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Generalized Kumaraswamy Generalized Power Gompertz Distribution: Statistical Properties, Application, and Validation Using a Modified Chi-Squared Goodness of Fit Test

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Abstract

A new six-parameter continuous distribution called the Generalized Kumaraswamy Generalized Power Gompertz (GKGPG) distribution is proposed in this study, a graphical illustration of the probability density function and cumulative distribution function is presented. The statistical features of the Generalized Kumaraswamy Generalized Power Gompertz distribution are systematically derived and adequately studied. The estimation of the model parameters in the absence of censoring and under-right censoring is performed using the method of maximum likelihood. The test statistic for right-censored data, criteria test for GKGPG distribution, estimated matrix $\hat{W}$, $\hat{C}$, and $\hat{G}$, criteria test Y_n^2, alongside the quadratic form of the test statistic is derived. Mean simulated values of maximum likelihood estimates $\hat{\gamma}$ and their corresponding square mean errors are presented and confirmed to agree closely with the true parameter values. Simulated levels of significance for $Y_n^2(\gamma)$ test for the GKGPG model against their theoretical values were recorded. We conclude that the null hypothesis for which simulated samples are fitted by GKGPG distribution is widely validated for the different levels of significance considered. From the summary of the results of the strength of a specific type of braided cord dataset on the GKGPG model, it is observed that the proposed GKGPG model fits the data set for a significance level $\varepsilon = 0.05$.

Keywords

Power Gompertz, Generalized Kumaraswamy-G, Modified Chi-Squared, the Goodness of Fit, Censoring

1. Introduction

The Gompertz distribution is a continuous probability distribution often applied in lifetime data analysis to describe the distribution of the science such as biology [1], gerontology [2], adult lifespans by demographers [3], actuaries [4], marketing [5], network theory [6] and computer science [7]. The Gompertz distribution has a convex hazard function. It is a flexible distribution, skewed to the right and the left, and a generalization of the exponential distribution.

To produce a more flexible distribution for a highly skewed dataset, new families of distributions are proposed daily. Some of these families of distributions include the Generalized Kumaraswamy generalized family by Nofal *et al.* [8], the Marshall-Olkin generalized family by Yousof *et al.* [9], the odd Dagum generalized family by Afify and Alizadeh [10], a new generalized Weibull-G family by Cordeiro *et al.* [11], a new Weibull-G family by Tahir *et al.* [12], the Gompertz generalized family by Alizadeh *et al.* [13], the Type II Power Topp-Leone generated family by Bantan *et al.* [14], the generated odd burr III family BY Hag *et al.* [15], Exponentiated-G (EG) by Cordeiro *et al.* [16], Weibull-X by Alzaatreh *et al.* [17], Weibull-G by Bourguignon *et al.* [18], Logistic-G by Torabi and Montazari [19], Gamma-X by Alzaatreh *et al.* [20], a Lomax-G family by Cordeiro *et al.* [21], Exponentiated T-X by Alzaghal *et al.* [22], a Beta Marshall-Olkin family of distributions by Alizadeh *et al.* [23], Logistic-X by Tahir *et al.* [24], the beta generalized family (Beta-G) by Eugene *et al.* [25], a Lindley G family by Cakmakyapan and Ozel [26], Odd Lindley-G family by Gomes-Silva *et al.* [27], Transmuted family of distributions by Shaw and Buckley [28], Gamma-G (type 1) by Zografos and Balakrishnan [29], the Kumaraswamy-G by Cordeiro and de Castro [30], McDonald-G by Alexander *et al.* [31], Gamma-G (type 2) by Ristic *et al.* [32], Gamma-G (type 3) by Torabi and Montazari [33], Log-gammaG by Amini *et al.* [34], and so on.

Statistics show that a powerful transformation is a series of functions used to create a monotonous data transformation using power functions. Applied to the random variable, the technique is useful in stabilizing variance, making the data more normal distribution-like, improving the validity of association measures like the Pearson correlation between variables, and providing a more flexible model by adding a new parameter named power parameter. The works of Ieren *et al.* [35], Ghitany *et al.* [36], and Rady *et al.* [37] prove this fact. Ieren *et al.* [35] proposed the power Gompertz distribution, and derived certain properties of the new distribution. Estimated parameters by Maximum Probability Estimate (MLE) were provided. The application of the proposed model with other existing distributions to a data set of remission times for a random sample of 128 patients with bladder cancer was done with the power Gompertz model providing better performance than the Gompertz model, Ghitany *et al.* [36] introduced the power Lindley distribution. This model provides more flexibility than Lindley distribution when applied to lifetime data, Rady *et al.* [37] proposed the Power Lomax distribution, when applied to bladder cancer data, the proposed Power Lomax distribution exhibited a much more flexible model than the Lomax distribution. To produce a more flexible distribution for a highly skewed dataset, our focus in this paper is to present an extension of the power Gompertz distribution using the generalized Kumaraswamy generalized family of distribution [8], the resulting distribution is a six-parameter continuous distribution called the generalized Kumaraswamy generalized power Gompertz distribution, various statistical properties will be looked at. The method of maximum likelihood is discussed for estimating the model parameter. We also construct and analyze the generalized Nikulin Rao-Robson goodness-of-fit statistic test Y_n^2 (Bagdonavicius and Nikulin [38], Bagdonavicius and Nikulin [39]) for the generalized Kumaraswamy generalized power Gompertz distribution based on censored data.

The remaining parts of this article are presented in sections as follows: formation of the new distribution is provided in Section 2. In Section 3, we analyzed the plots of the probability density and cumulative distribution function. Derivation of some properties of the new distribution such as asymptotic behavior, quantile function for median, Skewness and Kurtosis, and reliability analysis was discussed in Section 4. The distribution of order statistics in Section 5, estimation of parameters based on censored and uncensored random samples using Maximum Likelihood Estimation (MLE) is provided in Section 6. We evaluate the new goodness-of-fit statistic test Y_n^2, and investigate some criteria test for the generalized Kumaraswamy generalized power Gompertz distribution in Section 7, a simulation study was carried out in Section 8, and an application of the new model to the dataset is illustrated in Section 9.

2. Formation of the Generalized Kumaraswamy Generalized Power Gompertz Distribution (GKGPG)

The Power Gompertz (PG) distribution [35] with positive parameter α, β and θ has pdf and cdf given by:

$$g(x) = \alpha\theta x^{\theta-1} e^{\beta x^{\theta}} e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)} \quad (1)$$

and:

$$G(x) = 1 - e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)} \quad (2)$$

where $x > 0, \alpha > 0, \beta > 0, \theta > 0$

The cdf of the Generalized Kumaraswamy Generalized (GK-G) family is defined (for $x > 0$) by:

$$F(x)=\frac{1-\left[1-cG(x)^{a}\right]^{b}}{1-(1-c)^{b}} \tag{3}$$

The corresponding pdf of the GK-G family is given by:

$$f(x)=\frac{abcg(x)}{1-(1-c)^{b}}\left[G(x)\right]^{a-1}\left[1-cG(x)^{a}\right]^{b-1} \tag{4}$$

where $0<c\le 1$, $a>0$ and $b>0$ are shape parameters.

The hazard rate function (hrf) of the GK-G family is given by:

$$h(x)=\frac{abcg(x)\left[G(x)\right]^{a-1}\left[1-cG(x)^{a}\right]^{b-1}}{\left[1-cG(x)^{a}\right]^{b}-(1-c)^{b}} \tag{5}$$

Hence the pdf and cdf of the newly proposed Generalized Kumaraswamy Generalized Power Gompertz (GKGPG) distribution is given by:

$$f(x)=\frac{abc\alpha\theta x^{\theta-1}\mathrm{e}^{\beta x^{\theta}}\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta x^{\theta}}-1\right)}}{1-(1-c)^{b}}\left[1-\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta x^{\theta}}-1\right)}\right]^{a-1}\left[1-c\left(1-\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta x^{\theta}}-1\right)}\right)^{a}\right]^{b-1} \tag{6}$$

And:

$$F(x)=\frac{1-\left[1-c\left(1-\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta x^{\theta}}-1\right)}\right)^{a}\right]^{b}}{1-(1-c)^{b}} \tag{7}$$

where $x>0, 0<c\le 1, a>0, b>0, \alpha>0, \beta>0, \theta>0$.

3. Graphical Description of the Generalized Kumaraswamy Generalized Power Gompertz Distribution (GKGPG)

Here, we graphically illustrate the probability density function, and cumulative distribution function of the generalized kumaraswamy generalized power Gompertz distribution at different parameter values.

Remarks: Figure 1 represents the behavior of the density plot the effect of the different parameter values. The probability density function of the generalized kumaraswamy generalized power Gompertzdistribution is unimodal; it is also decreasing, and right skewed, depending on the indicated parameter values.

Remarks: Figure 2 represents the cdf plot, clearly, the cdf approaches one (1) as X tends to infinity and equals zero when X tends to zero.

4. Statistical Properties of the Generalized Kumaraswamy Generalized Power Gompertz Distribution (GKGPG)

4.1. *Asymptotic* Behavior

This section examines the limiting behavior of the GKGPG distribution as $X\to\infty$ and as $X\to 0$.

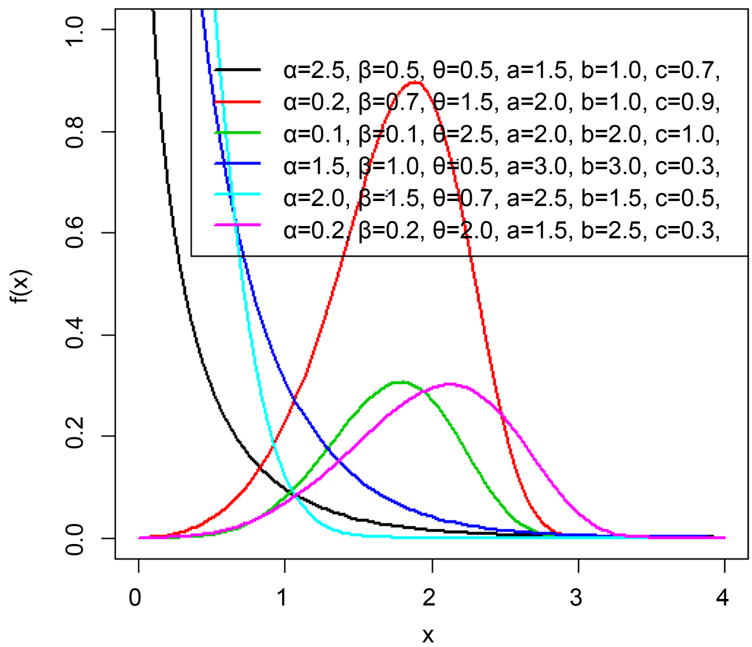

Figure 1. PDF plot of the Generalized Kumaraswamy Generalized Power Gompertz Distribution (GKGPG) at different parameter values.

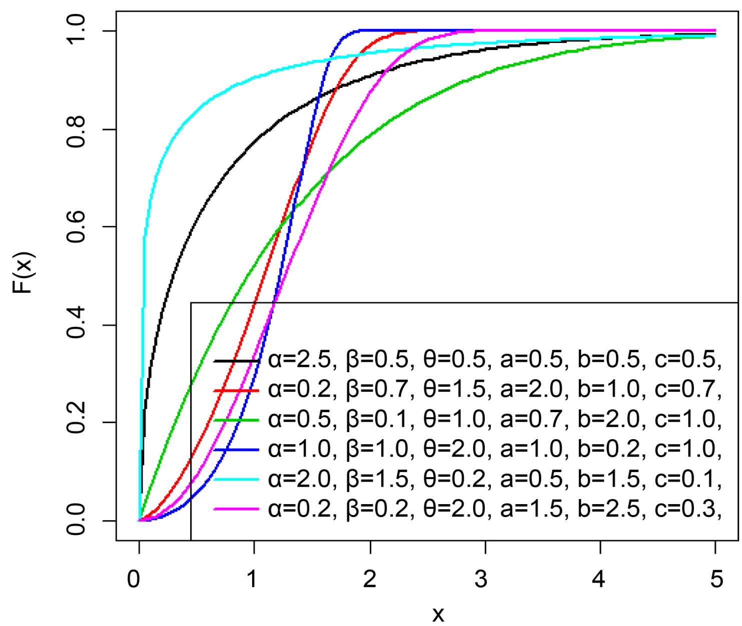

Figure 2. CDF plot of the Generalized Kumaraswamygeneralized Power Gompertz Distribution (GKGPG) at different parameter values.

For the pdf,

$$\lim_{x\to\infty} f(x)$$

$$=\lim_{x\to\infty}\left[\frac{abc\alpha\theta x^{\theta-1}\mathrm{e}^{\beta x^{\theta}}\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta x^{\theta}}-1\right)}}{1-(1-c)^{b}}\left[1-\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta x^{\theta}}-1\right)}\right]^{a-1}\left[1-c\left(1-\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta x^{\theta}}-1\right)}\right)^{a}\right]^{b-1}\right]$$

$$\lim_{x\to\infty} f(x) = \left[\frac{abc\alpha\theta\infty^{\theta-1}\mathrm{e}^{\beta\infty^{\theta}}\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta\infty^{\theta}}-1\right)}}{1-(1-c)^{b}}\left[1-\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta\infty^{\theta}}-1\right)}\right]^{a-1}\left[1-c\left(1-\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta\infty^{\theta}}-1\right)}\right)^{a}\right]^{b-1}\right]=0 \tag{8}$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[\frac{abc\alpha\theta x^{\theta-1}\mathrm{e}^{\beta x^{\theta}}\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta x^{\theta}}-1\right)}}{1-(1-c)^{b}}\left[1-\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta x^{\theta}}-1\right)}\right]^{a-1}\left[1-c\left(1-\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta x^{\theta}}-1\right)}\right)^{a}\right]^{b-1}\right]$$

$$\lim_{x\to 0} f(x) = \left[\frac{abc\alpha\theta 0^{\theta-1}\mathrm{e}^{\beta 0^{\theta}}\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta 0^{\theta}}-1\right)}}{1-(1-c)^{b}}\left[1-\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta 0^{\theta}}-1\right)}\right]^{a-1}\left[1-c\left(1-\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta 0^{\theta}}-1\right)}\right)^{a}\right]^{b-1}\right]=0 \tag{9}$$

For the cdf,

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty}\left[\frac{1-\left[1-c\left(1-\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta x^{\theta}}-1\right)}\right)^{a}\right]^{b}}{1-(1-c)^{b}}\right] = \frac{1-\left[1-c\left(1-\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta\infty^{\theta}}-1\right)}\right)^{a}\right]^{b}}{1-(1-c)^{b}}=1 \tag{10}$$

$$\lim_{x\to 0} F(x) = \lim_{x\to 0}\left[\frac{1-\left[1-c\left(1-\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta x^{\theta}}-1\right)}\right)^{a}\right]^{b}}{1-(1-c)^{b}}\right] = \frac{1-\left[1-c\left(1-\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta 0^{\theta}}-1\right)}\right)^{a}\right]^{b}}{1-(1-c)^{b}}=0 \tag{11}$$

4.2. Quantile Function

The quantile function (qf) of *X*, say $Q(u)=F^{-1}(u)$ can be obtained by inverting Equation (3) numerically, and it is given by:

$$Q(u)=G^{-1}\left\{c^{-1}\left[1-(1-ud)^{\frac{1}{b}}\right]\right\}^{\frac{1}{a}} \tag{12}$$

where $d=1-(1-c)^{b}$.

Ieren *et al.* (2019) defined the quantile function of the power Gompertz distribution as:

$$G^{-1}(u) = X_q = \left[\frac{1}{\beta}\log\left[1-\frac{\beta}{\alpha}\log(1-u)\right]\right]^{1/\theta} \tag{13}$$

By substituting Equations (12) in (13), we obtain the quantile function of the GKGPG distribution as:

$$Q(u) = \left[\frac{1}{\beta}\log\left[1-\frac{\beta}{\alpha}\log\left(1-\left\{c^{-1}\left[1-(1-ud)^{\frac{1}{b}}\right]\right\}^{\frac{1}{a}}\right)\right]\right]^{1/\theta} \tag{14}$$

This above derived function is used to obtain certain moments, such as Skewness and Kurtosis, as well as the median of the distribution and generation of random variables from the distribution concerned.

4.3. Skewness and Kurtosis

The analysis of the Skewness and Kurtosis variability on the shape parameters can be examined on the basis of quantile action. The weaknesses of the conventional measure of Kurtosis are well known. Kenney and Keeping [40] gives the Bowely Skewness based on quantiles as:

$$S_k = \frac{Q\left(\frac{3}{4}\right)-2Q\left(\frac{1}{2}\right)+Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right)-Q\left(\frac{1}{4}\right)} \tag{15}$$

Moors *et al.* [41] gave the Moors quantile based Kurtosis as:

$$K_u = \frac{Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)-Q\left(\frac{3}{8}\right)+Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{1}{8}\right)} \tag{16}$$

With $Q(.)$ is obtainable using the equation of the quantile function as given in Equation (14).

4.4. Reliability Analysis of the GKGPG Distribution

The Survival function of the generalized kumaraswamy generalized power Gompertz distribution is given as (**Figure 3**).

$$S(x) = 1-\left[\frac{1-\left[1-c\left(1-\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta x^{\theta}}-1\right)}\right)^{a}\right]^{b}}{1-(1-c)^{b}}\right] \tag{17}$$

where $x>0, 0<c\le 1, a>0, b>0, \alpha>0, \beta>0, \theta>0$.

The Hazard failure of the generalized kumaraswamy generalized power Gompertz distribution is given as (**Figure 4**).

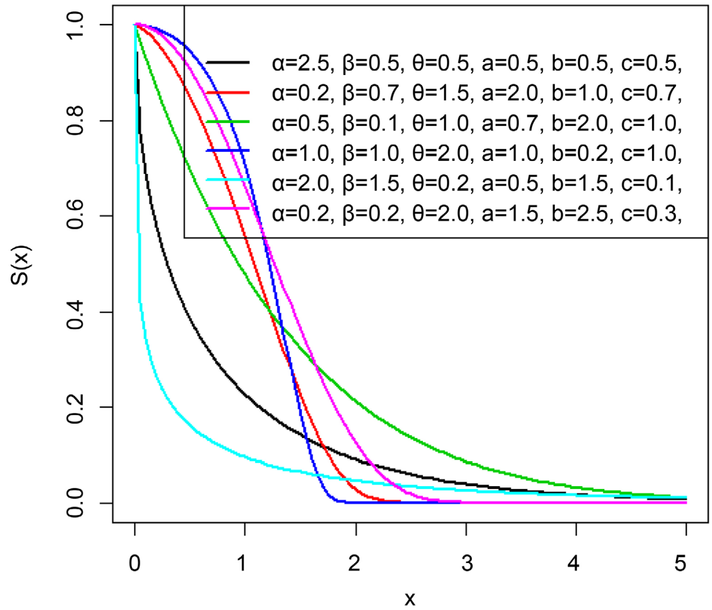

Figure 3. Survival plot of the Generalized Kumaraswamy generalized Power Gompertz Distribution (GKGPG) at different parameter values.

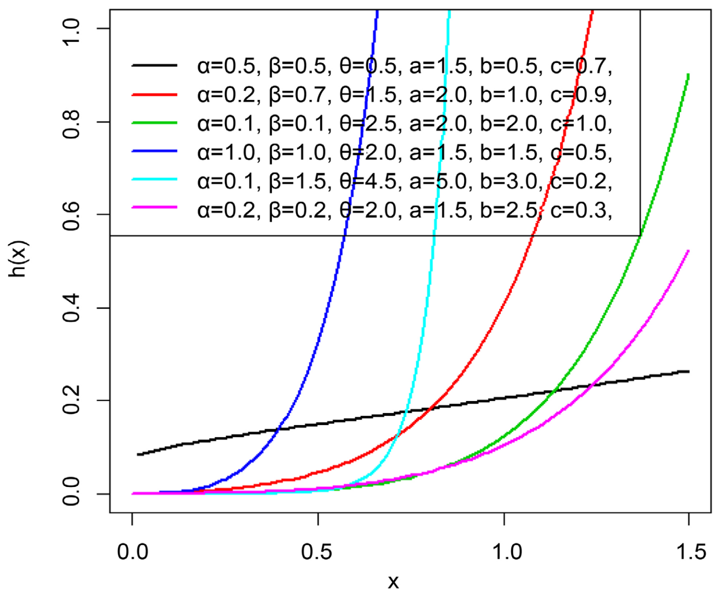

Figure 4. Hazard function plot of the Generalized Kumaraswamy generalized Power Gompertz Distribution (GKGPG) at different parameter values.

$$h(x)=\frac{abc\left[\alpha\theta x^{\theta-1}\mathrm{e}^{\beta x^{\theta}}\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta x^{\theta}}-1\right)}\right]\left[1-\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta x^{\theta}}-1\right)}\right]^{a-1}\left[1-c\left[1-\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta x^{\theta}}-1\right)}\right]^{a}\right]^{b-1}}{\left[1-c\left[1-\mathrm{e}^{-\frac{\alpha}{\beta}\left(\mathrm{e}^{\beta x^{\theta}}-1\right)}\right]^{a}\right]^{b}-(1-c)^{b}} \tag{18}$$

where $x>0, 0<c\leq 1, a>0, b>0, \alpha>0, \beta>0, \theta>0$.

5. Order Statistics

For $i = 1, \cdots, n$ from an independent and identically distributed random variables, let $X_1, \cdots, X_n$ denote a random sample from the Generalized Kumaraswamy generalized Power Gompertz Distribution with cdf $F(x)$, and pdf given by Equations (3) and (4) respectively. Then the probability density function $f_{i:n}(x)$ of the i^{th} order statistics of the GKGPG distribution is given by:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} f(x) F(x)^{K+i+1} \tag{19}$$

By substituting Equations (6) and (7) into the i^{th} order statistics of the GKGPG distribution, we have that:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} \frac{abc\alpha\theta x^{\theta-1} e^{\beta x^\theta} e^{-\frac{\alpha}{\beta}\left(e^{\beta x^\theta}-1\right)}}{1-(1-c)^b} \left[1 - e^{-\frac{\alpha}{\beta}\left(e^{\beta x^\theta}-1\right)}\right]^{a-1}$$
$$* \left[1 - c\left(1 - e^{-\frac{\alpha}{\beta}\left(e^{\beta x^\theta}-1\right)}\right)^a\right]^{b-1} \left[\frac{1 - \left[1 - c\left(1 - e^{-\frac{\alpha}{\beta}\left(e^{\beta x^\theta}-1\right)}\right)^a\right]^b}{1-(1-c)^b}\right]^{k+i-1} \tag{20}$$

Hence the minimum order statistics $X_{(1)}$ for the GKGPG distribution is given by:

$$f_{1:n}(x) = n \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{abc\alpha\theta x^{\theta-1} e^{\beta x^\theta} e^{-\frac{\alpha}{\beta}\left(e^{\beta x^\theta}-1\right)}}{1-(1-c)^b} \left[1 - e^{-\frac{\alpha}{\beta}\left(e^{\beta x^\theta}-1\right)}\right]^{a-1}$$
$$* \left[1 - c\left(1 - e^{-\frac{\alpha}{\beta}\left(e^{\beta x^\theta}-1\right)}\right)^a\right]^{b-1} \left[\frac{1 - \left[1 - c\left(1 - e^{-\frac{\alpha}{\beta}\left(e^{\beta x^\theta}-1\right)}\right)^a\right]^b}{1-(1-c)^b}\right]^{k} \tag{21}$$

Similarly, the maximum order statistics $X_{(n)}$ for the GKGPG distribution is given by:

$$f_{n:n}(x) = n \frac{abc\alpha\theta x^{\theta-1} e^{\beta x^\theta} e^{-\frac{\alpha}{\beta}\left(e^{\beta x^\theta}-1\right)}}{1-(1-c)^b} \left[1 - e^{-\frac{\alpha}{\beta}\left(e^{\beta x^\theta}-1\right)}\right]^{a-1}$$
$$* \left[1 - c\left(1 - e^{-\frac{\alpha}{\beta}\left(e^{\beta x^\theta}-1\right)}\right)^a\right]^{b-1} \left[\frac{1 - \left[1 - c\left(1 - e^{-\frac{\alpha}{\beta}\left(e^{\beta x^\theta}-1\right)}\right)^a\right]^b}{1-(1-c)^b}\right]^{n-1} \tag{22}$$

6. Parameter Estimation

6.1. Maximum Likelihood Estimation

Here, the parameters of the GKGPG distribution are estimated using the method of maximum likelihood. Let $X_1, X_2, \cdots, X_n$ be random samples distributed according to the GKGPG distribution, the likelihood function is obtained by the relationship:

$$l_n(\gamma) = \sum_{i=1}^{n} \ln f(X, \gamma) \tag{23}$$

$$\begin{aligned} l_n(\gamma) &= n\ln(abc\alpha\theta) + (\theta-1)\sum_{i=1}^{n}\ln(x_i) + \beta\sum_{i=1}^{n} x_i^{\theta} - \frac{\alpha}{\beta}\sum_{i=1}^{n}\left(e^{\beta x_i^{\theta}} - 1\right) \\ &\quad - \sum_{i=1}^{n}\ln(s_i) + (a-1)\sum_{i=1}^{n}\ln(1-\varphi_i) + (b-1)\sum_{i=1}^{n}\ln(\varpi_i) \end{aligned} \tag{24}$$

With $s_i = 1-(1-c)^b$, $\varpi_i = 1 - c\left(1 - e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}\right)^a$, $\varphi_i = e^{-\frac{\alpha}{\beta}\left(e^{\beta x^{\theta}}-1\right)}$, $v_i = e^{\beta x^{\theta}} - 1$.

The maximum likelihood estimators $\hat{a}, \hat{b}, \hat{c}, \hat{\alpha}, \hat{\beta}$ and $\hat{\theta}$ of the unknown parameters a, b, c, α, β and θ are derived from the nonlinear following score equations:

$$\frac{\partial L}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n}\ln(1-\varphi_i) - \sum_{i=1}^{n}\frac{c(b-1)(1-\varphi_i)^a\ln(1-\varphi_i)}{\varpi_i} \tag{25}$$

$$\frac{\partial L}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\frac{(1-c)^b\ln(1-c)}{s_i} + \sum_{i=1}^{n}\ln(\varpi_i) \tag{26}$$

$$\frac{\partial L}{\partial c} = \frac{n}{c} - \sum_{i=1}^{n}\frac{b(1-c)^{b-1}}{s_i} - \sum_{i=1}^{n}\frac{(b-1)(1-\varphi_i)^a}{\varpi_i} \tag{27}$$

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\frac{v_i}{\beta} + \sum_{i=1}^{n}\frac{(a-1)\varphi_i v_i}{\beta(1-\varphi_i)} - \sum_{i=1}^{n}\frac{ac(b-1)\varphi_i v_i(1-\varphi_i)^{a-1}}{\varpi_i} \tag{28}$$

$$\begin{aligned} \frac{\partial L}{\partial \beta} &= \sum_{i=1}^{n} x_i^{\theta} + \sum_{i=1}^{n}\frac{\alpha v_i}{\beta^2} - \frac{\alpha}{\beta}\sum_{i=1}^{n} x_i^{\theta} e^{\beta x_i^{\theta}} + \frac{\alpha(a-1)}{\beta^2}\sum_{i=1}^{n}\frac{\left(1 - e^{\beta x_i^{\theta}} + \beta x_i^{\theta} e^{\beta x_i^{\theta}}\right)\varphi_i}{1-\varphi_i} \\ &\quad + \frac{ac(b-1)}{\beta^2}\sum_{i=1}^{n}\frac{\left(1 - e^{\beta x_i^{\theta}} + \beta x_i^{\theta} e^{\beta x_i^{\theta}}\right)\varphi_i(1-\varphi_i)^{a-1}}{\varpi_i} \end{aligned} \tag{29}$$

$$\begin{aligned} \frac{\partial L}{\partial \theta} &= \frac{n}{\theta} + \sum_{i=1}^{n}\ln(x_i)\left(1 + \beta x_i^{\theta} - \alpha x_i^{\theta} e^{\beta x_i^{\theta}}\right) + \alpha(a-1)\sum_{i=1}^{n}\frac{x_i^{\theta}\ln(x_i) e^{\beta x_i^{\theta}}\varphi_i}{1-\varphi_i} \\ &\quad - a\alpha c(b-1)\sum_{i=1}^{n}\frac{x_i^{\theta}\ln(x_i) e^{\beta x_i^{\theta}}\varphi_i(1-\varphi_i)^{a-1}}{\varpi_i} \end{aligned} \tag{30}$$

6.2. Estimation under Right-Censored Data

The hypothesizing test will be discussed under complete and censored data, however, the MPS is only defined for complete data, since the MLE is usually considered for right-censored data, Let us consider $X_1, X_2, \cdots, X_n$ a random right censored sample obtained from the GKGPG distribution with the parameter vector $\gamma = (a,b,c,\alpha,\beta,\theta)^{\mathrm{T}}$. The censoring time τ is fixed. So, the observation X_i is equal to $X_i = (x_i, \delta_i)$ where:

$$\delta_i = \begin{cases} 0 & \text{if } x_i \text{ is a censoring time} \\ 1 & \text{if } x_i \text{ is a failure time} \end{cases} \tag{31}$$

In this case, the log-likelihood is obtained as follow:

$$L_n(\gamma) = \sum_{i=1}^{n} \delta_i \ln f(x_i,\gamma) + \sum_{i=1}^{n} (1-\delta_i) \ln S(x_i,\gamma) \tag{32}$$

$$\begin{aligned} L_n(\gamma) = \sum_{i=1}^{n} \delta_i \Bigg[& n\ln(abc\alpha\theta) + (\theta-1)\ln(x_i) + \beta x_i^{\theta} - \frac{\alpha v_i}{\beta} \\ & - \ln(s_i) + (a-1)\ln(1-\varphi_i) + (b-1)\ln(\varpi_i) \Bigg] \\ & + \sum_{i=1}^{n} (1-\delta_i)\ln(1-\delta_i)\ln\left(1 - \frac{1-\varpi_i^b}{s_i}\right) \end{aligned} \tag{33}$$

The maximum likelihood estimators $\hat{a}, \hat{b}, \hat{c}, \hat{\alpha}, \hat{\beta}$ and $\hat{\theta}$ of the unknown parameters a,b,c,α,β and θ are derived from the nonlinear following score equations:

$$\begin{aligned} \frac{\partial L}{\partial a} = & \sum_{i=1}^{n} \delta_i \left[\frac{1}{a} + \ln(1-\varphi_i) - \frac{c(b-1)(1-\varphi_i)^a \ln(1-\varphi_i)}{\varpi_i} \right] \\ & - bc \sum_{i=1}^{n} (1-\delta_i) \frac{(1-\varphi_i)^a \ln(1-\varphi_i)\varpi_i^{b-1}}{s_i - (1-\varpi_i^b)} \end{aligned} \tag{34}$$

$$\begin{aligned} \frac{\partial L}{\partial b} = & \sum_{i=1}^{n} \delta_i \left[\frac{1}{b} + \frac{(1-c)^b \ln(1-c)}{s_i} + \ln(\varpi_i) \right] \\ & + \sum_{i=1}^{n} (1-\delta_i) \left[\frac{s_i \varpi_i^b \ln(\varpi_i) - (1-c)^b \ln(1-c)(1-\varpi_i^b)}{s_i\left(s_i - (1-\varpi_i^b)\right)} \right] \end{aligned} \tag{35}$$

$$\begin{aligned} \frac{\partial L}{\partial c} = & \sum_{i=1}^{n} \delta_i \left[\frac{1}{c} - \frac{b(1-c)^{b-1}}{s_i} - \frac{(b-1)(1-\varphi_i)^a}{\varpi_i} \right] \\ & - \sum_{i=1}^{n} (1-\delta_i) \left[\frac{s_i b (1-\varphi_i)^a \varpi_i^{b-1} - b(1-c)^{b-1}(1-\varpi_i^b)}{s_i\left(s_i - (1-\varpi_i^b)\right)} \right] \end{aligned} \tag{36}$$

$$\begin{aligned} \frac{\partial L}{\partial \alpha} = & \sum_{i=1}^{n} \delta_i \left[\frac{1}{\alpha} - \frac{v_i}{\beta} + \frac{(a-1)\varphi_i v_i}{\beta(1-\varphi_i)} - \frac{ac(b-1)\varphi_i v_i (1-\varphi_i)^{a-1}}{\varpi_i} \right] \\ & - acb \sum_{i=1}^{n} (1-\delta_i) \frac{\varphi_i v_i (1-\varphi_i)^{a-1} \varpi_i^{b-1}}{s_i - (1-\varpi_i^b)} \end{aligned} \tag{37}$$

$$\frac{\partial L}{\partial \beta}=\sum_{i=1}^{n}\delta_i\left[x_i^{\theta}+\frac{\alpha v_i}{\beta^2}-\frac{\alpha}{\beta}x_i^{\theta}\mathrm{e}^{\beta x_i^{\theta}}+\frac{\alpha(a-1)}{\beta^2}\frac{\left(1-\mathrm{e}^{\beta x_i^{\theta}}+\beta x_i^{\theta}\mathrm{e}^{\beta x_i^{\theta}}\right)\varphi_i}{1-\varphi_i}\right.$$
$$\left.+\frac{ac(b-1)}{\beta^2}\frac{\left(1-\mathrm{e}^{\beta x_i^{\theta}}+\beta x_i^{\theta}\mathrm{e}^{\beta x_i^{\theta}}\right)\varphi_i(1-\varphi_i)^{a-1}}{\varpi_i}\right] \tag{38}$$
$$+\frac{acb}{\beta^2}\sum_{i=1}^{n}(1-\delta_i)\frac{\left(1-\mathrm{e}^{\beta x_i^{\theta}}+\beta x_i^{\theta}\mathrm{e}^{\beta x_i^{\theta}}\right)\varphi_i(1-\varphi_i)^{a-1}\varpi_i^{b-1}}{s_i-\left(1-\varpi_i^{b}\right)}$$

$$\frac{\partial L}{\partial \theta}=\sum_{i=1}^{n}\delta_i\left[\frac{1}{\theta}+\ln(x_i)\left(1+\beta x_i^{\theta}-\alpha x_i^{\theta}\mathrm{e}^{\beta x_i^{\theta}}\right)+\alpha(a-1)\frac{x_i^{\theta}\ln(x_i)\mathrm{e}^{\beta x_i^{\theta}}\varphi_i}{1-\varphi_i}\right.$$
$$\left.-a\alpha c(b-1)\frac{x_i^{\theta}\ln(x_i)\mathrm{e}^{\beta x_i^{\theta}}\varphi_i(1-\varphi_i)^{a-1}}{\varpi_i}\right] \tag{39}$$
$$-a\alpha cb\sum_{i=1}^{n}(1-\delta_i)\frac{x_i^{\theta}\ln(x_i)\mathrm{e}^{\beta x_i^{\theta}}\varphi_i(1-\varphi_i)^{a-1}\varpi_i^{b-1}}{s_i-\left(1-\varpi_i^{b}\right)}$$

Monte Carlo technique or other iterative methods can be used to determine the values of $\hat{a},\hat{b},\hat{c},\hat{\alpha},\hat{\beta}$ and $\hat{\theta}$.

7. Test Statistic for Right Censored Data

Let $X_1,\cdots,X_n$ be n i.i.d. random variables grouped into r classes I_i. To assess the adequacy of a parametric model F_0:

$$\mathrm{H}_0: P(X_i\le xH_0)=F_0(x;\gamma),\ x\ge 0,\ \gamma=(\gamma_1,\cdots,\gamma_s)^{\mathrm{T}}\in\Theta\subset R^s \tag{40}$$

When data are right censored and the parameter vector β is unknown, Bagdonavicius and Nikulin [38] proposed a statistic test Y^2 based on the vector:

$$Z_j=\frac{1}{\sqrt{n}}\left(U_j-e_j\right),\ j=1,2,\cdots,r\quad \text{with } r\succ s. \tag{41}$$

This one represents the differences between observed and expected numbers of failures (U_j and e_j) to fall into these grouping intervals $I_j=\left(p_{j-1},p_j\right]$ with $p_0=0$, $p_r=\tau$, where τ is a finite time. The authors considered p_j as random data functions such as the r intervals chosen have equal expected numbers of failures e_j.

The statistic test Y^2 is defined by:

$$Y^2=Z^{\mathrm{T}}\hat{\Sigma}^{-}Z=\sum_{i=1}^{r}\frac{\left(U_j-e_j\right)^2}{U_j}+Q \tag{42}$$

where $Z=\left(Z_1,\cdots,Z_r\right)^{\mathrm{T}}$ and $\hat{\Sigma}^{-}$ is a generalized inverse of the covariance matrix $\hat{\Sigma}$ and:

$$Q=W^{\mathrm{T}}\hat{G}^{-}W,\ \hat{A}_j=\frac{U_j}{n},U_j=\sum_{i:X_i\in I_j}\delta_i$$

$$W = \left(W_1, \cdots, W_s\right)^{\mathrm{T}},\ \hat{G} = \left[\hat{g}_{ll'}\right]_{s\times s},\ \hat{g}_{ll'} = \hat{i}_{ll'} - \sum_{j=1}^{r} \hat{C}_{lJ} \hat{G}_{l'J} \hat{A}_J^{-1}$$

$$\hat{C}_{lj} = \frac{1}{n} \sum_{i:X_i \in I_j} \delta_i \frac{\partial \ln h\left(x_i, \hat{\gamma}\right)}{\partial \gamma},\ \hat{i}_{ll'} = \frac{1}{n} \sum_{i=1}^{n} \delta_i \frac{\partial \ln h\left(x_i, \hat{\gamma}\right)}{\partial \gamma_l} \frac{\partial \ln h\left(x_i, \hat{\gamma}\right)}{\partial \gamma_{l'}}$$

$$\hat{W}_l = \sum_{j=1}^{r} \hat{C}_{lJ} \hat{A}_J^{-1} Z_j,\ l, l' = 1, \cdots, s$$

$\hat{\gamma}$ is the maximum likelihood estimator of γ on initial non-grouped data.

Under the null hypothesis H_0, the limit distribution of the statistic Y^2 is a chi-square with $r = rank\left(\Sigma\right)$ degrees of freedom. The description and applications of modified chi-square tests are discussed in Voinov *et al.* [42].

The interval limits p_j for grouping data into j classes I_j are considered as data functions and defined by:

$$\hat{p}_j = H^{-1}\left(\frac{E_j - \sum_{l=1}^{i-1} H\left(x_l, \hat{\gamma}\right)}{n-i+1}, \hat{\gamma}\right),\ \hat{p}_j = \max\left(X_{(n)}, \tau\right) \tag{43}$$

Such as the expected failure times e_j to fall into these intervals are $e_j = \frac{E_r}{r}$ for any *j*, with $E_r = \sum_{i=1}^{n} H\left(x_i, \gamma\right)$. The distribution of this statistic test Y_n^2 is chi-square (see Voinov *et al.*, 2013).

7.1. Criteria Test for GKGPG Distribution

For testing the null hypothesis H_0 that data belong to the GKGPG model, we construct a modified chi-squared type goodness-of-fit test based on the statistic Y^2. Suppose that τ is a finite time, and observed data are grouped into $r > s$ sub-intervals $I_j = \left(p_{j-1}, p_j\right]$ of $\left[0, \tau\right]$. The limit intervals p_j are considered as random variables such that the expected numbers of failures in each interval I_j are the same, so the expected numbers of failures e_j are obtained as:

$$E_j = -\frac{j}{r-1} \sum_{i=1}^{n} \ln\left(1 - \frac{1-\varpi_i^{b}}{s_i}\right),\ j = 1, \cdots, r-1 \tag{44}$$

7.2. Estimated Matrix $\hat{W}$ and $\hat{C}$

The components of the estimated matrix $\hat{\boldsymbol{W}}$ are derived from the estimated matrix $\hat{\boldsymbol{C}}$ which is given by:

$$\hat{C}_{1j} = \frac{1}{n} \sum_{i:x_i \in I_j}^{n} \delta_i \left[\frac{1}{a} + \ln\left(1-\varphi_i\right) - \frac{c\left(b-1\right)\left(1-\varphi_i\right)^a \ln\left(1-\varphi_i\right)}{\varpi_i} + \frac{bc\left(1-\varphi_i\right)^a \ln\left(1-\varphi_i\right) \varpi_i^{b-1}}{s_i - \left(1-\varpi_i^b\right)}\right] \tag{45}$$

$$\hat{C}_{2j} = \frac{1}{n} \sum_{i:x_i \in I_j}^{n} \delta_i \left[\frac{1}{b} + \frac{\left(1-c\right)^b \ln\left(1-c\right)}{s_i} + \ln\left(\varpi_i\right) - \frac{s_i \varpi_i^b \ln\left(\varpi_i\right) - \left(1-c\right)^b \ln\left(1-c\right)\left(1-\varpi_i^b\right)}{s_i\left(s_i - \left(1-\varpi_i^b\right)\right)}\right] \tag{46}$$

$$\hat{C}_{3j} = \frac{1}{n} \sum_{i:x_i \in I_j}^{n} \delta_i \left[\frac{1}{c} - \frac{b\left(1-c\right)^{b-1}}{s_i} - \frac{\left(b-1\right)\left(1-\varphi_i\right)^a}{\varpi_i} + \frac{s_i b\left(1-\varphi_i\right)^a \varpi_i^{b-1} - b\left(1-c\right)^{b-1}\left(1-\varpi_i^b\right)}{s_i\left(s_i - \left(1-\varpi_i^b\right)\right)}\right] \tag{47}$$

$$\hat{C}_{4j}=\frac{1}{n}\sum_{i:x_i\in I_j}^{n}\delta_i\left[\frac{1}{\alpha}-\frac{v_i}{\beta}+\frac{(a-1)\varphi_i v_i}{\beta(1-\varphi_i)}-\frac{ac(b-1)\varphi_i v_i(1-\varphi_i)^{a-1}}{\varpi_i}+\frac{abc\varphi_i v_i(1-\varphi_i)^{a-1}\varpi_i^{b-1}}{s_i-(1-\varpi_i^b)}\right] \quad (48)$$

$$\begin{aligned}\hat{C}_{5j}=\frac{1}{n}\sum_{i:x_i\in I_j}^{n}\delta_i&\left[x_i^\theta+\frac{\alpha(a-1)}{\beta^2}\frac{\left(1-e^{\beta x_i^\theta}+\beta x_i^\theta e^{\beta x_i^\theta}\right)\varphi_i}{1-\varphi_i}+\frac{ac(b-1)}{\beta^2}\frac{\left(1-e^{\beta x_i^\theta}+\beta x_i^\theta e^{\beta x_i^\theta}\right)\varphi_i(1-\varphi_i)^{a-1}}{\varpi_i}\right.\\&\left.+\frac{\alpha v_i}{\beta^2}-\frac{\alpha}{\beta}x_i^\theta e^{\beta x_i^\theta}-\frac{acb}{\beta^2}\frac{\left(1-e^{\beta x_i^\theta}+\beta x_i^\theta e^{\beta x_i^\theta}\right)\varphi_i(1-\varphi_i)^{a-1}\varpi_i^{b-1}}{s_i-(1-\varpi_i^b)}\right]\end{aligned} \quad (49)$$

$$\begin{aligned}\hat{C}_{6j}=\frac{1}{n}\sum_{i:x_i\in I_j}^{n}\delta_i&\left[\frac{1}{\theta}+\alpha(a-1)\frac{x_i^\theta\ln(x_i)e^{\beta x_i^\theta}\varphi_i}{1-\varphi_i}-a\alpha c(b-1)\frac{x_i^\theta\ln(x_i)e^{\beta x_i^\theta}\varphi_i(1-\varphi_i)^{a-1}}{\varpi_i}\right.\\&\left.+\ln(x_i)\left(1+\beta x_i^\theta-\alpha x_i^\theta e^{\beta x_i^\theta}\right)+\frac{a\alpha cbx_i^\theta\ln(x_i)e^{\beta x_i^\theta}\varphi_i(1-\varphi_i)^{a-1}\varpi_i^{b-1}}{s_i-(1-\varpi_i^b)}\right]\end{aligned} \quad (50)$$

And:

$$\hat{W}_l=\sum_{j=1}^{r}\hat{C}_{lJ}\hat{A}_J^{-1}Z_j,\ l,l'=1,2,3,4,5,6;\ j=1,\cdots,r$$

7.3. Estimated Matrix $\hat{G}$

The estimated matrix $\hat{G}=[\hat{g}_{ll'}]_{6\times 6}$ is defined by:

$$\hat{g}_{ll'}=\hat{i}_{ll'}-\sum_{j=1}^{r}\hat{C}_{lJ}\hat{G}_{l'J}\hat{A}_J^{-1}$$

where:

$$\hat{i}_{ll'}=\frac{1}{n}\sum_{i=1}^{n}\delta_i\frac{\partial\ln h(x_i,\hat{\gamma})}{\partial\gamma_l}\frac{\partial\ln h(x_i,\hat{\gamma})}{\partial\gamma_{l'}},$$
$$l,l'=1,2,3,4,5,6$$

Therefore the quadratic form of the test statistic can be obtained easily:

$$Y_n^2(\hat{\gamma})=\sum_{j=1}^{r}\frac{(U_j-e_j)^2}{U_j}+\hat{W}^{\mathrm{T}}\left[\hat{i}_{ll'}-\sum_{j=1}^{r}\hat{C}_{lJ}\hat{G}_{l'J}\hat{A}_J^{-1}\right]^{-1}\hat{W} \quad (51)$$

8. Simulations

8.1. Maximum Likelihood Estimation

We generated $N=10000$ right censored samples with different sizes ($n=25,50,130,350,500$) from the GKGPG model with parameters $a=2$, $b=1$, $c=0.9$, $\alpha=0.2$, $\beta=0.7$ and $\theta=1.5$. Using R statistical software and the Barzilai-Borwein (BB) algorithm (Ravi, [43]), we calculate the maximum likelihood estimators of the unknown parameters and their Mean Squared Errors (MSE). The results are given in Table 1.

The maximum likelihood estimated parameter values, presented in Table 1, agree closely with the true parameter values.

Table 1. Mean simulated values of MLEs $\hat{\gamma}$ their corresponding square mean errors.

$N = 10000$	$n = 25$	$n = 50$	$n = 130$	$n = 350$	$n = 500$
$\hat{a}$	1.9532 (0.0092)	1.9679 (0.0067)	1.9706 (0.0059)	1.9756 (0.0042)	1.9896 (0.0033)
$\hat{b}$	0.9686 (0.0079)	0.9713 (0.0052)	0.9876 (0.0038)	0.9903 (0.0025)	0.9982 (0.0012)
$\hat{c}$	0.9236 (0.0084)	0.9186 (0.0061)	0.9106 (0.0047)	0.9086 (0.0037)	0.9023 (0.0029)
$\hat{\alpha}$	0.1775 (0.0088)	0.1823 (0.0073)	0.1897 (0.0041)	0.1902 (0.0027)	0.1976 (0.0016)
$\hat{\beta}$	0.7361 (0.0098)	0.7253 (0.0079)	0.7126 (0.0053)	0.7098 (0.0034)	0.7012 (0.0018)
$\hat{\theta}$	1.5364 (0.0068)	1.5231 (0.0057)	1.5134 (0.0033)	1.5037 (0.0018)	1.5003 (0.0009)

8.2. Criteria Test Y_n^2

For testing the null hypothesis H_0 that right censored data become from GKGPG model, we compute the criteria statistic $Y_n^2(\gamma)$ as defined above for 10,000 simulated samples from the hypothezised distribution with different sizes (30, 50, 150, 350, 500). Then, we calculate empirical levels of significance, when $Y^2 > \chi_\varepsilon^2(r)$, corresponding to theoretical levels of significance ($\varepsilon = 0.10$, $\varepsilon = 0.05$, $\varepsilon = 0.01$), We choose $r = 7$. The results are reported in Table 2.

The null hypothesis H_0 for which simulated samples are fitted by GKGPG distribution is widely validated for the different levels of significance. Therefore, the test proposed in this work, can be used to fit data from this new distribution.

9. Application

In this section, we apply the results obtained through this study to real data set from reliability (Crowder *et al.* [44]), previously used by [45] [46] [47]. In an experiment to gain information on the strength of a certain type of braided cord after weathering, the strengths of 48 pieces of cord that had been weathered for a specified length of time were investigated. The observed right-censored strength-values are given below:

26.8*, 29.6*, 33.4*, 35*, 36.3, 40*, 41.7, 41.9*, 42.5*, 43.9, 49.9, 50.1, 50.8, 51.9, 52.1, 52.3, 52.3, 52.4, 52.6, 52.7, 53.1, 53.6, 53.6, 53.9, 53.9, 54.1, 54.6, 54.8, 54.8, 55.1, 55.4, 55.9, 56, 56.1, 56.5, 56.9, 57.1, 57.1, 57.3, 57.7, 57.8, 58.1, 58.9, 59, 59.1, 59.6, 60.4, 60.7

We use the statistic test provided above to verify if these data are modelled by GKGPG distribution, and at that end, we first calculate the maximum likelihood estimators of the unknown parameters:

$$\gamma = (a, b, c, \alpha, \beta, \theta)^{\mathrm{T}} = (2.5134, 1.6384, 0.9467, 0.3796, 0.5931, 1.7649)^{\mathrm{T}} \quad (52)$$

Table 2. Simulated levels of significance for $Y_n^2(\gamma)$ test for GKGPG model against their theoretical values ($\varepsilon = 0.01, 0.05, 0.10$).

$N = 10000$	$n_1 = 30$	$n_2 = 50$	$n_3 = 150$	$n_4 = 350$	$n_5 = 500$
$\varepsilon = 1\%$	0.0062	0.0067	0.0078	0.0086	0.0095
$\varepsilon = 5\%$	0.0412	0.0433	0.0442	0.0458	0.0476
$\varepsilon = 10\%$	0.0953	0.0972	0.0986	0.0998	0.1012

Table 3. Values of $p_j, e_j, U_j, \hat{C}_{1j}, \hat{C}_{2j}, \hat{C}_{3j}, \hat{C}_{4j}, \hat{C}_{5j}, \hat{C}_{6j}$.

p_j	43.5	51	52.5	53.5	54.5	56.7	58	60.7
U_j	9	4	5	3	5	9	6	7
e_j	0.9896	0.9896	0.9896	0.9896	0.9896	0.9896	0.9896	0.9896
$\hat{C}_{1j}$	1.1635	1.0856	−2.067	1.0856	−2.0345	1.8562	1.3462	1.0374
$\hat{C}_{2j}$	2.0845	1.5623	1.4326	0.9764	1.0844	0.9134	1.4393	1.0563
$\hat{C}_{3j}$	−2.1373	−3.5162	−1.846	−4.1862	−0.9463	−0.7485	−2.6314	−1.8462
$\hat{C}_{4j}$	0.9347	1.0236	−4.1632	1.0536	0.8326	−2.6351	−3.7486	1.0536
$\hat{C}_{5j}$	1.4963	2.0846	1.8631	0.9713	1.3719	1.6431	2.7931	2.1937
$\hat{C}_{6j}$	−0.9384	1.0746	2.0314	−1.5393	1.4639	1.7469	−1.0352	2.0845

Data are grouped into $r = 7$ intervals I_j. We give the necessary calculus in Table 3.

Then we obtain the value of the statistic test Y_n^2:

$$Y_n^2 = X^2 + Q = 5.6317 + 4.1237 = 10.7554 \tag{53}$$

For significance level $\varepsilon = 0.05$, the critical value $\chi_7^2 = 14.0671$ is superior than the value of $Y_n^2 = 10.7554$, so we can say that the proposed model GKGPGfit these data.

10. Conclusion

This research has successfully introduced and studied a six-parameter continuous distribution called the generalized Kumaraswamy generalized power Gompertz distribution. The plots of the probability density and cumulative distribution function have been analyzed. We have also derived some properties of the new distribution such as asymptotic behavior, quantile function for median, Skewness, and Kurtosis, and reliability analysis. The distribution of order statistics estimation of parameters based on censored and uncensored random samples using Maximum Likelihood Estimation (MLE) has been provided. We evaluated the new goodness-of-fit statistic test Y_n^2 and investigated some criteria tests for the generalized Kumaraswamy generalized power Gompertz distribution. A simulation study was carried out in applying the new model to datasets. The newly proposed model GKGPG adequately fits the data.

Formation of the Generalized Kumaraswamy Generalized

Defined in this paper has three shape parameters which control its Skewness, Kurtosis and tails. It can therefore be applied in more real-life situations. Maximum likelihood estimates are discussed, and modified chi-square goodness-of-fit tests for right censoring are constructed. The statistical test provided in this article can be used to fit unknown parameters and censorship into this model and its sub-models. The results and efficacy of the proposed test are shown in an important simulation study.

Conflicts of Interest

The authors declare no conflict of interest.

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