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# Investigation of Quantum Entanglement through a Trapped Three Level Ion Accompanied with Beyond Lamb-Dicke Regime

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## Abstract

In this study, our goal is to obtain the entanglement dynamics of trapped three-level ion interaction two laser beams in beyond Lamb-Dicke parameters. Three values of LDP, $\eta = 0.09$, $\eta = 0.2$ and $\eta = 0.3$ are given. We used the concurrence and the negativity to measure the amount of quantum entanglement created in the system. The interacting trapped ion led to the formation of phonons as a result of the coupling. In two quantum systems (ion-phonons), analytical formulas describing both these measurements are constructed. These formulas and probability coefficients include first order terms of final state vector. We report that long survival time of entanglement can be provided with two quantum measures. Negativity and concurrence maximum values are obtained N = 0.553 and for LDP = 0.3. As a similar, the other two values of LDP are determined and taken into account throughout this paper. For a more detailed understanding of entanglement measurement results, "contour plot" was preferred in Mathematica 8.

## Keywords

Entangled State, Trapped Three-Level Ion, Lamb-Dick Parameter, Rabi Frequency, Quantum Measures

## 1. Introduction

Quantum states as usual are evident in itself with laws in quantum information theory [1]. Entangled states are the proper kind of quantum correlation between two quantum system. Entanglement is an attractive physical phenomenon in which the overlap of two separable states is can be entangled state with photons. The widely read Einstein, Podolsky and Rosen (EPR) paper, contrary to what is known, has actually been published to criticize quantum mechanical laws [2]. In the same year, N. Bohr published a paper [3] with alike this EPR paper. The prominent article presented the entanglement with conversations on quantum theory. For the quantum theory, 1935 was an interesting year. In Erwin Schrödinger's article in Naturwissenschaften introducing "Verschränkung", where he advocated quantum theory [4].

Quantum entanglement has dramatically increased during the last two decades due to the emerging field of quantum information theory [5]. Entanglement is one of important features of quantum theory with no classical analog and quantum computing. Quantum measurement is discussed a local physical process [6]. Nonclassical nature of quantum entanglement has been long recognized [2] [7]. There has been an extensive research in the field of quantum communication which yields a variety of methods to distribute bipartite entanglement. It has reported an applying entanglement created the exchange interaction for many quantum information processing [8]. The maximally entangled states can be modelled physically by the states trapped atomic ions [9] [10] [11]. Trapped ions are between the most attractive implementations of quantum bits for applications in quantum information processing, due to their long coherence times [12]. Ions confined in a linear radio-frequency (Paul) trap are cooled to form a spatial array. Hilbert space of the composite quantum system considered in this paper can be written as

$$C^{d} = C^{d_{ION}} \otimes C^{d_{p}} \tag{1}$$

where $d_{ION} = 3$ and $d_{p} = 4$ represent the dimensions at three-level ion and photons, respectively. We characterize quantum correlations using concurrence ($C$) [13], negativity ($N$) [14], and quantum entropy [11] [15] [16] for time dependent interaction of a three-level trapped ion with two laser beams. Trapped ions systems are important for the entangled states Works. Quantum entanglement measurements are used to determine any known state is separable or entangled. Therefore, $C$ and $N$ are offered for pure states [17] [18]. $N$ and $C$ are an entanglement measure that a useful characterization in quantum information, commonly in ionic system. Product base and entangled base are shown generalization of Schmidt coefficients.

The deep Lamb-Dicke regime (LDR) described with LDP of small, $\eta \ll 1$. LD limit is not accordingly established with common experiments [19]. Such a way experiments act in named as beyond LDR here $\eta < 1$, for example $\eta = 0.2$ [20], such as this work. Entanglement of qutrit states [10] are testified by a quantum system for lower order terms of density matrix.

We report analytical results of quantum entanglement for system via $N$ and $C$ for the LDR and 12-Dimensional (D) of Hilbert space. We focus the quantum correllations in $N$ *and* $C$ *[10]* [11] [16] with respect to the total and the reduced density matrix. With respect to Ref. [9], we illustrated these evolutions of $N$ for trapped ion-phonons system.

The rest of the study is coordinated as follows. Section 2 discusses growth for two unentangled qubits and analitical solutions in the quantum system. Section 3 describes how to obtain highly $N$ and $C$ of two quantum systems by the LDR. The results and comments are given in Section 4.

## 2. A Quantum Solution of Ion-Phonons System and Its Theory

For section 2, flow chart is:

- In this section, the Hamiltonian and its dynamics are given between Equations (1)-(5).
- In $\Lambda$ configuration, U transformation matrix processes evolved in Equations (6)-(11).
- The initial state of the system has written by Equations (12)-(22).
- Equation (3) is the final state of the ion-phonons system.
- In Equations (24)-(32), the probability applitudes are given.

We propose a trapped atomic ion interacting with two laser beams. In this system, the Hilbert space dimension is 12. The quantum dynamics of trapped ion-phonons system is emerged by previous investigation [9] [21] [22]. The Hamiltonian of two quantum systems is $H_{total} = H_{Ion} + H_1 + H_2$, and $H_{Ion}$ indicates Hamiltonian of system ( $\hbar = 1$ ):

$$H_{Ion} = \omega_g |g\rangle\langle g| + \omega_r |r\rangle\langle r| + \omega_e |e\rangle\langle e| + \frac{p_x^2}{2m} + \frac{1}{2} m\upsilon^2 x_{Ion}^2 . \tag{2}$$

The e-level energy is $\omega_e = 0$, r-level is $\omega_r$, and g-level is $\omega_g$. The reason for $\omega_e$ to be zero is the following: As can be seen in Equation (12), the excited level $|e\rangle$ is removed in the first quantum state. Here $H_1$ and $H_2$ are Hamiltonians of these interactions for *excited-ground* and *excited-raman*:

$$H_{e-g} = H_1 = \frac{\Omega}{2} e^{i(k_1 x_{Ion} - \omega t)} |e\rangle\langle g| + h.c. \tag{3}$$

$$H_{e-r} = H_2 = \frac{\Omega}{2} e^{i(-k_2 x_{Ion} - \omega t)} |e\rangle\langle r| + h.c. \tag{4}$$

where $\hbar = 1$, $p_x$ and $x_{Ion}$ are momentum and the *x*-component of position of ion center of mass movement. The movement of ion in the system is along the *x*-axis (one-D). Atomic levels are shown: $|e\rangle \to$ trapped ion excited level, $|r\rangle \to$ raman level, and $|g\rangle \to$ ground level. Trapped ionmass center is given with standard harmonic-oscillator of $H_{ion}$ in $p_x = i\sqrt{\frac{1}{2} m\upsilon}\left(a^+ - a\right)$ and $x_{Ion} = \sqrt{\frac{1}{2m\upsilon}}\left(a + a^+\right)$. Here, $a$ is annihilation operator and $a^+$ creation operator for two laser beams. Laser frequencies are $\omega_1$ and $\omega_2$, and Rabi frequency is $\Omega$. Trapped ion-phonons total Hamiltonian is written ( $\hbar = 1$ ):

$$H = \left( \frac{\Omega}{2} e^{i\eta\left(a^+ + a\right)} |e\rangle\langle g| + \upsilon a^+ a - \delta |e\rangle\langle e| + \frac{\Omega}{2} e^{-i\eta\left(a + a^+\right)} |e\rangle\langle r| \right) + h.c., \tag{5}$$

here, LDP is $\eta = k/2m\upsilon$, $\upsilon$ is trap frequency of harmonic, and delta function is $\delta = \upsilon\eta^2$. We have taken the base vectors as follow:

$$|e\rangle=\begin{pmatrix}1\\0\\0\end{pmatrix},\ |r\rangle=\begin{pmatrix}0\\1\\0\end{pmatrix},\ |g\rangle=\begin{pmatrix}0\\0\\1\end{pmatrix} \tag{6}$$

In this study, important transformed Hamiltonian is $\tilde{H}=U^{+}HU$. Hamiltonian in Equation (5) is found after transmission action. $\Lambda$ model is given by a cascade $\Xi$ scheme in two phonons. Ion-two phonons system was covered by unitary transformation. Matrix of transformation, namely $U$ is performed [21],

$$U=\frac{1}{2}\begin{pmatrix}0 & \sqrt{2} & \sqrt{2}\\ -\sqrt{2}B[\eta] & B[\eta] & -B[\eta]\\ \sqrt{2}B[-\eta] & B[-\eta] & -B[-\eta]\end{pmatrix}. \tag{7}$$

Here displacement operators of Glauber, $B(\eta)=e^{\left(i\eta\left(a+a^{+}\right)\right)}$, $B(-\eta)=e^{\left(-i\eta\left(a+a^{+}\right)\right)}$ are achieved. $\tilde{H}$ is performed $\tilde{H}=\tilde{H}_0+\tilde{V}$, here

$$\tilde{H}_0=\upsilon\left(|r\rangle\langle r|-|g\rangle\langle g|\right)+\upsilon\eta^2+\upsilon a^{+}a \tag{8}$$

$$\tilde{V}=-i\frac{\sqrt{2}\delta\eta}{2}\left(a^{+}|e\rangle\langle r|-a^{+}|e\rangle\langle g|+h.c.\right). \tag{9}$$

In our system, the LDR is performed between the values 0.09 and 0.3 of LDP. By using unitary transformation method [21], an initial state $|\psi(0)\rangle$ is written in following form

$$|\psi(t)\rangle=U_0^{+}Ue^{-it\tilde{H}_0}K(t)U^{+}|\psi(0)\rangle, \tag{10}$$

where $K(t)$ is typical vector for time-independent Hamiltonian; $e^{\left(-it\tilde{H}_0\right)}$ is the exponencial function, and $U_0=\exp\left(-i\omega t|e\rangle\langle e|\right)$ is the transformation matrix [21]. Trapped ion two phonon states system acts for $\Lambda$ scheme. The propagator is performed

$$K(t)=\frac{1}{2}\begin{pmatrix}Cos(\Lambda t) & -\varepsilon Sa^{+} & -\varepsilon Sa\\ \varepsilon aS & 1+\varepsilon^2 aGa^{+} & \varepsilon^2 aGa\\ \varepsilon a^{+}S & \varepsilon^2 a^{+}Ga^{+} & 1+\varepsilon^2 a^{+}Ga\end{pmatrix}, \tag{11}$$

here $\varepsilon=\upsilon\eta/\sqrt{2}$, $\Lambda=\varepsilon\sqrt{2a^{+}a+1}$, $G=\frac{\cos(\Lambda t)}{\Lambda^2}$ and $S=\frac{\sin(\Lambda t)}{\Lambda}$. We take $\upsilon=10^6\,\text{Hz}$ and $\omega_{eg}=5\times10^{14}\,\text{Hz}$ for frequencies. In the system, we take $a=1$ and $b=0.005$. Normalization condition of ion is certainly $\left[\frac{1}{\sqrt{2}}\right]^2+\left[-\frac{1}{\sqrt{2}}\right]^2=1$, and normalization condition of two phonons is $\|a\|^2+\|b\|^2=|1|^2+|0.005|^5\cong1$, approximately. So, the earliest of trapped ion-phonon states system is given as

$$|\psi(0)\rangle=\frac{1}{\sqrt{2}}\left[|g\rangle-|r\rangle\right]\otimes\left(a|0\rangle+b|1\rangle\right), \tag{12}$$

here, the phonon levels are $\langle 0|=(1,0)$, and $\langle 1|=(0,1)$. $a$ and $b$ are the probability amplitudes of the first and the second phonon. New equation for ion-two phonons is performed as

$$\left|\psi(0)\right\rangle = \frac{1}{\sqrt{2}}\left[\left|g\right\rangle - \left|r\right\rangle\right] \otimes \left(\sum_{n=0}^{\infty} F_n(b)\left|n\right\rangle\right). \tag{13}$$

It is used by $\eta^0$ and $\eta^1$ are zero and first-order indication of LDP, respectively. Beside, both of them, $\eta^2$ and $\eta^3$ are ignored. Ion-phonons system is evolved to an initial unentangled state,

$$\left|\psi K(t)\right\rangle = \left|\tilde{\psi}(0)\right\rangle = U^{+}\left|\psi(0)\right\rangle = \sum_{\sigma,m} N_{\sigma,m}(t)\left|\sigma,m\right\rangle. \tag{14}$$

In Equation (12), our system is produced in respect of $\sum_{\sigma,m} N_{\sigma,m}(t)\left|\sigma,m\right\rangle$. As a result of advanced mathematical transformations between Equation (10)-(14), 12 of significiant coefficients are

$$scN_{e0}(t) = \left[\cos\left(\sqrt{\frac{1}{2}}t\right) + \frac{\eta i}{\sqrt{2}}\sin\left(\sqrt{\frac{1}{2}}t\right)\right]\exp\left[-ti/\eta\right] \tag{15}$$

$$scN_{e1}(t) = b\cos\left(\sqrt{\frac{3}{2}}t\right)\exp\left[-ti/\eta\right] \tag{16}$$

$$scN_{e2}(t) = -\frac{\eta i}{\sqrt{5}}\sin\left(\sqrt{\frac{5}{2}}t\right)\exp\left[-2ti/\eta\right] \tag{17}$$

$$scN_{r0}(t) = \frac{b}{\sqrt{3}}\sin\left(\sqrt{\frac{3}{2}}t\right)\exp\left[-ti/\eta\right] \tag{18}$$

$$scN_{r1}(t) = \frac{\eta i}{\sqrt{2}}\left[\frac{3}{2} + \frac{2}{5}\cos\left(\sqrt{\frac{5}{2}}t\right)\right]\exp\left[-2ti/\eta\right] \tag{19}$$

$$scN_{g1}(t) = \left[\sin\left(\sqrt{\frac{1}{2}}t\right) - \frac{\eta i}{\sqrt{2}}\cos\left(\sqrt{\frac{1}{2}}t\right)\right]\exp\left[-ti/\eta\right] \tag{20}$$

$$scN_{g2}(t) = b\sqrt{\frac{2}{3}}\sin\left(\sqrt{\frac{3}{2}}t\right)\exp\left[-ti/\eta\right] \tag{21}$$

$$scN_{g3}(t) = -\frac{\sqrt{3}}{5}\eta i\left[1 - \cos\left(\sqrt{\frac{5}{2}}t\right)\right]\exp\left[-2ti/\eta\right] \tag{22}$$

and four of significiant coefficients are zero:
$scN_{e3}(t) = scN_{r2}(t) = scN_{r3}(t) = scN_{g0}(t) = 0$. For Equations from (15) to (22), index $\sigma$ is positioned in the states of atomic $(g,r,e)$, index $m$ is positioned by vibrational numbers $(0,1,2,3)$. Vibrational phonon states are located by a Hilbert 4D-space $H_{phonons}$ and subsystem of trapped ion-phonons is located in a Hilbert 3D-space $H_{Ion}$. Thus, two quantum systems are in Hilbert 12D-space. Here, $t$ is dimensionless and scaled with $\upsilon\eta$. What does $\upsilon\eta$ dimensionless mean? Accordingly in **Figure 1**, time 1 equals to 5 ms (mikrosecond). The mathematical calculation is as follows; for $\eta = 0.2$, $\upsilon\eta = 0.2\times10^{6}$, $\frac{1}{\upsilon\eta} = 5\times10^{-6} = 5\ \text{ms}$. The state vector is

$$\left|\psi_{final}(t)\right\rangle = \sum_{m=0}^{3}\left(A_m(t)\left|e,m\right\rangle + B_m(t)\left|r,m\right\rangle + C_m(t)\left|g,m\right\rangle\right). \tag{23}$$

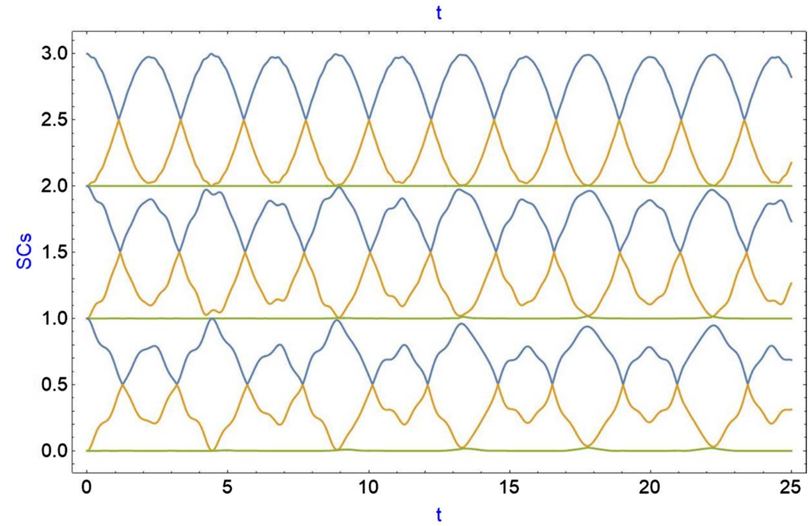


**Figure 1.** The time dependence of Schmidt coefficients, $\mu_1, \mu_2$ and $\mu_3$ for three LDP. Upper, middle and lower curves are for three LDP. The third SC, $\mu_3$ is green and small. Therefore, there are 9 functions in the figure. *t* is dimensionless scaled by $\upsilon\eta$. The earliest state of trapped ion and two phonons system is $\psi(0) = \frac{1}{\sqrt{2}}(|g\rangle - |r\rangle) \otimes (a|0\rangle - b|1\rangle)$ for $a = 1$, $b = 0.005$. These coupling parameters are written for $\upsilon = 1\,\text{MHz}$ and $\omega_{eg} = 5\times10^{14}\,\text{Hz}$.

The coefficients $A_m(t), B_m(t)$ and $C_m(t)$ are shown by state vector amplitudes of $\Lambda$ and $\Xi$ models. 12 of the probability amplitudes of the vector are

$$A_m(t) = \frac{1}{\sqrt{2}} e^{-i\omega t/\upsilon\eta} \left[ N_{rm}(t) + N_{gm}(t) \right], (m = 0,1,2,3), \tag{24}$$

$$B_0(t) = -\frac{1}{\sqrt{2}} N_{e0}(t) + \frac{1}{2} N_{r0}(t) - \frac{i\eta}{2} N_{g1}(t) \tag{25}$$

$$B_1(t) = -\frac{i\eta}{\sqrt{2}} N_{e0}(t) - \frac{1}{2} N_{r1}(t) + \frac{1}{2} N_{r1}(t) - \frac{1}{2} N_{g1}(t) \tag{26}$$

$$B_2(t) = -\frac{1}{\sqrt{2}} N_{e2}(t) - \frac{i\eta}{\sqrt{2}} N_{g1}(t) - \frac{1}{2} N_{g2}(t) \tag{27}$$

$$B_3(t) = -\frac{1}{2} N_{g3}(t) \tag{28}$$

$$C_0(t) = \frac{1}{\sqrt{2}} N_{e0}(t) + \frac{1}{2} N_{r0}(t) + \frac{i\eta}{2} N_{g1}(t) \tag{29}$$

$$C_1(t) = -\frac{i\eta}{\sqrt{2}} N_{e0}(t) + \frac{1}{2} N_{r1}(t) + \frac{1}{2} N_{r1}(t) - \frac{1}{2} N_{g1}(t) \tag{30}$$

$$C_2(t) = \frac{1}{\sqrt{2}} N_{e2}(t) + \frac{i\eta}{\sqrt{2}} N_{g1}(t) - \frac{1}{2} N_{g2}(t) \tag{31}$$

$$C_3(t) = -\frac{1}{2} N_{g3}(t) \tag{32}$$

here $\omega_{eg}$ is frequency e-g levels and $\omega = \omega_{eg} - \eta^2\upsilon$ for Equation (24). *i* is complex number, and i is ion index.

We plotted $N$ and $C$ of two quantum systems as $l \otimes l'(l \leq l')$ in **Figures 2-7**

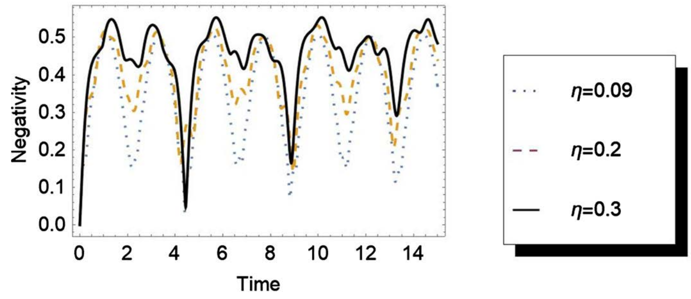


**Figure 2.** The time dependence of negativity, for three $\eta = 0.09$, $\eta = 0.2$ and $\eta = 0.3$. $t$ is dimensionless and scaled with $\upsilon\eta$, other assumptions parameters are the same as **Figure 1** in the system.

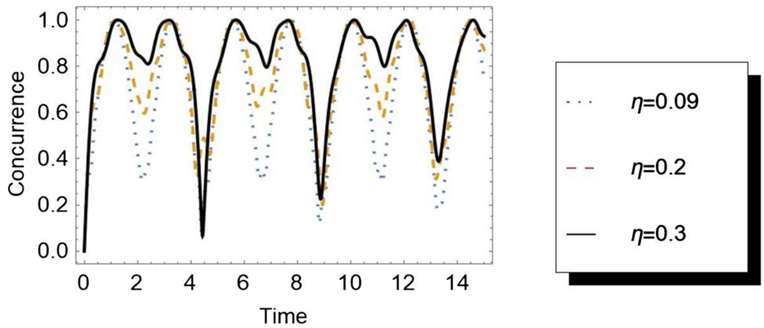


**Figure 3.** The time dependence of concurrence, for three $\eta$. Other assumptions parameters are the same as **Figure 1** in the system.

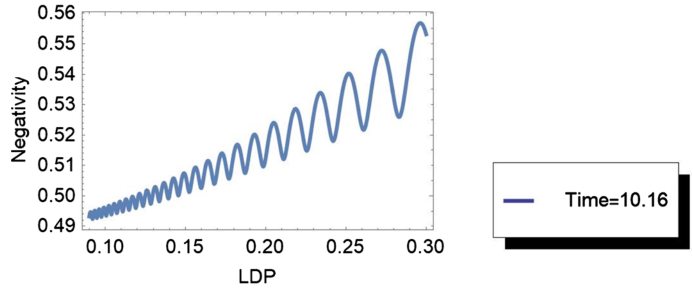


**Figure 4.** The LDP evolution of $N$ is given $t = 10.16$ s. Other assumptions parameters are the same as **Figure 1** in the system.

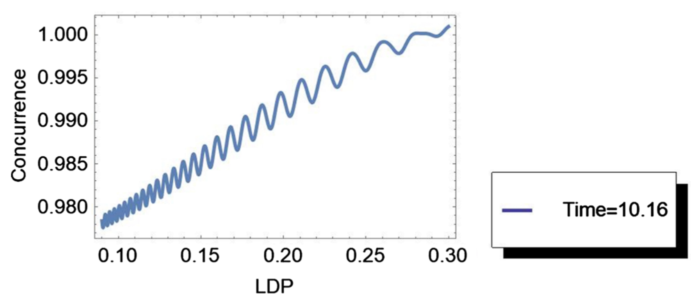


**Figure 5.** The LDP evolution of $C$ is given $t = 10.16$ s. Other assumptions parameters are the same as **Figure 1** in the system.

and Table 1. We found that final state vector $\left|\psi_{final}(t)\right\rangle$ is superposition of twelve function in Equations (24)-(32).

## 3. Two Measurements, Beyond LDR and Discussion

Hilbert spaces are $l=4$ for two-phonons, $l'=3$ for ion. It is used a simplified density matrix $\rho_{ion}=Tr_{phonon}\left(\rho_{ion-p}\right)$ by Equation (33). Fully density matrix $\rho_{ion-p}$ is performed with $12\times 12$ matrix with respect to the bases $|i,p\rangle$. With tracing, $3\times 3$ -simplified density matrix, $\rho_{ion}$ is performed

$$\rho_{ion}=Tr_p\left(\rho_{i-p}\right)=\begin{pmatrix} Tr|e\rangle\langle e| & Tr|e\rangle\langle r| & Tr|e\rangle\langle g| \\ Tr|r\rangle\langle e| & Tr|r\rangle\langle r| & Tr|r\rangle\langle g| \\ Tr|g\rangle\langle e| & Tr|g\rangle\langle r| & Tr|g\rangle\langle g| \end{pmatrix} \tag{33}$$

where diagonal terms, $|e\rangle\langle e|$, $|r\rangle\langle r|$ and $|g\rangle\langle g|$ are a $4\times 4$ -matrix. For help to Equation (32), fully density matrix of two quantum system is written as:

$$\rho_{ion-phonon}=\left(|Z\rangle\langle Z|\right) \tag{34}$$

where $|Z\rangle\langle Z|$ is a $12\times 12$ -square matrix and Hilbert 12-space in qauntum mechanic. The initial state in second section derive in Hilbert 12-space $H=H_i\otimes H_p$. In state vector $|\psi(t)\rangle$, fully density matrix of system is given by $\rho_{ion-phonon}=|\psi(t)\rangle\langle\psi(t)|=|Z\rangle\langle Z|$ in Equation (33). Negativity is first reported in literature as a quantum entanglement measurement in [20].

In this part, we examine if the state is entangled how much quantum entanglement it involves. They are analyzed quantum correlations with concurrence and negativity [17] [23]. The quantum state $\psi$ of a system such as $X$ and $Y$, with dimensions $k$ and $k'$, can be given

$$|\psi\rangle=\sum_j\sqrt{\mu_j}\left|x_j\right\rangle\left|y_j\right\rangle \tag{35}$$

where $\sqrt{\mu_j},(j=1,\cdots,k)$ are Schmidt coefficients abbreviated as SCs, $x_j$ and $y_j$ are orthogonal basis in $H_X$ and $H_Y$ [23]. We have given by Schmidt form for wave function.

Therefore, three SCs are the three eigenvalues of the matrix in Equation (33), $\mu_j$ [23]. Their time dependence is illustrated in Figure 1. Upper two curves are $\mu_1$ and $\mu_2$, while the lower curve, $\mu_3$ is the third SCs for $\eta=0.09$, $\eta=0.2$ and $\eta=0.3$. There are two ways to quantify the quantum entanglement. We work the entanglement of the solutions of our system by calculating negativity and concurrence.

Negativity of any quantum system is written as [23]

$$N(|\psi\rangle)=\frac{2}{k-1}\left(\sum_{i<j}\sqrt{\mu_i}\sqrt{\mu_j}\right) \tag{36}$$

$$N(|\psi\rangle)=\frac{2}{3-1}\left(\sqrt{\mu_1}\sqrt{\mu_2}+\sqrt{\mu_1}\sqrt{\mu_3}+\sqrt{\mu_2}\sqrt{\mu_3}\right) \tag{37}$$

Concurrence is developed as a quantum entanglement measurement for bipartite system of two qubits [13] [24]. The concurrence of bipartite system is given by [13] [24]

$$C(|\psi\rangle) = 2\left(\sum_{i<j}\sqrt{\mu_i\mu_j}\right) \tag{38}$$

$$C(|\psi\rangle) = 2\left(\sqrt{\mu_1\mu_2} + \sqrt{\mu_1\mu_3} + \sqrt{\mu_2\mu_3}\right) \tag{39}$$

As shown in **Figures 2-5**, LDPs are taken between 0.09 and 0.30. It is understood that taking these adjustable values of LDP is an appropriate choice, because the $N$ and $C$ values have seen with the maximums. This leads to higher dimensional entanglement with $\eta$. In **Figure 2** & **Figure 3**, time evolution of $N$ and $C$ is illustrated by $\eta = 0.09$, $\eta = 0.2$ and $\eta = 0.3$. We have obtained high amount of entanglement for three values of LDP.

We reported entanglement via negativity in the LDR discretely from other papers [9] [17]. The values of $N$ and $C$ in one ideal times are shown with **Table 1**. In **Figures 2-7**, a maximum value of $N$ is reported $N = 0.553$ for $\eta = 0.3$ in **Table 1**. A maximum value of $C$ is reported $C = 1.000$ for $\eta = 0.3$ in **Table 1**. The three values of $\eta$ are determined and taken into account throughout this study. In literature, we did not see that it has been worked with the value 0.09. We explain quantum dynamics of $N$ and $C$ according to time in **Figures 2-7**. The results of our former studies [9] [10] [17] [18] are in similar in **Figure 3** & **Figure 4**. $N$, $C$ and $E$, which are the other advanced measurements defining entanglement motion, have been worked out in literature [7] [11] [18] [19] [20].

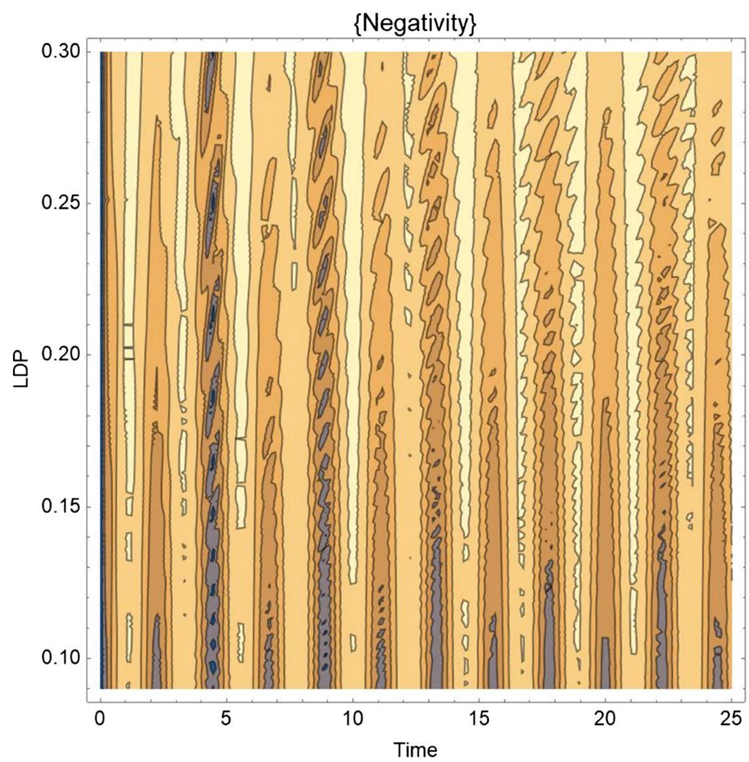


**Figure 6.** Contour plot of negativity for scaled time change of LDP to 0.3 from 0.09. Color scale from black to orange equals to 0.0 - 1.0. Other assumptions parameters are the same as **Figure 1** in the system.

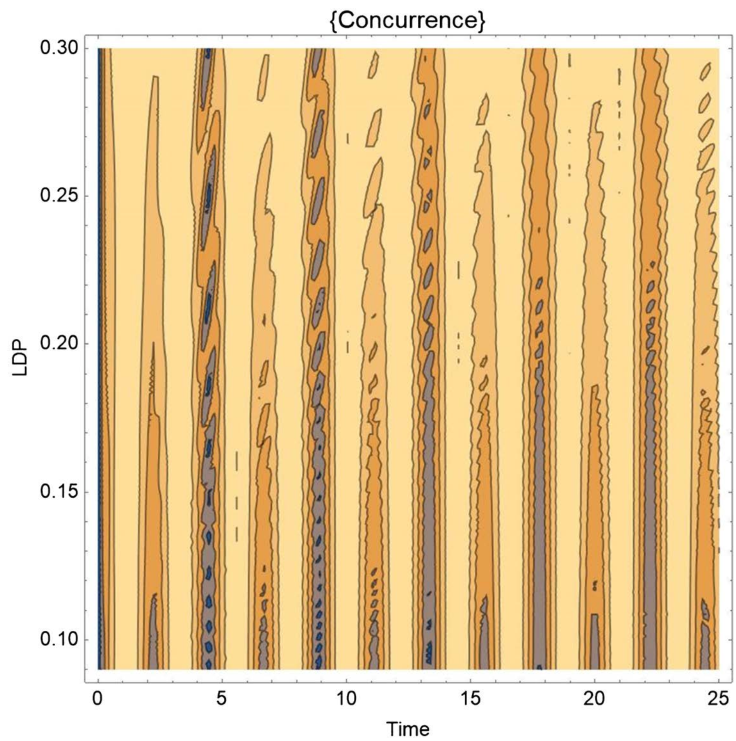


**Figure 7.** Contour plot of concurrence for scaled time change of LDP to 0.3 from 0.09. Color scale from black to orange equals to 0.0 - 1.0. Other assumptions parameters are the same as **Figure 1** in the system.

**Table 1.** Six values of negativity and concurrence within one ideal times, $t$ = 10.16 ms or $t$ = 10.16 scaled time, with respect to **Figure 2** and **Figure 3**.

| | $\eta = 0.09$ | $\eta = 0.2$ | $\eta = 0.3$ |
|---|---|---|---|
| Negativity, $t$ = 10.16 ms, **Figure 2** | 0.493 | 0.512 | 0.553 |
| Concurrence, $t$ = 10.16 ms, **Figure 3** | 0.978 | 0.992 | 1.000 |

We show the quantum correlations with $N$ and $C$ for coupling parameters. We found seperate dynamic features in $N$ in reaction to increasing $\eta$. In **Figure 2**, $N$ oscillates between the values of minimum $N = 0$ and highest rate $N = 0.553$ at $t = 10.16$ ms for $\eta = 0.3$. The variations between the maximum and the minimum values of negativity are regular with time. In **Figure 3**, $C$ oscillates between the values of minimum $C = 0$ and highest rate $C = 1.000$ at $t = 10.16$ ms for $\eta = 0.3$. The presence of long lived entanglement in trapped ion and phonons system has been recognized by **Figure 6** & **Figure 7**. We explore with $N$ *and* $C$ that measurement degrees have a flash crop entangled state up in parallel to raising $\eta$ and this is in comparison to the previous observations [14] [15] [16] [17] [25]. Similar quantum correlations exist between the $N$ and the $C$: see **Figure 6** & **Figure 7**. The color domain is from White to orange. The lower $N$ and $C$ obtain the darker colored domains. However, the system is disentangled some scaled times in **Figure 6** & **Figure 7**. The existence of quantum entanglement is shown by entropy calculations in subatomic particles such as electron, proton and quark [26]. It is investigated the dynamical and stationary properties of the entanglement entropy after a quench from initial states [27]. The entanglament between measured qubit and memory qubit has been inspected via von Neumann entropy [28]. Quantum entanglement has demonstrated with certain statement in time-dependent fifteen-dimensional Hilbert space [29]. Quantum linearity is theoretically characterized by the second order terms of the LDP [30].

## 4. Conclusions

We concentrated on quantum entanglement of two quantum systems in the Hilbert 12-space. We investigate the negativity through the definition of variance LDR. Some physical correlations have been that we measure $N$ and $C$. Our analysis has discovered maximally entangled state. The family is equal to a group of quantum measurements. To more detailed understanding of entanglement measurement results $N$ and $C$, "contour plot" was preferred in Mathematica 8 in **Figure 6** & **Figure 7**.

These plots are obtained by $N$ and $C$ with quantum corrections. Entanglement is compared and is analyzed by two quantum measures which are $N$ *and* $C$. Quantum correlations and interactions between ion and two phonons are investigated. Because, the discussion on physical properties of trapped ion-two phonos interaction is an important subject for quantum information.

The main contribution and novelty of my work has been explained with concluding remarks shown below:

- In our system, quantum entanglement is shown to have the capacity and degree of N and C are $N = 0.553$, $C = 1.000$;
- $N$ bases on three different LDPs;
- This extracts that such entanglement is connected with $\eta$. We achieved long-lived entanglement in LDR;
- Maximally entangled states as presented by means of ion-two phonons system can be important for researchers with trapped ions;
- Extending the life time can be succeeded by using Rabi frequencies and $\eta$. This study and similar studies based on quantum measurement will lead to a better understanding of quantum physics and quantum entanglement.

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## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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