

Time Intervals of the Energy Emission in Quantum Systems Obtained from the Conservation Rule of the Electron Momentum

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Abstract

The paper presents a non-probabilistic approach to the time interval associated with the energy emission produced by the electron transition in a quantum system. The calculations were performed for the hydrogen atom and the electron particle in a one-dimensional potential box. In both cases, the rule of conservation of the electron momentum has been applied. The results, limited to the time intervals of transitions between two neighbouring quantum energy levels, occur to be much similar to those obtained earlier with the aid of the Joule-Lenz energy emission theory.

Keywords

Non-Probabilistic Approach to the Electron Transition Time, Conservation Rule of the Electron Momentum, Joule-Lenz Energy Emission Theory

1. Introduction

In principle, we assume that some interval of time should accompany any quantum process in which a change of a quantum physical system does occur. In the previous approach to such processes, a probabilistic analysis accompanied any electron transition phenomenon (see e.g. [1] [2] [3] [4]), leaving unknown the corresponding interval, or intervals, of time. In general, such situation did not change in the modern quantum theory [5] [6].

A different, viz. non-probabilistic situation, took place when the classical Joule-Lenz theorem for the energy emission (see e.g. [7]) has been adapted in calculating the transition time of an electron between two quantum energy levels [8]-[14]. In this case, a very simple rule coupling the distance between two quantum energy levels with the size of the time interval for the electron transi-

tion could be found. Nevertheless, the limitations of that rule became rather evident. In consequence, an alternative approach to the time interval of the electron transition seemed to be of use. Such approach is outlined in the next Sections of the present paper.

2. Electron Momentum, Its Change and a Use of the Postulate Concerning Conservation of the Momentum. The Case of the Hydrogen Atom Taken as an Example

In principle any change of the electron energy, say obtained in effect of the electron transition between two quantum levels, can be associated with a corresponding change of the electron momentum. By taking into account the bound electron states of the hydrogen atom, the state n of the energy is given by [15] [16]

$$E_n = -E_{\text{kin}}^{(n)} = -\frac{m}{2} v_n^2, \quad (1)$$

where the first equation in (1) is due to the virial theorem

$$2E_{\text{kin}}^{(n)} + E_{\text{pot}}^{(n)} = 0 \quad (2)$$

in result of which the total electron energy of the atom equal to a sum of the kinetic and potential parts becomes

$$E_{\text{tot}}^{(n)} = E_{\text{kin}}^{(n)} + E_{\text{pot}}^{(n)} = E_n. \quad (3)$$

The electron velocity entering (1) is [17]

$$v_n = \frac{e^2}{n\hbar} \quad (4)$$

and the electron momentum in state n becomes

$$p^n = m v_n. \quad (5)$$

Another approach to p^n can be obtained from the quanta of the electron angular momentum

$$L_n = m v_n r_n = \frac{m e^2 n^2 \hbar^2}{n\hbar m e^2} = n\hbar. \quad (6)$$

which holds because

$$r_n = \frac{n^2 \hbar^2}{m e^2} \quad (7)$$

is the radius of the electron circular orbit in the hydrogen atom [17].

A final result for the quanta of energy in (1) is

$$E_n = -\frac{m e^4}{2 n^2 \hbar^2} \quad (8)$$

and the energy change

$$\Delta E = E_{n+1} - E_n \quad (9)$$

due to the change of the quantum state is equal to

$$\Delta E = \frac{m e^4}{2 \hbar^2} \left[\frac{1}{n^2} - \frac{1}{(n+1)^2} \right] = \frac{m e^4}{2 \hbar^2} \frac{(n+1)^2 - n^2}{n^2 (n+1)^2} = \frac{m e^4}{\hbar^2} \frac{n+1/2}{n^2 (n+1)^2} \cong \frac{m e^4}{\hbar^2} \frac{1}{n^3}. \quad (10)$$

The last step in (10) is valid on condition n is a large number.

Respectively to (9) we have the momentum change

$$\Delta p^n = p^{n+1} - p^n = \frac{m e^2}{\hbar} \left(\frac{1}{n+1} - \frac{1}{n} \right) = -\frac{m e^2}{\hbar} \frac{1}{(n+1)n} \approx -\frac{m e^2}{\hbar n^2} \quad (11)$$

which provides us evidently with a smaller electron momentum in state $n+1$ than in state n .

If the momentum in states $n+1$ and n should be conserved, the negative difference in (11) has to be compensated by the momentum supplement resulting from the orbit change, viz.

$$\Delta r_n = r_{n+1} - r_n, \quad (12)$$

in effect of which we obtain the momentum change

$$m \frac{\Delta r_n}{\Delta t} = m \frac{r_{n+1} - r_n}{\Delta t} = \frac{m}{\Delta t} \left[\frac{(n+1)^2 \hbar^2}{m e^2} - \frac{n^2 \hbar^2}{m e^2} \right] = \frac{(2n+1) \hbar^2}{\Delta t e^2} \approx \frac{2n \hbar^2}{\Delta t e^2}. \quad (13)$$

We postulate that the sum of (11) and (13) has to be zero, so

$$\Delta p_n + m \frac{\Delta r_n}{\Delta t} = 0 \quad (14)$$

or

$$-\frac{m e^2}{\hbar n^2} + \frac{2n \hbar^2}{\Delta t e^2} = 0. \quad (15)$$

It should be noted that the momenta balance postulated in (15) concerns solely the momenta values and not directions of the vectors. The requirement in (15) gives

$$\frac{m e^2}{\hbar n^2} = \frac{2n \hbar^2}{\Delta t e^2} \quad (16)$$

from which

$$\Delta t = \frac{2n^3 \hbar^3}{m e^4}. \quad (17)$$

This Δt is a time interval necessary to provide us with a conservation of momentum represented by the formula (14).

3. Comparison with the Joule-Lenz Law [8]-[14]

According to that law the time interval

$$\Delta t^{(JL)} \quad (18)$$

should approximately satisfy the formula

$$\Delta E \Delta t^{(JL)} = \frac{m e^4}{\hbar n^3} \Delta t^{(JL)} = h, \quad (19)$$

where the interval ΔE is taken from (10). In virtue of (19) we have

$$\Delta t^{(JL)} = \frac{2\pi\hbar^3 n^3}{me^4}. \quad (20)$$

In result we find that the Joule-Lenz emission time (20) differs from the time interval obtained in (17) solely by the factor of π .

4. Time Interval Connected with the Absorption of Energy Compared with the Time of the Emission Process

Both the absorption and emission processes are of a semiclassical nature. Therefore if in case of absorption we have a change of quantum indices

$$n \rightarrow n+1 \quad (21)$$

the result for the time interval Δt becomes equal to that for the case of emission between the levels

$$n+1 \rightarrow n \quad (22)$$

A different situation can be obtained when the emission change of states

$$n \rightarrow n-1 \quad (23)$$

is compared with the absorption change which is for example

$$n \rightarrow n+1. \quad (24)$$

In the case of (23) we have the momentum balance given by the condition

$$\begin{aligned} p^{n-1} - p^n + m \frac{r_{n-1} - r_n}{\Delta t} &= \frac{me^2}{\hbar} \left(\frac{1}{n-1} - \frac{1}{n} \right) + m \frac{(n-1)^2 - n^2}{\Delta t} \frac{\hbar^2}{me^2} \\ &= \frac{me^2}{\hbar} \frac{1}{(n-1)n} + \frac{m}{\Delta t} (-2n+1) \frac{\hbar^2}{me^2} = 0 \end{aligned} \quad (25)$$

from which we obtain the equation

$$\frac{me^2}{\hbar} \frac{1}{(n-1)n} = \frac{m}{\Delta t} (2n-1) \frac{\hbar^2}{me^2} \quad (26)$$

or

$$\Delta t = \Delta t_{em} = \frac{\hbar^3}{me^4} (2n-1)(n-1)n \cong \frac{\hbar^3}{me^4} 2n^3 \quad (27)$$

where the last step holds for the large n .

On the other hand, for the absorption process in (24), we have the balance

$$\begin{aligned} p^{n+1} - p^n + \frac{m}{\Delta t} \left[(n+1)^2 - n^2 \right] \frac{\hbar^2}{me^2} \\ = \frac{me^2}{\hbar} \left(\frac{1}{n+1} - \frac{1}{n} \right) + \frac{m}{\Delta t} (2n+1) \frac{\hbar^2}{me^2} = 0 \end{aligned} \quad (28)$$

from which

$$\frac{me^2}{\hbar} \frac{-1}{(n+1)n} = -\frac{m}{\Delta t} (2n+1) \frac{\hbar^2}{me^2} \quad (29)$$

or

$$\Delta t = \Delta t_{abs} = \frac{\hbar^3}{me^4} (2n+1)(n+1)n \approx \frac{\hbar^3}{me^4} 2n^3, \quad (30)$$

where the last step holds for the large n .

Evidently

$$\Delta t_{abs} - \Delta t_{em} = \frac{\hbar^3 n}{me^4} [(2n+1)(n+1) - (2n-1)(n-1)] = \frac{\hbar^3}{me^4} 6n^2. \quad (31)$$

For large n the difference (31) becomes only a small fraction of Δt in (27) or (30).

5. Electron Particle Moving in a One-Dimensional Potential Box and Its Transition Process

A reasoning similar to that developed for the electron in the hydrogen atom can be applied also in case of the electron particle moving in a one-dimensional potential box.

Let the box has the length L . The electron quantum states for the energy are [18]

$$E_n = \frac{n^2 \hbar^2}{8mL^2} = \frac{m}{2} v_n^2 \quad (32)$$

or

$$E_n = \frac{p_n^2}{2m} \quad (32a)$$

where v_n are the electron velocities:

$$v_n = \frac{nh}{2mL} \quad (33)$$

and p_n are the electron momenta

$$p_n = \frac{nh}{2L}. \quad (33a)$$

The electron can have the momenta in both motion directions along the box, so p_n in (33a) can have both positive and negative values. Let the energy emission produces the difference of momenta equal to that in the states n and $n-1$. This difference becomes:

$$\Delta p_n = p_n - p_{n-1} = \frac{\hbar}{2L}. \quad (34)$$

Due to the momentum conservation the difference (34) should be cancelled by the momentum

$$m \frac{2L}{\Delta t} \quad (35)$$

where Δt is the time interval of the electron motion along the box length, first from zero to L , next from L to zero. This implies

$$\frac{2L}{\Delta t} = v_n \quad (36)$$

which is the velocity of the particle. This velocity has been obtained from the electron energy in the formula (33).

The momentum balance provides us with the equation

$$-\Delta p_n + m \frac{2L}{\Delta t} = 0 \quad (37)$$

which gives

$$\Delta p_n = \frac{h}{2L} = m \frac{2L}{\Delta t} = mv_n \quad (38)$$

where the last step is due to (36).

From (38) we obtain the relation for Δt :

$$\Delta t = \frac{2L}{v_n} = 2L \frac{2mL}{nh} = \frac{4mL^2}{nh} \quad (39)$$

identical to that calculated from [19]:

$$\Delta t = \frac{\Delta q}{\partial E_n / \partial p_n} = \frac{2L}{p_n / m} = \frac{2L}{v_n} \quad (39a)$$

because the distance $\Delta q = 2L$. This result can be compared with the formula represented by the Joule-Lenz law, viz.

$$\Delta E \Delta t = h. \quad (40)$$

Since ΔE in (40) becomes for a free particle

$$\Delta E = E_n - E_{n-1} = \frac{n^2 - (n-1)^2}{8mL^2} h^2 \approx \frac{2nh^2}{8mL^2} = \frac{nh^2}{4mL^2} \quad (41)$$

we obtain from (41) the time interval

$$\Delta t = \frac{h}{\Delta E} = h \frac{4mL^2}{nh^2} = \frac{4mL^2}{nh}. \quad (42)$$

This is a result identical to Δt in (39) and (39a).

6. Size Limits of Mechanical Parameters Entering Simple Quantum Systems

Conservation of momentum suggests to calculate the limits of mechanical parameters like energy, velocity, distance and time entering the examined simple quantum systems. These limits can be obtained in an equally simple way.

Beginning with the hydrogen atom, the relativistic limit of the electron velocity leads to requirement

$$v_n \Big|_{n=1} = v_1 = \frac{e^2}{\hbar} < c \quad (43)$$

which gives

$$1 < \frac{\hbar c}{e^2} \cong 137. \quad (44)$$

A limit for the absolute value of the electron energy is represented by the formula

$$E_1 = E_{\text{kin}}^{(n=1)} = \frac{mv_1^2}{2} = \frac{me^4}{2\hbar^2} < mc^2 \quad (45)$$

which gives the relation

$$\frac{e^4}{2\hbar^2} < c^2. \quad (46)$$

This relation is equivalent to the formula

$$1 < \frac{2\hbar^2 c^2}{e^4} \cong 2 \cdot 137^2. \quad (47)$$

The result in (47) is an extension of that presented in (44).

The properties connected with the radius limit of the electron orbit which for the quantum number $n = 1$ is equal to

$$r_n = r_1 = \frac{\hbar^2}{me^2} \quad (48)$$

can be deduced from the virial theorem

$$2E_{\text{kin}} + E_{\text{pot}} = 0. \quad (49)$$

This gives the largest negative size of the electron potential energy equal to

$$E_{\text{pot}}^{(n=1)} = -\frac{e^2}{r_1} = -2E_{\text{kin}}^{(n=1)} \quad (50)$$

coupled with the largest kinetic electron energy value. From (50) we have the relation

$$\frac{e^2}{r_1} = 2E_{\text{kin}}^{(n=1)} = mv_1^2. \quad (51)$$

Since $v_1 < c$ we obtain

$$\frac{e^2}{r_1} < mc^2 \quad (52)$$

or

$$r_1 > \frac{e^2}{mc^2}. \quad (52a)$$

By taking into account (48), we obtain from (52a):

$$\frac{\hbar^2}{me^2} > \frac{e^2}{mc^2} \quad (53)$$

according to which

$$1 > \frac{e^4}{\hbar^2 c^2} \cong \frac{1}{137^2}. \quad (54)$$

There remains still the condition satisfied by the time interval Δt of the electron transition. We have the formula

$$\Delta r_n = r_{n+1} - r_n = \frac{(n+1)^2 - n^2}{me^2} \hbar^2 = \frac{2n+1}{me^2} \hbar^2 \quad (55)$$

which gives the change of the radius of the electron orbit. For two neighbouring quantum numbers, n and $n+1$, we have

$$\frac{\Delta r_n}{\Delta t} = \Delta v_n, \quad (56)$$

where the velocity change satisfies the condition:

$$|\Delta v_n| = \left| \frac{e^2}{(n+1)\hbar} - \frac{e^2}{n\hbar} \right| = \frac{n+1-n}{(n+1)n} \frac{e^2}{\hbar} \approx \frac{1}{n^2} \frac{e^2}{\hbar} < \frac{e^2}{\hbar} < c \quad (57)$$

where the last steps hold for any large n . An alternative formula for the last step in (57) is:

$$\frac{\hbar}{e^2} > \frac{1}{c}. \quad (58)$$

We obtain

$$\Delta t = \frac{\Delta r_n}{|\Delta v_n|} = \frac{2n+1}{me^2} \hbar^2 \frac{n^2 \hbar}{e^2} \approx \frac{2n^3 \hbar^3}{me^4}. \quad (59)$$

For very low n , say $n=1$, relation (59) for Δt becomes

$$\Delta t = \frac{3\hbar^3}{me^4} > \frac{3\hbar}{mc^2} \quad (60)$$

on condition (58) does hold.

A similar reasoning can be performed for the electron particle moving in a one-dimensional potential box. In the first step, from the requirement that the kinetic energy on the quantum level $n=1$ is smaller than the rest energy of the electron particle, we obtain the formula:

$$E_{n=1} = \frac{\hbar^2 l^2}{8mL^2} = \frac{mv_1^2}{2} < mc^2 \quad (61)$$

which gives the requirement

$$v_1 < 2^{1/2} c. \quad (62)$$

Therefore, with the aid of the first equation given in (61), we obtain:

$$v_1 = \frac{\hbar}{2mL} < 2^{1/2} c. \quad (63)$$

In effect it should be

$$\frac{\hbar}{2^{3/2} mc} < L. \quad (64)$$

The limits obtained for L and v_1 can provide us with the size of the interval Δt according to the formula

$$\Delta t v_1 \cong 2L, \quad (65)$$

so a maximal size of the interval Δt for the electron oscillation in the box be-

comes:

$$\Delta t \cong \frac{2L}{v_1} = \frac{2L \cdot 2mL}{h} = \frac{4mL^2}{h}. \quad (66)$$

Another approach applies L calculated in (64) and v_1 in (63):

$$\Delta t \cong \frac{2L}{v_1} \cong \frac{h}{2^{1/2} mc} \cdot \frac{1}{2^{1/2} c} = \frac{h}{2mc^2}. \quad (67)$$

7. Summary

In the paper, the transition time between the nearest quantum energy levels is examined for the case of the Bohr hydrogen atom and the electron particle enclosed in a one-dimensional potential box. In both cases, the calculations are based on the assumption that the electron momentum in course of the electron transition is conserved.

It is found that the time intervals of the electron transitions obtained in this way are much similar to those calculated on the basis of the Joule-Lenz law for the energy emission: in the case of hydrogen, a difference between the results of both kinds is represented by a constant factor π ; for the electron particle moving in a one-dimensional potential box there exists an identity of the results for Δt calculated in both ways. The limiting sizes of the mechanical parameters characterizing the quantum states in the systems considered in the paper have been also calculated.

It should be noted that the electron transition time Δt considered in the paper does not correspond, in general, to the reciprocal time of the frequency $\nu_{n+q,n}$ joining the energy difference of two quantum states E_{n+q} and E_n by the formula

$$E_{n+q} - E_n = h\nu_{n+q,n} = \frac{h}{T_{n+q,n}}. \quad (68)$$

Only for the case of

$$q = 1 \quad (69)$$

we have proved the formula

$$T_{n+1,n} = \Delta t; \quad (70)$$

see [8] [9] [10] [11].

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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