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On the Pólya Enumeration Theorem

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Abstract: Simple formulas for the number of different cyclic and dihedral necklaces containing n_j beads of the j-th color, $j \le m$ and $\sum_{j=1}^{m} n_j = N$, are derived, using the Pólya enumeration theorem.

Keywords: permutations and cyclic invariance, cycle index, Pólya enumeration theorem

Among a vast number of counting problems one of the most popular is *a necklace* enumeration. A *cyclic necklace* is a coloring in m colors of the vertices of a regular N--gon, where two colorings are equivalent if one can be obtained from the other by a cyclic symmetry C_N, e.g. colored beads are placed on a circle, and the circle may be rotated (without reflections). A basic enumeration problem is then: for given m and $N = \sum_{j=1}^{m} n_j$, how many different cyclic necklaces containing n_j beads of the j-th color are there. The answer follows by an application of the Pólya's theorem [1]: the number γ (C_N, n^m) of different cyclic necklaces is the coefficient of $x_1^{n_1} \cdot \ldots \cdot x_m^{n_m}$ in the cycle index

$$Z_{C_N}(x_i) = \frac{1}{N}\sum_{g|N}\phi(g) X_g^{N/g}\,, \quad X_g = x_1^g + \ldots + x_m^g\,, \quad (1)$$

where $\phi(g)$ denotes the Euler totient function and n^m denotes a tuple $(n_1, \ldots, n_m)$.

Since γ (C_N, n^m) is not available in closed form in standard and advanced textbooks [2–6] we found it worthwhile to derive this number from (1). In this article we prove that

$$\gamma\left(C_N, n^m\right) = \frac{1}{N}\sum_{d|\Delta}\phi(d) P\left(k^m\right),$$
$$\text{where} \quad P\left(k^m\right) = \frac{(k_1 + \ldots + k_m)!}{\prod_{j=1}^{m} k_j!}\,, \quad k_j = \frac{n_j}{d}\,, \quad (2)$$

and Δ denotes a great common divisor gcd n^m of the tuple n^m. We denote also $k^m = (k_1, \ldots, k_m)$.

Note that the term $x_1^{n_1} \cdot \ldots \cdot x_m^{n_m}$ does appear only once in *the multinomial series expansion* (MSE) of (1) with a weight $P\left(n^m\right)$ when $g = 1$,

$$X_1^N \to P\left(n^m\right) x_1^{n_1} \cdot \ldots \cdot x_m^{n_m}\,, \quad \text{where} \quad N = n_1 + \ldots + n_m\,. \quad (3)$$

Show that for $g > 1$ the polynomial $Z_{C_N}(x_i)$ contributes in γ (C_N, n^m) if and only if $\Delta > 1$. We prove that if N is divisible by g and Δ is not divisible by g then the term $x_1^{n_1} \cdot \ldots \cdot x_m^{n_m}$ does not appear in MSE of (1).

Denote $N/g = L$, $1 < L < N$ and consider MSE of (1)

$$X_g^L = \sum_{\substack{l_1 + \ldots + l_m = L \\ l_i \ge 0}} P\left(l^m\right) x_1^{g l_1} \cdot \ldots \cdot x_m^{g l_m}\,, \quad (4)$$

where l^m denotes a tuple (l_1,…,l_m). However MSE in (4) does not contribute in γ (C_N, n^m) since Δ is not divisible by g, i.e. we cannot provide such g that $g l_i = n_i$ holds for all $i = 1, \ldots, m$. Thus, we have reduced expression (1) by summing only over the divisors d of Δ,

$$Z_{C_N}(x_i) = \frac{1}{N}\sum_{d|\Delta}\phi(d) X_d^{N/d}\,. \quad (5)$$

Denoting $k_j = n_j/d$, $N/d = K = k_1 + \ldots + k_m$, and considering MSE of (5) we obtain

$$X_d^K \to P\left(k^m\right) x_1^{d k_1} \cdot \ldots \cdot x_m^{d k_m} = P\left(k^m\right) x_1^{n_1} \cdot \ldots \cdot x_m^{n_m}\,. \quad (6)$$

Combining (5) and (6) we arrive at (2).

It is easy to extend the explicit Formula (2) to the case of *dihedral necklaces* where two colorings are equivalent if one can be obtained from the other by a dihedral symmetry D_N, e.g. colored beads are placed on a circle, and the circle may be rotated and reflected. Start with the cycle indices [5]

$$2Z_{D_N}(x_i) = Z_{C_N}(x_i) + \begin{cases} X_1 X_2^L, \ if \quad N = 2L+1, \\ \frac{1}{2}\left(X_1^2 X_2^{L-1} + X_2^L\right), \ if \quad N = 2L \end{cases} \quad (7)$$

If $N = 2L+1$ we have to distinguish two different cases.

1) There is one odd integer $n_j = 2a_j + 1 \in n^m$, while the rest of n_i are even, $n_i = 2a_i$, $L = \sum_{i=1}^m a_i$,

$$\gamma\left(D_N, n^m\right) = \frac{1}{2}\left[P\left(n^m\right) + P\left(a^m\right)\right], \quad where \quad a^m = (a_1, \ldots, a_j, \ldots, a_m). \quad (8)$$

2) There is more than one odd integer $n_j = 2a_j + 1 \in n^m$, $1 \le j \le m$,

$$\gamma\left(D_N, n^m\right) = \frac{1}{2}\gamma\left(C_N, n^m\right). \quad (9)$$

If $N = 2L$ we have to distinguish three different cases.

1) All integers $n_j \in n^m, j < m$ are even, $n_j = 2b_j$, and $L = \sum_{i=1}^m b_i$, $b^m = (b_1, \ldots, b_m)$,

$$\gamma\left(D_N, n^m\right) = \frac{1}{2}\gamma\left(C_N, n^m\right) + \frac{1}{4}\sum_{q=1}^m P\left(b_q^m\right) + \frac{1}{4}P\left(b^m\right), \quad where \quad (10)$$

$b_1^m = (b_1 - 1, b_2, b_3, \ldots, b_m)$, $b_2^m = (b_1, b_2 - 1, b_3, \ldots, b_m)$, $\ldots$, $b_m^m = (b_1, b_2, b_3, \ldots, b_m - 1)$.

2) There is one pair of odd integers, $n_{j_1, j_2} = 2c_{j_1, j_2} + 1 \in n^m$, while the rest of n_i are even, $n_i = 2c_i$,

$$\gamma\left(D_N, n^m\right) = \frac{1}{2}\left[P\left(n^m\right) + P\left(c^m\right)\right], \quad where \quad c^m = (c_1, \ldots, c_{j_1}, \ldots, c_{j_2}, \ldots, c_m), \quad (11)$$

and $L = 1 + c_1 + \ldots + c_{j_1} + \ldots + c_{j_2} + \ldots + c_m$.

3) There is more than one pair of odd integers $n_{j_1, j_2} = 2c_{j_1, j_2} + 1 \in n^m$, $1 \le j_1, j_2 \le m$,

$$\gamma\left(D_N, n^m\right) = \frac{1}{2}\gamma\left(C_N, n^m\right). \quad (12)$$

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