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Alternate Cooling Model vs Newton's Cooling

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Abstract

It is customary to apply Newton's cooling as the standard model investigating the temperature profile of a hot substance exposed to a cool ambient. The rate of change of temperature in Newton's model is simplistically related to linear-temperature difference of the two e.g. [1]. In our research flavored investigation, we consider a fresh model, cooling that depends to the difference of temperature-squared conducive to similar results. Utilizing a Computer Algebra System (CAS), especially *Mathematica* [2] we show the equivalency of the two.

Keywords

Newton's Cooling, Thermo-Physics, Computer Algebra System (CAS), *Mathematica*

1. Introduction, Motivations and Goals

We consider a thermo-physics problem that its proposed solution augments the traditional methodology routine providing a variant analyzing the cooling process in general. The proposed practical problem poses as: How long it takes a hot object exposed to cold ambient reach a certain temperature? The traditional solution of this problem relies on Newton's cooling as discussed in textbooks and frequently online published references e.g. [1] [3] [4]. The rate of change of temperature is set in proportion to the linear-temperature difference of the time varying temperature of the hot object and the ambient. The simplicity of the model is appealing. Its mathematical analysis conducive to a reasonable temperature profile is convincing as well. Textbooks and published articles are flooded with hypothetical numeric man-made examples. The lack of correspondence between the model and data makes the model less unique. In other words, alternative cooling mechanisms may also be proposed. In this research flavored article, we have considered a fresh logical model. As it is shown in the next section the output of the analysis is as good as the Newton's cooling. Although on the face our model mathematically is different from the Newton's model but in the conclusions section, we show within a reasonable approximation these two may be considered comparable.

2. Physics of the Problem and Its Proposed Solution

We begin with the well-known Newton's cooling model. The scenario is a hot object with initial given temperature is exposed to a colder surrounding at lower constant temperature. The temperature difference of the two runs the heat flow changing the temperature over time. To determine the temperature profile, it is written,

Model 1, Newton's cooling

$$\frac{\mathrm{d}T(t)}{\mathrm{d}t} = -k\left[T(t) - T_a\right], \tag{1}$$

where $T(t)$ is the time-dependent temperature of the hot object, T_a is the cooler constant ambient temperature and k is the proportionality constant. Straight forward solution of this trivial ODE without utilizing a CAS is,

$$T(t) = T_a + (T_i - T_a)\mathrm{e}^{-kt}, \tag{2}$$

here, T_i is the initial temperature of the hot object.

The value of the k may be determined e.g., utilizing Equation (2) by assigning temperature to the hot object at a certain intermediate state. This is a routine procedure practiced in references. For instance, the hot object is a brownie or a pie with initial temperature hot out of the oven at 350˚F. The ambient temperature, e.g., kitchen is at 75˚F. Furthermore, we envision the brownie within 15 minutes cools to a 150˚F. Equation (2) utilizing this data yields the k-value of k = 0.086 minutes^{-1}. To determine the time needed to cool the brownie to eatable temperature, e.g., 80˚F, we set $T(t)$ = 80 and applying *Mathematica* solving Equation (2) we arrive at t = 46.6 minutes.

values={Ti→350, Ta→75, Tf→80};

rightSide1=(Ta + (Ti − Ta)E^{-k1t})/.k1→0.086/.values;

t1=Solve[rightSide1==Tf/.values,t];

{{t→46.59}}.

So, the time needed to wait to eat the brownie is 46.6 minutes. The **values** is the list of the mentioned temperatures needed to run the *Mathematica* code. To furthering the analysis, it is insightful plotting the temperature profile conducive to visual display of the temperature profile and the graphic solution of the Equation (2).

plotT1=Plot[{80, rightSide1}, {t, 0, 120}(*, AxesOrigin→{0, 60}*), PlotStyle;

→{{Black, Dashing[0.01]}, Black}, AxesLabel;

→{"t(minutes)", "Tem(F)"}, GridLines;

→Automatic, PlotRange→{0,All}].

Figure 1 displays the exponential decay temperature of the cooling object. It also shows the crossing of the expression given by Equation (2) with the aimed 80 degrees temperature. The abscissa of the intersection is about 46 minutes. The tail of the profile is flat with the value approaching to the ambient ultimate temperature.

The numeric waiting period, 46.6 minutes is the result based on the applied assumed mentioned intermediate state namely, $\{15\ \text{minutes}, 150^{\circ}\text{F}\}$. Of course, one might assign different set of temperature and time to the intermediate state leading to different waiting periods. We craft a robust code capable of striving the same final desired temperature, Tf utilizing various intermediate states. These are tabulated in Table 1.

The second column is the desired targeted common Tf, the third column is the coordinate of the paired intermediate states, {time, temperature} yielding evaluating the k-value. And the fourth column is the required waiting period. As shown and logically making sense the waiting period (fourth column) increases as the time coordinate of the intermediate state increases with its paired lowered temperature.

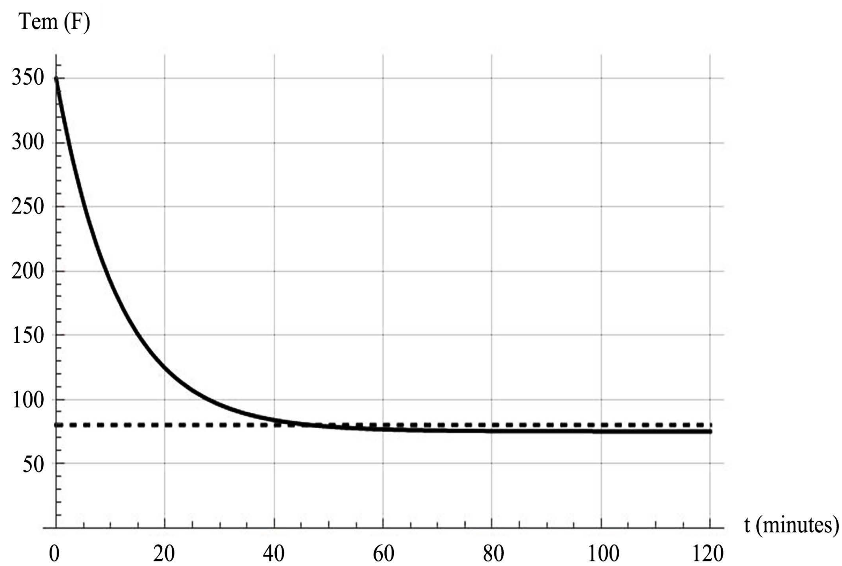

Figure 1. Plots of the temperature profile of cooling of the hot object (solid black), and the constant final aimed temperature (dashed black), respectively.

Table 1. Waiting periods and their associated intermediate states.

	80˚ F	data	time (minutes)
1	80	15,150	46.26
2	80	16,148	48.34
3	80	17,143	48.75
4	80	18,139	49.47
5	80	20,130	49.79

Model 2, A fresh Idea.

Intrigued by the simplicity of the Newton's cooling model we propose a variant cooling mechanism. The lack of data justifying the validity of the Newton's model sets opportunities considering alternate models. We consider a scenario where the rate of change of temperature is in proportion to the squared-temperature difference of the specimen and the surrounding.

$$\frac{\mathrm{d}T(t)}{\mathrm{d}t} = -k\left[T(t)^2 - T_a^2\right], \tag{3}$$

Notations are the same as Equation (1). At the first attempt applying *Mathematica* to solving Equation (3) yields a complicated symbolic output; it is a hyperbolic trigonometric function! However, by inspection it is obvious that it can be manipulated to a solvable expression. Replacing the RHS of Equation (3) applying partial fraction we write,

$$\frac{1}{T^2 - T_a^2} = \frac{1}{2T_a}\left(\frac{1}{T - T_a} - \frac{1}{T + T_a}\right), \tag{4}$$

therefore, each term of the RHS may be integrated directly yielding an expression for the k, and T(t), namely,

$$k = -\frac{1}{t}\frac{1}{2T_a}\ln\left[\frac{T(t) - T_a}{T(t) + T_a}\frac{T_i + T_a}{T_i - T_a}\right], \tag{5}$$

$$T(t) = T_a\frac{1 + \beta \mathrm{e}^{-\alpha t}}{1 - \beta \mathrm{e}^{-\alpha t}}, \tag{6}$$

where the two constants are,

$$\{\beta, \alpha\} = \left\{\frac{T_i - T_a}{T_i + T_a}, 2kT_a\right\}, \tag{7}$$

Utilizing the same input data as used in Model 1, Equation (5) yields the value for $k = 0.00029\left(\text{minutes} \cdot {}^{\circ}\text{F}\right)^{-1}$. Applying the k value, we then determine the values of the two constants given by Equation (7) yielding the temperature profile, Equation (6). Solving Equation (6) applying the same coordinate used in Model 1 applying *Mathematica* we arrive at t = 67.8 minutes.

T=Solve[Evaluate[Ta(1 + βE^{−αt})/(1 − βE^{−αt})==Tf/.values], t].

{t→67.81}.

This shows according to our model it will take 21.2 minutes longer than the Model 1 to get to the target temperature.

It is insightful plotting the temperature profile of the Models together to observe the similarities and the differences of the models. This is shown in Figure 2.

It appears our model, Model 2, has a little sharper decay rate at the beginning and later on it decays somewhat softer vs. the Model 1. Overall, the temperature profile as shown has the correct mathematical behavior, it decays exponentially, and its tail asymptotically reaches the ambient temperature as expected. This is an important feature. More on this in the Conclusions. These two models also

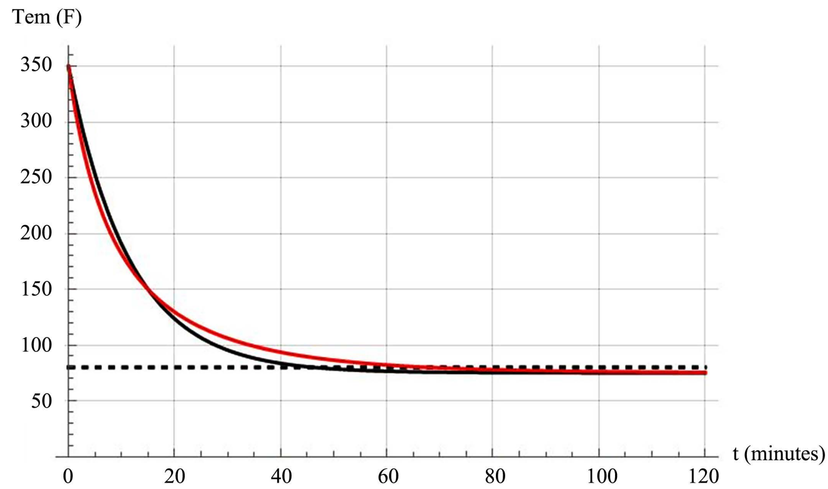

Figure 2. The solid red curve is the temperature profile of Model 2, Equation (5). The black curve is the profile of Model 1, Equation (2).

have the same common coordinate at the imposed intermediate state shown by the crossing coordinate of the two. **Figure 2** may be utilized reading off the intersection of the temperature of Model 2 with the targeted Tf, the dashed black line yielding to the approximate time given by the code of the last paragraph.

3. Discussions and Conclusions

We began by considering the Newton's cooling mechanism as being the "gold standard" of cooling process within the context of the problem at hand. In this model the rate of change of temperature of a hot specimen is set in proportion to the linear-temperature difference between the specimen and the ambient. Solution of the simple ODE describing the process is conducive to temperature profile, *i.e.*, the graph of the temperature of the hot specimen vs. time. Literature search reveals the same cooling mechanism identically is applied to the same physics problem at least twice. What distinguishes the difference of these references is the applied CAS to solving the implied ODE! Reference [5] applies *Mathematica* while [6] utilizes Maple! However, neither one of these references as well as e.g. [1] [3] [4] suggest a fresh, new physical cooling mechanism. This "gold standard", The Newton's cooling mechanism needs to be tested against data! The lack of data makes the mechanism vulnerable to establishing basis to proposing fresh mechanisms. Our current proposal falls in the latter category.

Inspired by the simplicity of Newton's process we assume the rate of change of temperature of the specimen is in proportion to the squared-temperature difference of the two.

As mentioned in the text we think the compatibility of these two models should have been expected. The reason is the RHS of our model given by Equation (3) can be written as,

$$\left[T(t)-T_a\right]\left[T(t)+T_a\right] \tag{8}$$

The first term of Equation (8) is the Newton's cooling model and the second term may be considered as an "average". To a degree and for the sake of argument if we assume the "average" does not fluctuate severely utilizing our model should yield to about the Newton's result, as it does. Although tempted to apply a CAS to solve the needed equation, *i.e.*, Equation (3) as we have shown with straight forward algebraic manipulation, we were able to solve the equation longhand deducing a simple analytic equation describing $T(t)$, *i.e.*, the temperature profile. Our investigation shows alternate potential models such as, $\sqrt{T(t)}-\sqrt{T_a}$ and $T(t)^n - T_a^n$ for $n > 2$ yield to non-physical temperature profiles. Also, we note that a common assumption about the unchangeable temperature of the ambient because of its large heat capacity is just. Our proposed model gives an acceptable temperature profile, yielding to agreeable result as the Newton's model. Especially, the temperature profile decays exponentially, and its tail asymptotically reaches the ambient temperature. The other tested mentioned models lack these features. In our analysis we have shown simple longhand calculation yields the needed output. The numeric computation and graphs are carried out utilizing *Mathematica*. To this end interested readers may find [7] [8] resourceful.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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