R_\oplus$ and is equal to the measured circumference of the equator of the Earth in meters divided by 2π.

We assign to the particle a mass m and assume it behaves as a normal classical particle.

For e we have the expression $e = \sqrt{1 + \dfrac{2EL^2}{m^3 G^2 M_\oplus^2}}$.

In this expression $M_\oplus$ represents the mass of the Earth, $E = \dfrac{1}{2} m v^2 - G \dfrac{m M_\oplus}{r}$ the constant sum of the particle's kinetic and potential energy and L the particle's constant angular momentum with respect to the center of the Earth. The gravitational constant $G = 6.7 \times 10^{-11}\ \mathrm{m}^3 \cdot \mathrm{kg}^{-1} \cdot \mathrm{s}^{-2}$.

At infinity the particle has only the kinetic energy $E = \dfrac{1}{2} m c^2$. At P it has the kinetic plus potential energy $E = \dfrac{1}{2} m c_p^2 - G \dfrac{m M_\oplus}{R_\oplus}$.

From the law of conservation of energy we have

$$E = \frac{1}{2} m c^2 = \frac{1}{2} m c_p^2 - G \frac{m M_\oplus}{R_\oplus} \rightarrow c_p = c \sqrt{1 + \frac{2GM_\oplus}{c^2 R_\oplus}}.$$

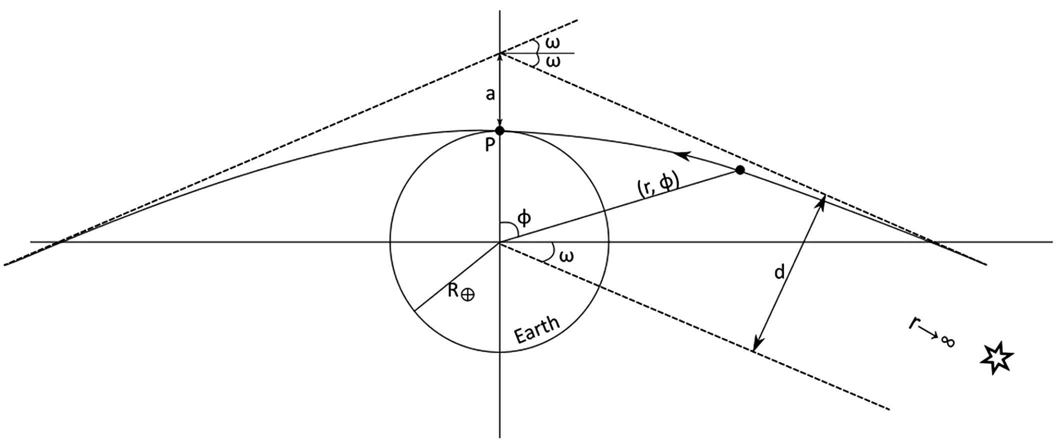

Figure 1. The geometrical situation for the classical derivation. For illustrational purposes the bending is hugely exaggerated.

The quantity $\frac{2GM_{\oplus}}{c^2 R_{\oplus}}$ is very small compared to 1, so in good approximation c_p equals *c*.

At infinity the particle has angular momentum $L = mcd$. For the meaning of d see **Figure 1**. At P the particle has angular momentum $L = mc_p R_{\oplus}$.

From the law of conservation of angular momentum we have

$$L = mcd = mc_p R_{\oplus} \to d = \frac{c_p}{c} R_{\oplus} = \sqrt{1 + \frac{2GM_{\oplus}}{c^2 R_{\oplus}}} R_{\oplus}.$$

The quantity $\frac{2GM_{\oplus}}{c^2 R_{\oplus}}$ is very small compared to 1, so in good approximation d equals $R_{\oplus}$. For the eccentricity of the hyperbolic path we then find

$$e = \sqrt{1 + \frac{2\left(\frac{1}{2}mc^2\right)(mcd)^2}{m^3 G^2 M_{\oplus}^2}} = \sqrt{1 + \frac{c^4 d^2}{G^2 M_{\oplus}^2}}$$

$$\approx \frac{1}{GM_{\oplus}} \sqrt{G^2 M_{\oplus}^2 + c^4 R_{\oplus}^2} \approx \frac{c^2 R_{\oplus}}{GM_{\oplus}} \sqrt{1 + \frac{G^2 M_{\oplus}^2}{c^4 R_{\oplus}^2}}$$

So in good approximation we find $e = \frac{c^2 R_{\oplus}}{GM_{\oplus}}$.

For $r \to \infty$ we find from $r(\phi)$ that $1 + e\cos(\phi) \to 0$.

From **Figure 1** we also see that for $r \to \infty$ the particle's angular coordinate $\phi \to \frac{\pi}{2} + \omega$.

We then have $e\cos\left(\frac{\pi}{2} + \omega\right) = -1$ and thus $\sin(\omega) = \frac{1}{e} = \frac{GM_{\oplus}}{c^2 R_{\oplus}}$.

For small ω we can set $\sin(\omega) \approx \omega$ and then we find that in good approximation $\omega = \frac{GM_{\oplus}}{c^2 R_{\oplus}}$.

So for the deflection for one arm of the path we have in good approximation $\omega = \frac{GM_{\oplus}}{c^2 R_{\oplus}}$.

This is the deflection as observed by an astronomer on Earth. For the deflection for both arms of the path we then have in good approximation $\omega = \frac{2GM_{\oplus}}{c^2 R_{\oplus}}$. This is the deflection as would be observed by an astronomer (infinitely) far from Earth.

Soldner was only interested in deflections observed on Earth, as we can infer from his text and also from the drawing in his paper which represents only one arm of the path.

With the Sun as the deflecting body we must replace the mass of the Earth by the mass of the Sun.

The gravitational deflection by the Earth is observed together with the much larger refraction by the Earth's atmosphere. In case of the Sun there could be an additional refraction by the Sun's Corona.

2.2. The GR Derivation

We follow the method used by McVittie (1965) [3]. Accordingly we have for the path of a lightray grazing the surface of the Earth the formula

$$R(\phi) = R_\oplus \left(\frac{R_s}{R_\oplus} + \left(1 - \frac{1}{2}\frac{R_s}{R_\oplus}\right)\cos(\phi) - \frac{1}{2}\frac{R_s}{R_\oplus}\cos^2(\phi) \right)^{-1}.$$

In this formula the constant R_s is an integration constant following from solving the equations of GR. It is called the Schwarzschild Radius of the Earth and is in GR determined to be $R_s = \frac{2GM_\oplus}{c^2}$. The value of R_s for the Earth equals 0.887 cm and for the Sun 2.956 km.

$R_\oplus$ again is the measured circumference of the equator of the Earth in meters divided by 2π. In GR it is the value of the radial coordinate of the surface of the Earth and corresponds to the height $h = r - R_\oplus = 0$ in the classical context. The radial coordinate R at height h is in GR equal to the measured circumference in meters of a circle around the Earth at that height divided by 2π.

When the particle is at infinity, thus when $R \to \infty$ we see from the formula for $R(\phi)$ that

$$\frac{1}{2}\frac{R_s}{R_\oplus}\cos^2(\phi) - \left(1 - \frac{1}{2}\frac{R_s}{R_\oplus}\right)\cos(\phi) - \frac{R_s}{R_\oplus} \to 0.$$

Solving this quadratic equation in $\cos(\phi)$ in the usual way we find

$$\cos(\phi) = -\frac{R_s}{R_\oplus}\left(1 + \frac{1}{2}\frac{R_s}{R_\oplus}\right).$$

From **Figure 2** we also see that for $R \to \infty$ the particle's angular coordinate $\phi \to \frac{\pi}{2} + \omega$.

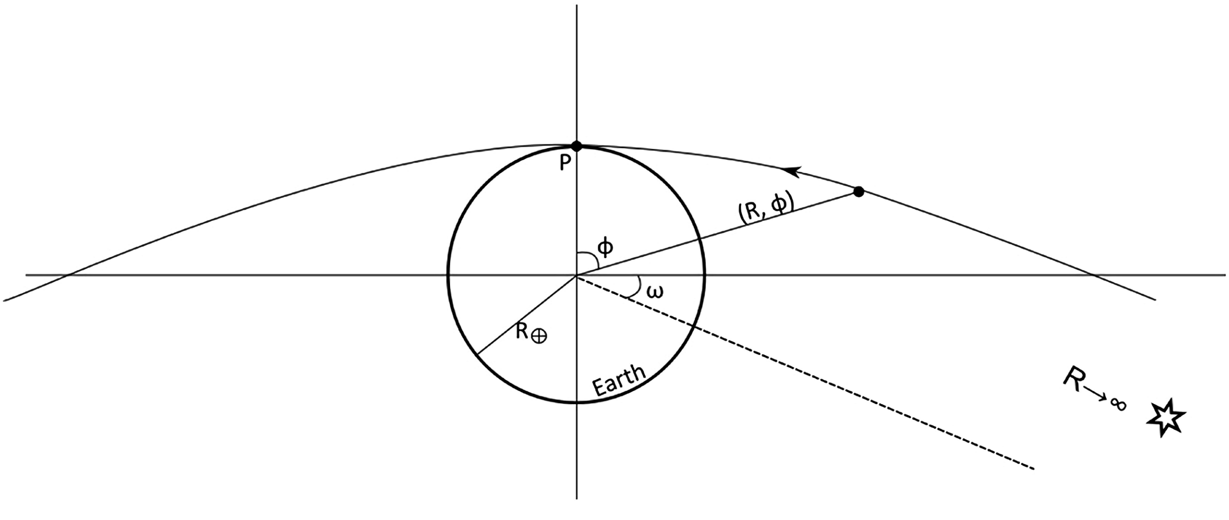

Figure 2. The geometrical situation for the GR derivation. For illustrational purposes the bending is hugely exaggerated.

We then have $\cos\left(\frac{\pi}{2}+\omega\right)=-\sin(\omega)=-\frac{R_s}{R_\oplus}\left(1+\frac{1}{2}\frac{R_s}{R_\oplus}\right)$. For small ω this reduces in good approximation to $\omega=\frac{R_s}{R_\oplus}$. When we insert $R_s=\frac{2GM_\oplus}{c^2}$ we find $\omega=\frac{2GM_\oplus}{c^2R_\oplus}$.

So for the deflection for one arm of the path we have in good approximation $\omega=\frac{2GM_\oplus}{c^2R_\oplus}$. This is the deflection as observed by an astronomer on Earth. For the deflection for both arms of the path we then have in good approximation $\omega=\frac{4GM_\oplus}{c^2R_\oplus}$. This is the deflection as would be observed by an astronomer (infinitely) far from Earth.

We see the GR deflection is twice the classical deflection as derived in (2.1).

Remark

We remark that in [4] [5] and [6] according to $E = hf = mc^2$ to the particle is assigned a mass $m = hf/c^2$ and a linear momentum $p = hf/c$, thus turning a photon into a particle. In the classical context a particle with mass m should at infinity have a kinetic energy $\frac{1}{2}mc^2=\frac{1}{2}hf$ and a linear momentum $p = mc = hf/c$.

Because in the derivation given in (2.1) the mass drops out of the equation, putting the kinetic energy $E=\frac{1}{2}mc^2$ and $p = mc$ with $m = hf/c^2$ would still lead to the classical result.

When in the classical context we put the kinetic energy $E = mc^2$ we should, it seems, from the classical relation between the kinetic energy and the linear momentum $p = 2E/c$, have to put $p = 2mc$. This would lead to twice the classical result.

Interesting that may be, in this way for the particle an ad hoc hypothesis is introduced that for Soldner and his contemporaries would have turned the particle into a rather "strange" particle.

2.3. The derivation by Soldner

Soldner used the concept of "gravitational acceleration" at the surface of the deflecting body. This concept is different from the present-day concept of "surface gravity". That can be confusing and complicates the analysis. We will give a plausible interpretation to what Soldner meant.

- ***The entity* $2g$**

When Soldner speaks of the gravitational acceleration g he means simply, following a convention of those days, the distance traversed in unit time by a particle falling freely from rest near the surface [7]. That convention may be related to the use of the pendulum as measuring device.

For the Earth that distance equals 4.9 m for one second and is thus equal to half the numerical value of the surface gravity of 9.8 m/s^2. We therefore identify the numerical value of $2g$ as the numerical value of the surface gravity of the Earth $\frac{GM_\oplus}{R_\oplus^2}$ according to Newton's Law of Gravity.

- ***The entity r***

Soldner wrote: "The force, by which the light…will be attracted…will be $2gr^{-2}$." At first sight we see r as the real distance in meters to the center of the Earth, as our radial coordinate in **Figure 1**. However, with the numerical identification of $2g$ as 9.8 we have for the surface of the Earth the number $r = 1$. For that reason we prefer to identify r as the dimensionless ratio of the distance to the radius of the Earth. For example, for $r = 4$ the gravitational force is 16 times smaller than at the surface.

- ***The dimensions of 2g and r***

With this numerical identification of $2g$ we would have the dimension length $\times$ time^{-2} for the entity $2g$ determining the surface gravity of the Earth and r would be a mere number determining the distance to the center of the Earth.

With a different interpretation of r as a real distance with the dimension length, the entity $2g$ would have the dimension length^3 $\times$ time^{-2} following Tilman Sauer (2021) [7]. That looks rather complex compared to the simple concept of gravitational acceleration we had initially. This point needs closer attention.

In doing the actual calculations we focus primarily on numerical values.

- ***Changing the scale of the units of length and time***

After having introduced the entities g and $2g$ Soldner declares he takes the radius of the Earth from then on as the unit of length, changing the scale of his drawing. We will see in (3.1) that Soldner in the numerical calculation for the Earth indeed divides the experimental value of g by the numerical value of $R_\oplus$. He however still designated the entity $g/R_\oplus$ by the symbol g. That can easy lead to confusion in applying his formula. For that reason we must in the actual numerical calculations keep this in mind.

In (3.1) we will see that Soldner used a unit of time other than the second, the so called "decimal second".

- ***The velocity of light***

For the determination of the velocity of light Soldner used the time of travel of light from the Sun to the Earth. He did not give a source for the numerical value he used.

To determine the velocity of light Soldner had to use some reliable value for the distance from the Sun to the Earth. Soldner did however not mention that value. We come back on this point in (3.1).

We remark that in this way Soldner determined the mean velocity of light. He did not specify the velocity of light as a function of the distance to the Earth. According to (2.1) that does not significantly influence the result, because the velocity of the particle hardly changes on its path.

By taking the radius of the Earth as the unit of length the numerical value for the velocity of light v changes into $v/R_\oplus$. As in his treatment of $2g$ he still designated the entity $v/R_{\oplus}$ by the symbol v. Again, to avoid confusion we must in the actual numerical calculations keep this in mind.

- ***Soldner's derivation and formula***

After applying Newton's Law of Gravity Soldner concluded that the particle follows a hyperbolic path due to its large velocity. Then he used that form of the path and in the end obtains for one arm of the path the relation $\tan(\omega) = \frac{2g}{v\sqrt{v^2 - 4g}}$.

For small ω we have $\tan(\omega) \approx \omega$ and for large v we can state $\frac{2g}{v\sqrt{v^2 - 4g}} \approx \frac{2g}{v^2}$.

From this we see that for one arm of the path Soldner derived in good approximation the deflection $\omega = \frac{2g}{v^2}$.

For two arms of the path he thus derived in good approximation the deflection $\omega = \frac{4g}{v^2}$.

- ***Transformation of the formula to the classical formula***

When we replace in the formula of Soldner $2g$ by $2g/R_{\oplus}$ and v by $c/R_{\oplus}$ we regain the classical expressions derived in (2.1) and find that the formula of Soldner is equivalent to the classical formula.

We note that in Lotze and Simionato (2021) [8] this equivalence is also shown, be it with different replacement rules. These different replacement rules are presumably related to a different interpretation of the entities $2g$ and r. This point needs closer attention.

3. The Numerical Calculations by Soldner

In calculating the numerical value of the deflection by the Earth and by the Sun Soldner had, of course, to use the data known to him in those days. Because he did not mention all the values needed for the calculation, we made some plausible assumptions.

In his paper Soldner presented some very precise numerical values. For that reason we also used those values in the calculations and not the more rounded values, unless where appropriate.

3.1. Calculation of the Deflection by the Earth

Soldner did not mention the numerical value he used for the velocity of light nor the distance from the Sun to the Earth. He only mentioned the time of travel of light from the Sun to the Earth. He further used the so called "decimal second" as the unit of time.

- ***The decimal second***

The decimal second is a unit in use since 1792 for a few years during the French Revolution. One decimal second equals 0.864 seconds. This point can easy be missed and lead to confusion.

- ***The velocity of light***

Soldner assumed that light requires 564.8 decimal seconds to come from the Sun to the Earth. This corresponds to 488 seconds. It is not clear from which source this value came and how it was determined. Newton had estimated a value of around seven to eight minutes. Bradley gave in the year 1728 a value of 492 seconds. The present-day value equals 500 seconds.

In the year 1672 Cassini and Flamsteed determined from observations of Mars, then being in opposition, independent from one another the distance from the Sun to the Earth as 140×10^9 m. In the year 1769 Lalande determined from data of transits of Venus the distance as 153×10^9 m. The present-day value equals 150×10^9 m.

Soldner took from Laplace the radius of the Earth as $R_\oplus = 6.369514 \times 10^6$ m. The present-day value equals 6.371×10^6 m.

Expressed in Earth radii the distance from the Sun to the Earth equals 2.198×10^4 Earth radii in case Soldner used the value of Cassini and Flamsteed.

We then find for the velocity of light the value $v/R_\oplus = 38.916$ Earth radii per decimal second. This value corresponds to 2.48×10^8 meter per decimal second and to 2.87×10^8 m/s.

Expressed in Earth radii the distance from the Sun to the Earth equals 2.402×10^4 Earth radii in case Soldner used the value of Lalande.

We then find for the velocity of light the value $v/R_\oplus = 42.529$ Earth radii per decimal second. This value corresponds to 2.71×10^8 meter per decimal second and to 3.14×10^8 m/s.

- ***The mistaken value v for a lightray grazing the surface of the Earth***

Soldner then however declared that from his data follows $v/R_\oplus = 15.562085$ Earth radii per decimal second, corresponding to a velocity of 1.15×10^8 m/s.

In relation to the correct value $v/R_\oplus = 38.916$ Earth radii per decimal second (Cassini and Flamsteed) a factor 2.5007 (or the inverse 0.3999) is missing.

In relation to the correct value $v/R_\oplus = 42.529$ Earth radii per decimal second (Lalande) a factor 2.7329 (or the inverse 0.3659) is missing.

This missing factor is also noticed by Tilman Sauer (2021) [7], who estimates a missing factor of around 2.6.

Trying to explain this missing factor we consider that Soldner derived the velocity of light from the time of travel from the Sun to the Earth. In that context we suspect Soldner calculated the component of the velocity parallel to the surface of the Earth by taking the sinus of the Axial Tilt (Obliquity) of the Earth. Taking for that Axial Tilt the value of 23.5˚, we have sin(23.5˚) = 0.3999, very precise the missing factor in case Soldner had used the value of Cassini and Flamsteed. In case Soldner had used the value of Lalande the missing factor is 0.3659 and this explanation does not quite fit.

This remarkable precise correspondence can be a mere numerical coincidence, although that would be very coincidental in case of a mere technical calculational error.

For this reason we guess Soldner used the value of Cassini and Flamsteed. An other reason we can think of is the circumstance that Cassini and Flamsteed independent from each other determined that value. That must have given that value reliability. Further that value was surely to be found in books on the subject and must have been widely known. Or may be Soldner did not know the value determined by Lalande, or may be he did put more trust in the value of Cassini and Flamsteed.

We remark that in Ginoux (2021) [9] a connection is made with the latitude of 35˚16’ from Soldner’s paper. That connection seems to be made more in a descriptive sense than classifying it as a conceptual mistake. At that latitude however not the sinus but the square of the sinus is mentioned to be 1/3. That square is probably the factor determining the so called Gravity Anomaly of the Earth due to its rotation. The factor 1/3 = 0.3333 nor the sinus of the mentioned latitude do quite fit as an explanation for the factor 0.3999.

Although for these reasons the connection with the latitude of 35˚16’ does not seem to work, it originated the idea to try the Axial Tilt of the Earth as an explanation.

Be it as it is, Solder used for $v/R_\oplus$ the numerical value 15.562085.

- ***The value of $2g$ for the Earth***

Soldner mentioned from Laplace for g the value 3.66394 m using the decimal second. For $2g$ we then have the numerical value 7.32788. When we divide this numerical value by the numerical value for the radius of the Earth we find for $2g/R_\oplus$ the numerical value 1.150461 × 10^{-6}.

- ***The result of Soldner’s calculation for the Earth***

With these numerical values $v/R_\oplus$ = 15.562085 and $2g/R_\oplus$ = 1.150461 × 10^{-6} Solder calculated with his formula ω = 0.004750 rad. By multiplying with the number of angular seconds per radial (1 rad = 206265”) we find, as Soldner did, ω = 0.0009798”.

Had he used his correct numerical value $v/R_\oplus$ = 38.916 he would have found ω = 0.0001576”, which is a factor 6.25 smaller than he actually calculated.

When we use the present-day data we find ω = 0.0001435”, which is a factor 6.9 smaller than the value Soldner calculated.

The value ω = 0.00014” was already given in 1921 in a footnote by Robert Trumpler (1923) [10]. Trumpler calls this a mistake in the calculation due to the units used.

The discrepancy is however for the larger part (factor 6.25) rooted in the mistaken value Soldner used for the velocity of light and can, we suspect, be explained by a conceptual mistake. That conceptual mistake, if any, Soldner certainly had not made in case the velocity of light had been measured in the laboratory, as Fizeau did nearly half a century later.

3.2. Calculation of the Deflection by the Sun

After he calculated the deflection by the Earth Soldner paid attention “to what extent a lightray is deflected by the Moon when it passes the Moon and travels to the Earth”.

In this new observational context Soldner states that "double the value that was found by the formula must be taken because of the two arms of the hyperbola". So his intentions are clear.

Then he calculates the deflection by the Sun. So for the Sun he intended clearly to calculate the deflection due to both arms of the hyperbola. He certainly had not in mind being on the surface of the Sun observing a star rising from the Sun's horizon.

But the way Soldner wrote it down ("If we substitute into the formula for tan(ω) the acceleration of gravity on the surface of the sun we find $\omega = 0.84''$.") raises questions. The symbol ω he used and depicted in his drawing refers to only one arm of the path. This has led on the one hand to speculations about misprints as in [11] and on the other hand to the idea that Soldner derived by classical means the GR result as in [6].

As we will see Soldner must first have calculated ω according to his formula and after that took double the value that was found.

In the new observational context Soldner had for clarity better made a second drawing. He also could have given a more general definition of the term deflection as the angular displacement observed by an astronomer on Earth. That definition would be valid for all the observational contexts of those days. He could then have assigned to that angular displacement the symbol ω. But he did not do all of that. The angle ω in his drawing and in his formula now formally represents the angle of displacement for one arm of the path.

We guess Soldner just forgot to write down the number 2 on paper after he had finished his calculation; or for himself it was obvious because he had already explained how to proceed.

We further note that in the new observational context the lightray, already been deflected by the Sun, generally speaking does not arrive at the Earth at grazing incidence. In the one special case of perpendicular incidence there is no additional gravitational deflection by the Earth. In the other special case of grazing incidence the additional deflection by the Earth is ignorable as Soldner already had calculated. Soldner did not find it necessary to mention this (minor) point.

- ***The velocity v of a "particle of light" grazing the surface of the Sun***

Soldner did not give any further information regarding the data he used in case of the Sun. He only gave 0.84" as the result.

At first thought it seems obvious he should have used the same mistaken value of the velocity of light as he used for the Earth. At second thought, assuming he did indeed make a conceptual mistake with the Axial Tilt of the Earth, there is no reason for him to make that mistake in case of the Sun.

Therefore we assume in case of the Sun he used his correct value $v = 2.48 \times 10^8$ meter per decimal second, using the value of Cassini and Flamsteed for the distance from the Sun to the Earth.

In the units used he next divided v by the value of the radius $R_\odot$ of the Sun.

Because he did not mention the value of $R_\odot$, we must first make an assumption.

We assume Soldner used the accurate angular measurement of 964.8” regarding the radius of the Sun by Picard in the year 1670. Using 1” = 4.8481×10^{-6} rad this angle corresponds to 0.004677 rad.

Combined with the distance from the Sun to the Earth of 140×10^9 m according to Cassini and Flamsteed, we thus find $R_\odot = 6.55 \times 10^8$ m by taking $(140 \times 10^9) \times \tan(0.004677)$.

With these values for $v = 2.48 \times 10^8$ meter per decimal second and $R_\odot = 6.55 \times 10^8$ m we find $v/R_\odot = 0.379$ Sun radii per decimal second.

- ***The acceleration of gravity g for the Sun***

With the present-day (rounded) data for the Earth and the Sun

Mass of the Earth $M_\oplus = 6.0 \times 10^{24}$ kg

Radius of the Earth $R_\oplus = 6.4 \times 10^6$ m

Mass of the Sun $M_\odot = 2.0 \times 10^{30}$ kg

Radius of the Sun $R_\odot = 7.0 \times 10^8$ m

we find that the mass-radius ratio of the Sun is 3048 times larger the mass-radius ratio of the Earth. From the classical formula we therefore conclude that the deflection by the Sun must be around 3048 times larger than the deflection by the Earth.

Soldner could not look up those values as simple as we can. Cavendish only recently determined the mass of the Earth in his laboratory. Soldner could, of course, not experimentally determine the surface acceleration of the Sun with a pendulum as in case of the Earth. So the question is what value Soldner used for the acceleration of gravity g of the Sun.

From his result of 0.84” we can work our way back to the value he must have used.

From $2\omega = 0.84”$ we have $2\omega = 4.072 \times 10^{-6}$ rad (using 1” = 4.8481×10^{-6} rad).

We thus have $\omega = 2.036 \times 10^{-6}$ rad.

For $v/R_\odot$ we already found the numerical value 0.379.

From the formula $2\omega = \dfrac{4g}{v^2}$ we then have $2g = \omega \times v^2$. As mentioned in (3.1) we must use in this formula for g and v the numerical values for $g/R_\odot$ and $v/R_\odot$.

We thus find $2g/R_\odot = (2.036 \times 10^{-6}) \times (0.379)^2 = 0.2925 \times 10^{-6}$.

From this we find $2g = (0.2925 \times 10^{-6}) \times (6.55 \times 10^8) = 192$. Transformed from the decimal second to the second we find for $2g$ the numerical value 257.

From our interpretation of $2g$ as the numerical value of the surface gravity we conclude, given the assumptions made, that Soldner used in his calculation the value 257 m/s^2 for the surface gravity of the Sun, close to the present-day value 274 m/s^2.

Had Soldner calculated the value 0.84” for only one arm of the path, he should have used a value of 514 m/s^2. That would not have been a realistic value. In Lotze and Simionato (2021) [8] is illustrated how Soldner in his days could relatively easy have determined the surface gravity of the Sun. Newton already gave a value for the ratio of the "weights towards the Sun and the Earth" as 10,000 to 435 in his Principia (Book 3, Proposition 8, Theorem 8) [12]. Assuming Newton meant by this the ratio of the surface gravities, his value for the surface gravity of the Sun was 225 m/s^2.

We finally remark that to the gravitational deflection of the Sun must be added the refractional effect, if any, of the Corona of the Sun. At the time of the 1919 Eclipse (see **Appendix 3**) that refractional effect was not quantitatively known. From this the conclusiveness of the measurements of the 1919 Eclipse could on logical grounds be questioned [13].

4. Conclusions

We conclude Soldner derived the correct classical formula for the gravitational deflection. Because he used a different concept for the strength of the gravitational field than we do today, his formula looks different, but is shown to be equivalent.

Soldner mentioned Laplace as the source of his data for the radius of the Earth and for what he called the "acceleration of gravity" of the Earth. He did not mention the source of his value for the time of travel of light from the Sun to the Earth.

Soldner did not mention the value he used for the distance from the Sun to the Earth. Because that distance is needed to calculate the velocity of light we made a plausible assumption for that distance.

We found Soldner calculated the velocity of light grazing the surface of the Earth a factor 2.5 smaller then he should according to his data. A conceptual mistake in relation to the Axial Tilt of the Earth can account very precise for that factor.

As a consequence he calculated the deflection by the Earth a factor 6.25 too large given the data used. Compared to a calculation with present-day data the deflection he calculated is a factor 6.9 too large.

Soldner did not give any further information regarding his calculation for the Sun. He only mentioned the result as 0.84", close to the present-day classical value of 0.87".

Working our way back from his result we found Soldner used for the surface gravity of the Sun the value 257 m/s^2, close to the present-day value of 274 m/s^2. Based on the plausibility of this value, Soldner's motive for doing the calculations and his description of the observational context, we conclude Soldners value of 0.84" represents the deflection for both arms of the path.

In hindsight he had done well to add a separate drawing for the observational context he had in mind in relation to the calculation for the Sun. Such a drawing would also indicate that the value observed by an astronomer on Earth would in theory be slightly less than 2ω due to the finite distance of the Earth to the Sun, as Soldner already remarked.

In those days the spread of knowledge and information did not go by the velocity of light as today and not always in a straight line [14]. A closer look at the historical context (as in larger strokes already is given in [15]), such as the precise astronomical data known at the time in which and at the place where he wrote his paper, is asked for to put further to the test our suspicion of the conceptual mistake in the calculation for the Earth.

As with writing a historical novel, some things we know, some things we speculate and fantasize about, but some things we will never know because only Soldner could tell us.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix 1. Calculation of the Height Close to Perigee

- **the Euclidean case**

In case of no gravitational field the particle of light and the photon have according to Euclidean Geometry for small values of ϕ the height $KN = \frac{1}{2} R_{\oplus} \phi^2$ (Figure A1)

- **the classical calculation**

For the classical value of the eccentricity we found in (2.1) that $e = \frac{c^2 R_{\oplus}}{GM_{\oplus}}$.

For small values of ϕ we can set $\cos(\phi) \approx 1 - \frac{\phi^2}{2}$.

When we insert these values for e and $\cos(\phi)$ in the classical formula of the path $h(\phi) = r(\phi) - R_{\oplus} = \frac{R_{\oplus}(e+1)}{1 + e\cos(\phi)} - R_{\oplus}$, we find for the height KM of the particle of light $KM = h(\phi) \approx \frac{1}{2} R_{\oplus} \phi^2 - \frac{GM_{\oplus}}{2c^2} \phi^2$. We note that $\frac{1}{2} R_{\oplus} \phi^2$ is equal to the height KN in the Euclidean case.

- **the GR calculation**

When we substitute $\cos(\phi) \approx 1 - \frac{\phi^2}{2}$ for small values of ϕ in the GR formula of the path $R(\phi) = R_{\oplus} \left(\frac{R_s}{R_{\oplus}} + \left(1 - \frac{1}{2} \frac{R_s}{R_{\oplus}}\right) \cos(\phi) - \frac{1}{2} \frac{R_s}{R_{\oplus}} \cos^2(\phi) \right)^{-1}$, we find for the difference ΔR of the radial coordinates of the points L and K that $\Delta R \approx \frac{1}{2} \left(R_{\oplus} - \frac{3}{2} R_s \right) \phi^2$.

We must now transform ΔR to the height h of the point L.

Between the coordinates Δh and ΔR exists close to the surface according to GR the relation $\Delta h \approx \frac{\Delta R}{\sqrt{1 - \frac{R_s}{R_{\oplus}}}}$. When we use a Taylor expansion for $\frac{1}{\sqrt{1 - \frac{R_s}{R_{\oplus}}}}$

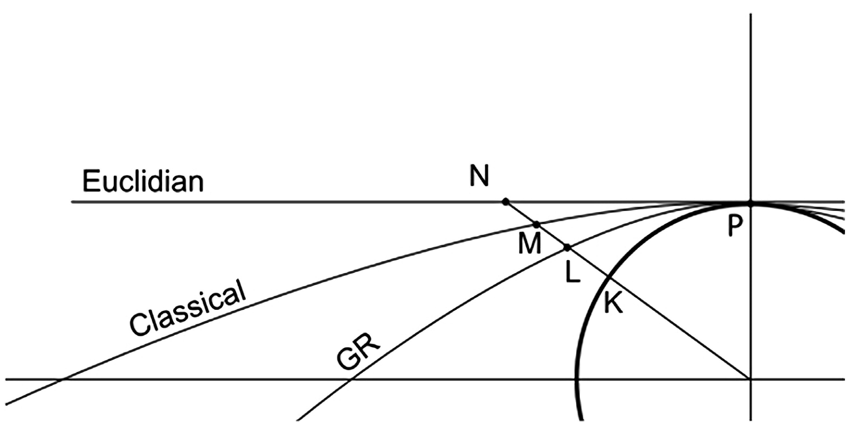

Figure A1. The different heights close to Perigee. For illustrational purposes the bending is hugely exaggerated.

according to $\frac{1}{\sqrt{1-x}} \approx 1+\frac{1}{2}x+\frac{3}{8}x^2$, we find for the height KL of the photon

$$KL = h(\phi) \approx \frac{1}{2}R_{\oplus}\phi^2 - \frac{1}{2}R_s\phi^2 .$$

After inserting $R_s = \frac{2GM_{\oplus}}{c^2}$ we find for the height KL of the photon

$$KL = h(\phi) \approx \frac{1}{2}R_{\oplus}\phi^2 - \frac{GM_{\oplus}}{c^2}\phi^2 .$$

We note again that $\frac{1}{2}R_{\oplus}\phi^2$ is equal to the height KN in the Euclidean case.

When we compare for a small value of ϕ the classical height with the GR height we see that the photon is attracted by gravity "twice as much" as the particle of light.

We can see this by comparing with the height KN, thus with the Euclidean straight line representing the path in case there were no gravitational field. This is rather loosely formulated, because it is not intended to say that the acceleration is twice as much. In GR the concepts of velocity and acceleration are complicated because the involvement of the time-coordinate.

It must also not be concluded that the particle of light and the photon reach the successive values of ϕ simultaneously. To compare their real behavior we should let the particle of light and the photon run against each other in a simulation, after first solving their trajectories as a function of time.

Appendix 2. Calculation of the Distance at Semi Latus Rectum

- **the classical calculation**

When we set $\phi = \frac{\pi}{2}$ in the classical formula of the path we find for the height or distance to the surface of the Earth of the point W (**Figure A2**)

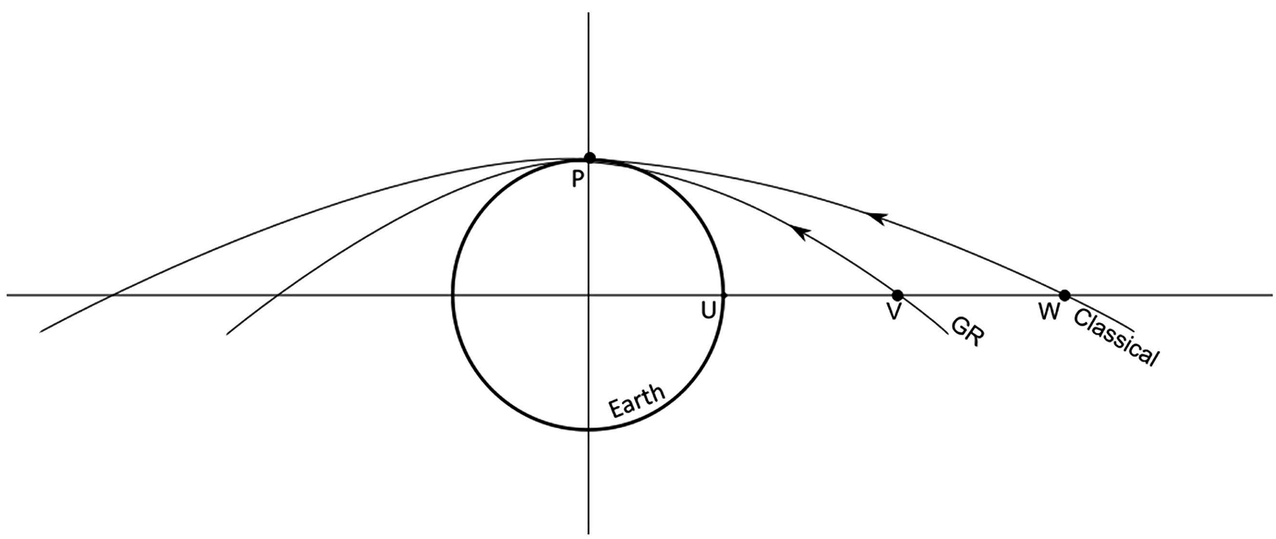

Figure A2. The different distances to the surface of the Earth at Semi Latus Rectum. For illustrational purposes the bending is hugely exaggerated.

$$UW = h\left(\frac{\pi}{2}\right) = r\left(\frac{\pi}{2}\right) - R_\oplus = \frac{R_\oplus(e+1)}{1+e\cos\left(\frac{\pi}{2}\right)} - R_\oplus = eR_\oplus + R_\oplus - R_\oplus = eR_\oplus \approx \frac{c^2R_\oplus^2}{GM_\oplus}.$$

- **the GR calculation**

When we set $\phi = \frac{\pi}{2}$ in the GR formula of the path, we find for the point V the radial GR coordinate $R\left(\frac{\pi}{2}\right) = \frac{R_\oplus^2}{R_s}$.

After inserting $R_s = \frac{2GM_\oplus}{c^2}$ we find $R\left(\frac{\pi}{2}\right) = \frac{c^2R_\oplus^2}{2GM_\oplus}$.

We must now transform the coordinate R to the distance h. Between the coordinates h and R exists in GR the relation:

$$UV = h\left(\frac{\pi}{2}\right) = \int_{R_\oplus}^{R\left(\frac{\pi}{2}\right)} \frac{1}{\sqrt{1-\frac{R_s}{R}}}\mathrm{d}R \approx \int_{R_\oplus}^{R\left(\frac{\pi}{2}\right)} \left(1+\frac{1}{2}\frac{R_s}{R}\right)\mathrm{d}R$$

$$\approx R\left(\frac{\pi}{2}\right) - R_\oplus + \frac{1}{2}R_s \ln\left(\frac{R\left(\frac{\pi}{2}\right)}{R_\oplus}\right)$$

$$\approx \frac{c^2R_\oplus^2}{2GM_\oplus} - R_\oplus + \frac{1}{2}R_s \ln\left(\frac{c^2R_\oplus}{2GM_\oplus}\right) \approx \frac{c^2R_\oplus^2}{2GM_\oplus}$$

Comparing the results we see that for $\phi = \frac{\pi}{2}$ the GR distance is approximately half the classical distance.

For the Earth the GR distance at Semi Latus Rectum is approximately 0.46 lightyear and for the Sun 0.017 lightyear. The distance to the nearest star Proxima Centauri is approximately 4.2 lightyear.

Appendix 3. The Measurement at the 1919 Eclipse

Paying closer attention to the angles being measured at the 1919 eclipse we note that the formulas for the classical deflection and the GR deflection are based on lightrays emanating from stars supposed to be at infinity. In the measurement at the 1919 eclipse the stars were, as always, far away, but not at infinity. Also the position of the Earth is such that a large part of the path is left out. The geometry of the situation in which the measurements were made, is rather complicated, as illustrated in **Figure A3**.

It must further be noted that the lightrays from the stars being photographed at the 1919 eclipse were close to the rim of the Sun, but not exactly grazing the surface of the Sun.

A more detailed treatment of these points can be found in [3].

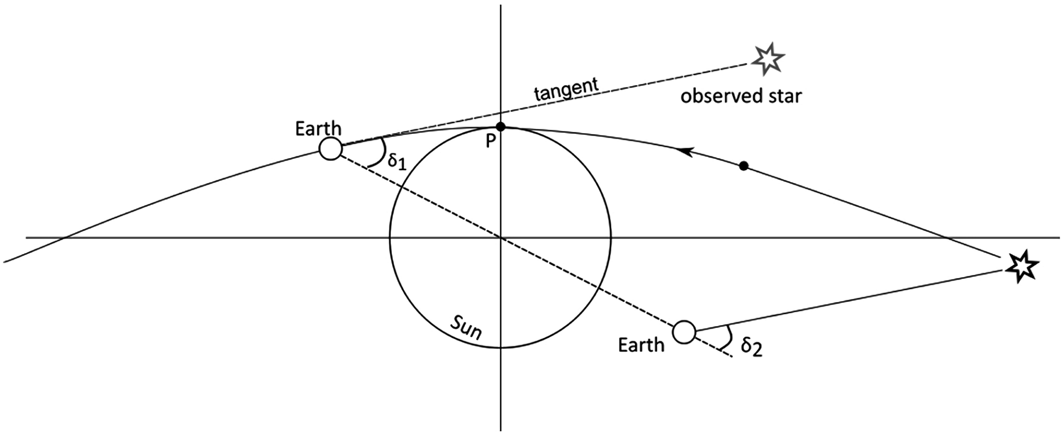

Figure A3. The star is observed at an angle δ_1 at a different position on the sky than its "true" position δ_2, measured half a year later. For illustrational purposes the bending is hugely exaggerated. A more detailed and on-scale figure can be found in [3].