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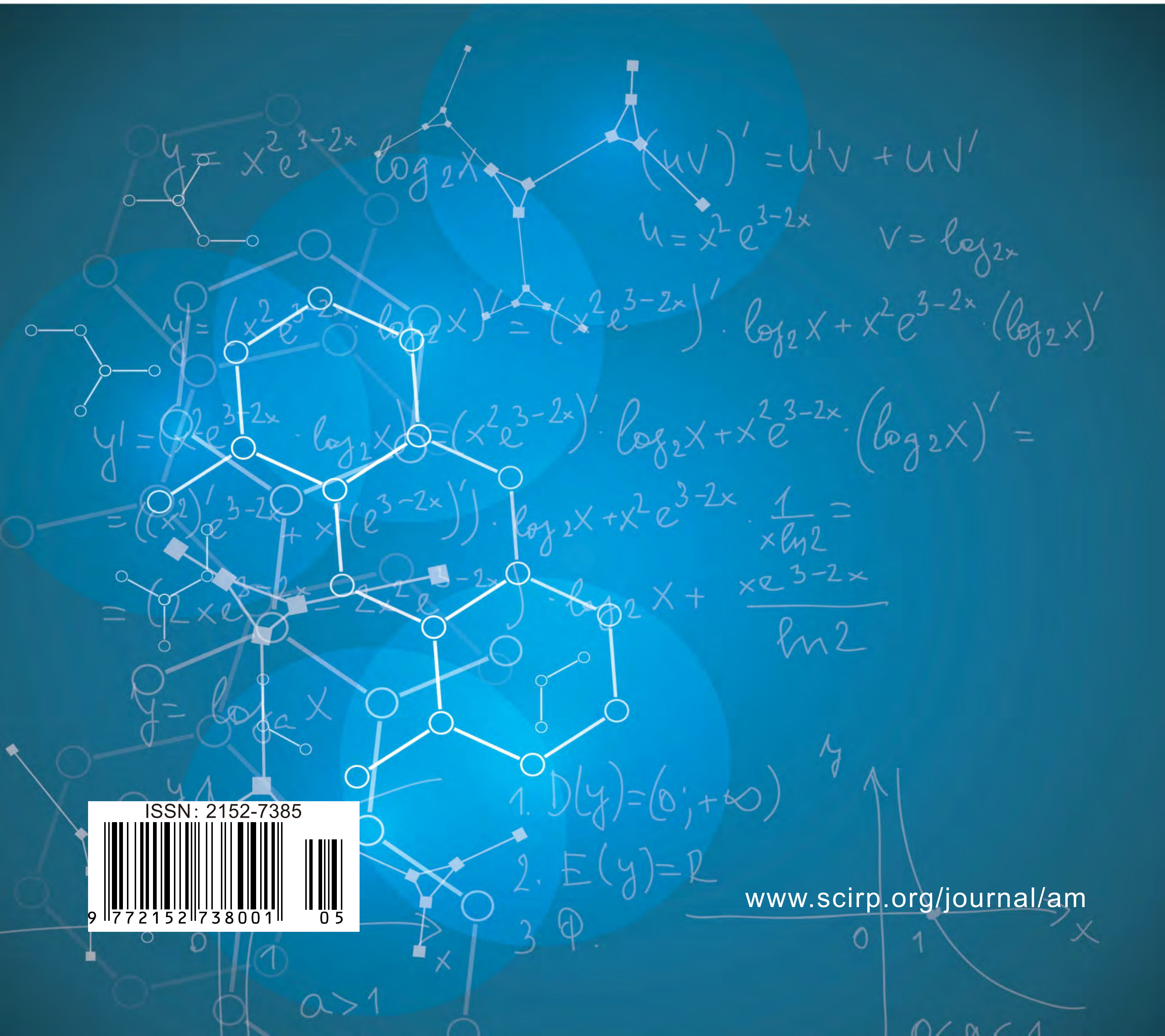

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On q-Analogues of Laplace Type Integral Transforms of q^2-Bessel Functions

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Abstract

The present paper deals with the evaluation of the q-Analogues of Laplece transforms of a product of basic analogues of q^2-special functions. We apply these transforms to three families of q-Bessel functions. Several special cases have been deducted.

Keywords

q-Extensions of Bessel Functions, q-Analogues of Laplace Type Integrals Transforms, q-Analogues of Gamma Function, q-Shift Factorials

1. Introduction

In the second half of twentieth century, there was a significant increase of activity in the area of the q-calculus mainly due to its application in mathematics, statistics and physics. In literature, several aspects of q-calculus were given to enlighten the strong inter disciplinary as well as mathematical character of this subject. Specifically, there have been many q-analogues and q-series representations of various kinds of special functions. In the case of q-Bessel function, there are two related q-Bessel functions introduced by Jackson [1] and denoted by Ismail [2] as

$$J_\mu^{(1)}(z;q)=\left(\frac{z}{2}\right)^\mu \sum_{n=0}^{\infty}\frac{\left(\frac{-z^2}{4}\right)^n}{(q,q)_{\mu+n}(q;q)_n},|z|<2 \tag{1}$$

$$J_\mu^{(2)}(z;q)=\left(\frac{z}{2}\right)^\mu \sum_{n=0}^{\infty}\frac{q^{n(n+\mu)}\left(\frac{-z^2}{4}\right)^n}{(q,q)_{\mu+n}(q;q)_n},z\in\mathbb{C} \tag{2}$$

The third related q-Bessel function $J_{\mu}^{(3)}(z;q)$ was introduced in a full case as [3]

$$J_{\mu}^{(3)}(z;q)=z^{\mu}\sum_{n=0}^{\infty}\frac{(-1)^{n}q^{\frac{n(n-1)}{2}}(qz^{2})^{n}}{(q,q)_{\mu+n}(q;q)_{n}},z\in\mathbb{C} \tag{3}$$

A certain type of Laplace transforms, which is called L_2-transform, was introduced by Yürekli and Sadek [4]. Then these transforms were studied in more details by Yürekli [5], [6]. Purohit and Kalla applied the q-Laplace transforms to a product of basic analogues of the Bessel function [7].

On the same manner, integral transforms have different q-analogues in the theory of q-calculus. The q-analogue of the Laplace type integral of the first kind is defined by [8] as

$${}_{q}L_{2}\left(f(\xi);y\right)=\frac{1}{1-q^{2}}\int_{0}^{y^{-1}}\xi E_{q^{2}}\left(q^{2}y^{2}\xi^{2}\right)f(\xi)\mathrm{d}\xi \tag{4}$$

and expressed in terms of series representation as

$${}_{q}L_{2}\left(f(\xi);y\right)=\frac{(q^{2};q^{2})_{\infty}}{[2]_{q}y^{2}}\sum_{i=0}^{\infty}\frac{q^{2i}}{(q^{2};q^{2})_{i}}f\left(q^{i}y^{-1}\right). \tag{5}$$

On the other hand, the q-analogue of the Laplace type integral of the second kind is defined by [8] as

$${}_{q}\ell_{2}\left(f(\xi);y\right)=\frac{1}{1-q^{2}}\int_{0}^{\infty}\xi e_{q^{2}}\left(-y^{2}\xi^{2}\right)f(\xi)\mathrm{d}_{q}\xi \tag{6}$$

whose q-series representation expressed as

$${}_{q}\ell_{2}\left(f(\xi);y\right)=\frac{1}{[2]_{2}(-y^{2};q^{2})_{\infty}}\sum_{i\in\mathbb{Z}}q^{2i}f\left(q^{i}\right)\left(-y^{2};q^{2}\right)_{i}. \tag{7}$$

In this paper we build upon analysis of [8]. Following [9], we discuss the q-Laplace type integral transforms (4) and (7) on the q-Bessel functions $J_{\mu}^{(1)}(z;q)$, $J_{\mu}^{(2)}(z;q)$ and $J_{\mu}^{(3)}(z;q)$, respectively. In Section 2, we recall some notions and definitions from the q-calculus. In Section 3, we give the main results to evaluate the q-analogue of Laplace transformation of q^2-Basel function. In Section 4, we discuss some special cases.

2. Definitions and Preliminaries

In this section, we recall some usual notions and notations used in the q-theory. It is assumed in this paper wherever it appears that $0<q<1$. For a complex number a, the q-analogue of a is introduced as $[a]_{q}=\frac{1-q^{a}}{1-q}$. Also, by fixing $a\in\mathbb{C}$, the q-shifted factorials are defined as

$$(a;q)_{0}=1;(a,q)_{n}=\prod_{k=0}^{n-1}\left(1-aq^{k}\right),n=1,2,\cdots;(a;q)_{\infty}=\lim_{n\to\infty}(a;q)_{n}. \tag{8}$$

This indeed lead to the conclusion

$$\left(\left[n\right]_q\right)! = \frac{(q;q)_n}{(1-q)^n}, n \in \mathbb{N} \text{ and } (a;q)_x = \frac{(a;q)_\infty}{(aq^x;q)_\infty}. \tag{9}$$

The q-analogue of the exponential function of first and second type are respectively given in [10] by

$$e_q(x) = \sum_0^\infty \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty}, |x| < 1. \tag{10}$$

and

$$E_q(x) = \sum_0^\infty \frac{(-1)^n q^{n\frac{n-1}{2}} x^n}{(q;q)_n}, x \in \mathbb{C}. \tag{11}$$

Indeed it has been shown that

$$e_q(x) = \frac{1}{(x;q)_\infty}, |x| < 1 \text{ and } E_q(x) = (x,q)_\infty, x \in \mathbb{C} \tag{12}$$

The finite q-Jackson and improper integrals are respectively defined by [11]

$$\int_0^x f(t)\,\mathrm{d}_q t = x(1-q)\sum_{k=0}^\infty q^k f\left(xq^k\right) \tag{13}$$

and

$$\int_0^{\infty/A} f(t)\,\mathrm{d}_q t = (1-q)\sum_{k\in\mathbb{Z}} \frac{q^k}{A} f\left(\frac{q^k}{A}\right). \tag{14}$$

The q-analogues of the gamma function of first and second type are respectively defined in [9] as

$$\Gamma_q(\alpha) = \int_0^{1/(1-q)} x^{\alpha-1} E_q\left(q(1-q)x\right)\mathrm{d}_q x, (\alpha > 0) \tag{15}$$

and

$${}_q\Gamma(\alpha) = K(A;\alpha)\int_0^{\infty/A(1-q)} x^{\alpha-1} e_q\left(-(1-q)x\right)\mathrm{d}_q x \tag{16}$$

where, $\alpha_1 > 0$, where $K(A;\alpha)$ is the function given by

$$K(A;\alpha) = A^{\alpha-1}\frac{(-q/\alpha;q)_\infty(-\alpha;q)_\infty}{(-q^t/\alpha;q)_\infty(-\alpha q^{1-t};q)_\infty}. \tag{17}$$

Some useful results, for $x \neq 0,-1,-2,\cdots$, we use here are given by

$$\Gamma_q(\alpha) = \frac{(q;q)_\infty}{(1-q)^{\alpha-1}}\sum_{k=0}^\infty \frac{q^{k\alpha}}{(q;q)_k} = \frac{(q;q)_\infty}{(q^\alpha;q)_\infty}(1-q)^{1-x}, \tag{18}$$

and

$${}_q\Gamma(\alpha) = \frac{K(A;\alpha)}{(1-q)^{\alpha-1}\left(-\frac{1}{A};q\right)_\infty}\sum_{k\in\mathbb{Z}}\left(\frac{q^k}{A}\right)\left(-\frac{1}{A};q\right)_k. \tag{19}$$

3. Main Theorems

Theorem 1. *Let* $J_{2\mu_1}^{(1)}\left(2\sqrt{a_1 t};q^2\right),\cdots,J_{2\mu_n}^{(1)}\left(2\sqrt{a_n t};q^2\right)$ *be a set of first kind of*

q^2*-Bessel* functions, $f(t)=t^{\Delta-1}\prod_{j=1}^{n}J_{2\mu_j}^{(1)}\left(2\sqrt{a_j t};q^2\right)$, *where* Δ, a_j *and* μ_j *for* $j=1,2,\cdots,n$ *are constants, then the q-analogue of Lablace transformation* ${}_qL_2$ *of* $f(t)$ *is given as:*

$$
\begin{aligned}
&{}_qL_2\left(f(t);s\right)\\
&=A_\Delta\prod_{j=1}^{n}\left(\frac{a_j}{s}\right)^{\mu_j}\sum_{m_j=0}^{\infty}\left(\frac{-a_j}{s}\right)^{m_j}B_{m_j}\left(q^2\right)\Gamma_{q^2}\left(\frac{m_j+\mu_j+\Delta+1}{2}\right)
\end{aligned}
\tag{20}
$$

and the q-analogue of Laplace transformation ${}_ql_2$ of $f(t)$ is given as:

$$
\begin{aligned}
&{}_ql_2\left(f(t);s\right)\\
&=A_\Delta\prod_{j=1}^{n}\left(\frac{a_j}{s}\right)^{\mu_j}\sum_{m_j=0}^{\infty}\frac{\left(\frac{-a_j}{s}\right)^{m_j}B_{m_j}\left(q^2\right)_{q^2}\Gamma\left(\frac{m_j+\mu_j+\Delta+1}{2}\right)}{K\left(\frac{1}{s^2};\frac{m_j+\mu_j+\Delta+1}{2}\right)}.
\end{aligned}
\tag{21}
$$

where $Re(s)>0$, $Re(\Delta)>0$ and

$$
A_\Delta=\frac{\left(1-q^2\right)^{\Delta/2}}{[2]s^{\Delta+1}\left(q^2;q^2\right)_\infty},\ B_{m_j}\left(q^2\right)=\frac{\left(q^{2\mu_j+m_j+2};q^2\right)_\infty\left(1-q^2\right)\frac{m_j+\mu_j-1}{2}}{\left(q^2;q^2\right)_{m_j}}
$$

Proof. Now,

$$
{}_qL_2\left(f(t);s\right)=\frac{\left(q^2;q^2\right)_\infty}{[2]s^2}\sum_{k=0}^{\infty}\frac{q^{2k}f\left(q^k s^{-1}\right)}{\left(q^2;q^2\right)_k}
$$

since

$$
J_{2\mu_j}^{(1)}\left(2\sqrt{a_j t};q^2\right)=\left(\frac{2\sqrt{a_j t}}{2}\right)^{2\mu_j}\sum_{m_j=0}^{\infty}\frac{\left(\frac{-\left(2\sqrt{a_j t}\right)^2}{4}\right)^{m_j}}{\left(q^2;q^2\right)_{2\mu_j+m_j}\left(q^2;q^2\right)_{m_j}}
$$

so

$$
\begin{aligned}
&{}_qL_a\left(f(t);s\right)\\
&=\frac{\left(q^2;q^2\right)_\infty}{[2]s^2}\sum_{k=0}^{\infty}\frac{q^{2k}}{\left(q^2;q^2\right)_k}\left(q^k s^{-1}\right)^{\Delta-1}\prod_{j=1}^{n}\left(\sqrt{a_j q^k s^{-1}}\right)^{2\mu_j}\\
&\cdot\sum_{m_j=0}^{\infty}\frac{(-1)^{m_j}\left(a_j q^k s^{-1}\right)^{m_j}}{\left(q^2;q^2\right)_{2\mu_j+m_j}\left(q^2;q^2\right)_{m_j}}\\
&=\frac{\left(q^2;q^2\right)_\infty}{[2]s^{\Delta+1}}\sum_{k=0}^{\infty}\frac{q^{k(\Delta+1)}}{\left(q^2;q^2\right)_k}\prod_{j=1}^{n}\left(\frac{a_j q^k}{s}\right)^{\mu_j}\sum_{m_j=0}^{\infty}\frac{(-1)^{m_j}\left(\frac{a_j q^k}{s}\right)^{m_j}}{\left(q^2;q^2\right)_{2\mu_j+m_j}\left(q^2;q^2\right)_{m_j}}\\
&=\frac{\left(q^2;q^2\right)_\infty}{[2]s^{\Delta+1}}\prod_{j=1}^{n}\left(\frac{a_j}{s}\right)^{\mu_j}\sum_{m_j=0}^{\infty}\frac{(-1)^{m_j}\left(\frac{a_j}{s}\right)^{m_j}}{\left(q^2;q^2\right)_{2\mu_j+m_j}\left(q^2;q^2\right)_{m_j}}\sum_{k=0}^{\infty}\frac{q^{k\left(\Delta+1+m_j+\mu_j\right)}}{\left(q^2;q^2\right)_k}
\end{aligned}
\tag{22}
$$

Since

$$\Gamma_{q^2}(\alpha)=\frac{\left(q^2;q^2\right)_\infty}{\left(1-q^2\right)^{\alpha-1}}\sum_{k=0}^{\infty}\frac{q^{2k\alpha}}{\left(q^2;q^2\right)_k}$$

putting $\alpha=\frac{1+\Delta+m_j+\mu_j}{2}$, so (22) becomes:

$$_qL_s\left(f(t);s\right)=\frac{1}{[2]s^{\Delta+1}}\prod_{j=1}^{n}\left(\frac{a_j}{s}\right)^{\mu_j}\sum_{m_j=0}^{\infty}\frac{(-1)^{m_j}\left(\frac{a_j}{s}\right)^{m_j}\left(1-q^2\right)^{\frac{1+\Delta+m_j+\mu_j}{2}}}{\left(q^2;q^2\right)_{2\mu_j+m_j}\left(q^2;q^2\right)_{m_j}}.$$

$$\Gamma_{q^2}\left(\frac{m_j+\mu_j+\Delta+1}{2}\right)\tag{23}$$

Since

$$\left(q^2;q^2\right)_{2\mu_j+m_j}=\frac{\left(q^2;q^2\right)_\infty}{\left(q^2q^{2\mu_j+m_j};q^2\right)_\infty}$$

so (23) becomes:

$$_qL_2\left(f(t);s\right)=\frac{\left(1-q^2\right)^{\Delta/2}}{[2]s^{\Delta+1}\left(q^2;q^2\right)_\infty}\prod_{j=1}^{n}\left(\frac{a_j}{s}\right)^{\mu_j}$$

$$\cdot\sum_{m_j=0}^{\infty}\frac{\left(\frac{a_j}{s}\right)(-1)^{m_j}\left(q^{2\mu_j+m_j+2};q^2\right)_\infty\left(1-q^2\right)^{\frac{m_j+\mu_j-1}{2}}}{\left(q^2;q^2\right)_{m_j}}.$$

$$\Gamma_{q^2}\left(\frac{m_j+\mu_j+\Delta+1}{2}\right)$$

$$=A_\Delta\prod_{j=1}^{n}\left(\frac{a_j}{s}\right)^{\mu_j}\sum_{m_j=0}^{\infty}\left(\frac{a_j}{s}\right)(-1)^{m_j}B_{m_j}\left(q^2\right)\Gamma_{q^2}\frac{\left(m_j+\mu_j+\Delta+1\right)}{2}$$

Similarly we have

$$_ql_2\left(f(t);s\right)=\frac{1}{[2]}\frac{1}{\left(-s^2;q^2\right)_\infty}\sum_{k=0}^{\infty}q^{2k}\left(-s^2;q^2\right)_k\left(q^k\right)^{\Delta-1}\prod_{j=1}^{n}J_{2\mu_j}^{(1)}\left(2\sqrt{a_jq^k};q^2\right)$$

$$=\frac{1}{[2]}\frac{1}{\left(-s^2;q^2\right)_\infty}\sum_{k=0}^{\infty}q^{2k}\left(-s^2;q^2\right)_k\left(q^k\right)^{\Delta-1}\prod_{j=1}^{n}\left(a_jq^k\right)^{\mu_j}.$$

$$\sum_{m_j=0}^{\infty}\frac{\left(-a_jq^k\right)^{m_j}}{\left(q^2;q^2\right)_{m_j}+2\mu_j}$$

$$=\prod_{j=1}^{n}\frac{\left(a_j\right)^{\mu_j}}{[2]}\sum_{m_j=0}^{\infty}\frac{\left(-a_j\right)^{m_j}}{\left(q^2;q^2\right)_{m_j+2\mu_j}\left(q^2;q^2\right)_{m_j}}\sum_{k=0}^{\infty}\frac{\left(-s^2;q^2\right)_kq^{k\left(m_j+\mu_j+\Delta+1\right)}}{\left(-s^2;q^2\right)_\infty}$$

Now using

$$ {}_{q^2}\Gamma(\alpha) = \frac{K(A;\alpha)}{\left(1-q^2\right)^{\alpha-1}\left(-\frac{1}{A};q^2\right)_\infty} \sum_{k\in Z}\left(\frac{q^K}{A}\right)^\alpha \left(-\frac{1}{A};q^2\right)_K $$

with $A=\frac{1}{s^2}$, $\alpha=\frac{m_j+\mu_j+\Delta+1}{2}$ we get

$$ {}_q l_2(f(t);s) $$

$$ =\prod_{j=1}^{m}\frac{(a_j)^{\mu_j}}{[2]s^{\mu_j+\Delta+1}}\sum_{m_j=0}^{\infty}\frac{\left(\frac{-a_j}{s}\right)^{m_j}\left(1-q^2\right)^{\frac{m_j+\mu_j+\Delta+1}{2}} q^2\Gamma\left(\frac{m_j+\mu_j+\Delta+1}{2}\right)}{K\left(\frac{1}{s^2};\frac{m_j+\mu_j+\Delta+1}{2}\right)\left(q^2;q^2\right)_{m_j+2\mu_j}\left(q^2;q^2\right)_{m_j}} $$

$$ =\frac{\left(1-q^2\right)^{\frac{\Delta}{2}}}{[2]s^{\Delta+1}\left(q^2;q^2\right)_\infty}\prod_{j=1}^{m}\left(\frac{a_j}{s}\right)^{\mu_j}\sum_{m_j=0}^{\infty}\frac{\left(\frac{-a_j}{s}\right)^{m_j}\left(1-q^2\right)^{\frac{m_j+\mu_j-1}{2}}\left(q^{m_j+2\mu_j+2};q^2\right)_\infty}{K\left(\frac{1}{s^2},\frac{m_j+\mu_j+\Delta+1}{2}\right)\left(q^2;q^2\right)_{m_j}}. $$

$$ {}_{q^2}\Gamma\left(\frac{m_j+\mu_j+\Delta+1}{2}\right) $$

$$ =A_\Delta\prod_{j=1}^{m}\left(\frac{a_j}{s}\right)^{\mu_j}\sum_{m_j=0}^{\infty}\frac{\left(\frac{-a_j}{s}\right)^{m_j}}{K\left(\frac{1}{s^2};\frac{m_j+\mu_j+\Delta+1}{2}\right)}B_{m_j}\left(q^2\right){}_{q^2}\Gamma\left(m_j+\mu_j+\Delta+1\right) $$

Theorem 2. *Let* $J_{2\mu_1}^{(2)}\left(2\sqrt{a,t};q^2\right),\cdots,J_{2\mu_n}^{(2)}\left(2\sqrt{at};q^2\right)$ *be a set of second order* q^2*-Bessel function,* $f(t)=t^{\Delta-1}\prod_{j=1}^{n}J_{2\mu_j}^{(2)}\left(2\sqrt{a_jt};q^2\right)$ *where* Δ,a_j *and* μ_j *for* $j=1,2,\cdots,n$ *are constants then* ${}_qL_2$*-transform of* $f(t)$ *is given as:*

$$ {}_qL_2(f(t),s)=A_\Delta\prod_{j=1}^{n}\left(\frac{a_j}{s}\right)^{\mu_j}\sum_{m_j=0}^{\infty}(-1)^{m_j}q^{2m_j(m_j+2\mu_j)}\left(\frac{a_j}{s}\right)^{m_j+\mu_j} \cdot B_{m_j}\left(q^2\right)\Gamma_{q^2}\left(m_j+\mu_j+\Delta+1\right) \quad (24) $$

and the q-analogue of Laplace transformation ${}_ql_2$ of $f(t)$ is given as:

$$ {}_ql_2(f(t);s)=A_\Delta\prod_{j=1}^{n}\left(\frac{a_j}{s}\right)^{\mu_j}\sum_{m_j=0}^{\infty}\frac{\left(\frac{-a_j}{s}\right)^{m_j}q^{2m_j(m_j+2\mu_j)}}{K\left(\frac{1}{s^2};\frac{m_j+\mu_j+\Delta+1}{2}\right)} \cdot B_{m_j}\left(q^2\right){}_{q^2}\Gamma\left(\frac{m_j+\mu_j+\Delta+1}{2}\right) \quad (25) $$

Proof. Now,

$$ J_{2\mu_j}^{(2)}\left(2\sqrt{a_jt};q^2\right)=\left(\frac{2\sqrt{a_jt}}{2}\right)^{2\mu_j}\sum_{m_j}^{\infty}\frac{\left(-\frac{\left(2\sqrt{a_jt}\right)^2}{4}\right)^{m_j}q^{2m_j(m_j+2a_j)}}{\left(q^2;q^2\right)_{2\mu_j+m_j}\left(q^2;q^2\right)_{m_j}} $$

so

$$
{}_qL_2\left(f(t);s\right)=\frac{\left(q^2;q^2\right)_\infty}{[2]s^2}\sum_{k=0}^{\infty}\frac{q^{2k}}{\left(q^2;q^2\right)_k}\left(q^ks^{-1}\right)^{\Delta-1}\prod_{j=1}^{n}\left(\frac{2\sqrt{a_jq^ks^{-1}}}{2}\right)^{2\mu_j}
\cdot\sum_{m_j=0}^{\infty}\frac{\left(-\frac{\left(2\sqrt{a_jq^ks^{-1}}\right)^2}{4}\right)^{m_j}q^{2m_j\left(m_j+2\mu_j\right)}}{\left(q^2;q^2\right)_{2\mu_j+m_j}\left(q^2;q^2\right)_{m_j}} \tag{26}
$$

By the same argument we can write (26) as

$$
{}_qL_2\left(f(t);s\right)=\frac{\left(q^2;q^2\right)_\infty}{[2]s^{\Delta+1}\left(q^2;q^2\right)_\infty}\prod_{j=1}^{n}\sum_{m_j=0}^{\infty}\frac{(-1)^{m_j}q^{2m_j\left(m_j+2\mu_j\right)}}{\left(q^2;q^2\right)_{m_j}}
\cdot\left(\frac{a_j}{s}\right)^{m_j+\mu_j}\left(q^{2\mu_j+m_j+2};q^2\right)_\infty\sum_{k=0}^{\infty}\frac{q^{k\left(m_j+\mu_j+1+\Delta\right)}}{\left(q^2;q^2\right)_k}
$$

put $\alpha=\dfrac{m_j+\mu_j+\Delta+1}{2}$ in $\Gamma_{q^2}(\alpha)$, then

So (25) becomes:

$$
{}_qL_2\left(f(t);s\right)=A_\Delta\prod_{j=1}^{n}\left(\frac{a_j}{s}\right)^{\mu_j}\sum_{m_j=0}^{\infty}(-1)^{m_j}q^{2m_j\left(m_j+2\mu_j\right)}\left(\frac{a_j}{s}\right)^{m_j+\mu_j}
\cdot B_{m_j}\left(q^2\right)\Gamma_{q^2}\left(m_j+\mu_j+\Delta+1\right)
$$

Similarly

$$
{}_ql_2\left(f(t);s\right)=\frac{1}{[2]}\frac{1}{\left(-s^2;q^2\right)_\infty}\sum_{k=0}^{\infty}q^{2k}\left(-s^2;q^2\right)_k\left(q^k\right)^{\Delta-1}\prod_{j=1}^{n}\left(a_jq^k\right)^{\mu_j}
\cdot\sum_{m_j=0}^{\infty}\frac{\left(-a_jq^k\right)^{m_j}q^{2m_j\left(m_j+2\mu_j\right)}}{\left(q^2;q^2\right)_{m_j+2\mu_j}\left(q^2;q^2\right)_{m_j}}
$$

Put $A=\dfrac{1}{s^2},\alpha=\dfrac{m_j+\mu_j+\Delta+1}{2}$ we get

$$
{}_ql_2\left(f(t);s\right)
$$

$$
=\frac{1}{[2]}\prod_{j=1}^{n}\left(a_j\right)^{\mu_j}\sum_{m_j=0}^{\infty}\frac{\left(-a_j\right)^{m_j}q^{2m_j\left(m_j+2\mu_j\right)}\left(1-q^2\right)^{\frac{m_j+\mu_j+\Delta+1}{2}}q^2\Gamma\left(\frac{m_j+\mu_j+\Delta+1}{2}\right)}{\left(q^2;q^2\right)_{m_j+2\mu_j}K\left(\frac{1}{s^2};\frac{m_j+\mu_j+\Delta+1}{2}\right)s^{m_j+\mu_j+\Delta+1}}
$$

$$
=A_\Delta\prod_{m_j=0}^{\infty}\frac{\left(\frac{-a_j}{s}\right)^{m_j}q^{2m_j\left(m_j+2\mu_j\right)}}{K\left(\frac{1}{s^2};\frac{m_j+\mu_j+\Delta+1}{2}\right)}B_{m_j}\left(q^2\right)_{q^2}\Gamma\left(\frac{m_j+\mu_j+\Delta+1}{2}\right)
$$

Theorem 3. *Let* $J^{(3)}_{2\mu_j}\left(\sqrt{q^{-1}a_1t};q^2\right),\cdots,J^{(3)}_{2\mu_n}\left(\sqrt{q^{-1}a_nt};q^2\right)$ *be s set of* q^2*-Bessel* *functions,* $f(t)=t^{\Delta-1}\prod_{j=1}^{n}J_{2\mu_j}^{(3)}\left(\sqrt{q^{-1}a_j t};q^2\right)$ *where* Δ, a_j *and* μ_j *for* $j=1,2,\cdots,n$ *are constants. Then we have*

$$\begin{aligned}{}_qL_2\left(f(t);s\right)=A_\Delta\prod_{j=1}^{n}\left(\frac{a_j}{qs}\right)^{\mu_j}\sum_{m_j=0}^{\infty}(-1)^{m_j}q^{m_j(m_j-1)}\left(\frac{a_jq}{s}\right)^{m_j}\\ \cdot B_{m_j}\left(q^2\right)\Gamma_{q^2}\left(\frac{m_j+\mu_j+\Delta+1}{2}\right)\end{aligned} \tag{27}$$

and the q-analogue of Laplace transformation ${}_ql_2$ of $f(t)$ is given by:

$$\begin{aligned}{}_ql_2\left(f(t);s\right)=A_\Delta\prod_{j=1}^{n}\left(\frac{a_j}{qs}\right)^{\mu_j}\sum_{m_j=0}^{\infty}\frac{\left(\frac{-a_jq}{s}\right)^{m_j}q^{m_j(m_j-1)}}{K\left(\frac{1}{s^2};\frac{m_j+\mu_j+\Delta+1}{2}\right)}\\ \cdot B_{m_j}\left(q^2\right){}_{q^2}\Gamma\frac{\left(m_j+\mu_j+\Delta+1\right)}{2}\end{aligned} \tag{28}$$

Proof. Now

$$J_{2\mu_j}^{(3)}\left(\sqrt{a_jq^{k-1}s^{-1}};q^2\right)=\left(\sqrt{a_jq^{k-1}s^{-1}}\right)^{2\mu_j}\sum_{m_j=0}^{\infty}(-1)^{m_j}\frac{q^{\frac{2m_j(m_j-1)}{2}}\left(q^2a_jq^{k-1}s^{-1}\right)^{m_j}}{\left(q^2;q^2\right)_{m_j+2\mu_j}\left(q^2;q^2\right)_{m_j}}$$

$$\begin{aligned}{}_qL_2\left(f(t);s\right)=\frac{\left(q^2;q^2\right)_\infty}{[2]s^2}\sum_{k=0}^{\infty}\frac{q^{2k}\left(q^ks^{-1}\right)^{\Delta-1}}{\left(q^2;q^2\right)_k}\prod_{j=1}^{n}\left(a_jq^{k-1}s^{-1}\right)^{\mu_j}\\ \cdot\sum_{m_j=0}^{\infty}\frac{(-1)^{m_j}q^{m_j(m_j-1)}\left(q^2a_jq^{k-1}s^{-1}\right)^{m_j}}{\left(q^2;q^2\right)_{m_j+2\mu_j}\left(q^2;q^2\right)_{m_j}}\end{aligned}$$

put $\alpha=\frac{m_j+\mu_j+\Delta+1}{2}$, we get

$$\begin{aligned}&{}_qL_2\left(f(t);s\right)\\&=A_\Delta\prod_{j=1}^{n}\left(\frac{a_j}{qs}\right)^{\mu_j}\sum_{m_j=0}^{\infty}(-1)^{m_j}q^{m_j(m_j-1)}\left(\frac{a_jq}{s}\right)^{m_j}B_{m_j}\left(q^2\right)\Gamma_{q^2}\left(\frac{m_j+\mu_j+\Delta+1}{2}\right)\end{aligned}$$

Similarly

$$\begin{aligned}&{}_ql_2\left(f(t);s\right)\\&=\frac{1}{[2]}\frac{1}{\left(-s^2;q^2\right)_\infty}\prod_{j=1}^{n}\left(q^{k-1}\right)^{\mu_j}\left(a_j\right)^{\mu_j}\sum_{m_j=0}^{\infty}\frac{(-1)^{m_j}q^{m_j(m_j-1)}\left(q^{km_j}\right)\left(qa_j\right)^{m_j}}{\left(q^2;q^2\right)_{m_j+\mu_2}\left(q^2;q^2\right)_{m_j}}\\&\cdot\sum_{k=0}^{\infty}q^{k(\Delta+1)}\left(-s^2;q^2\right)_k.\end{aligned}$$

Put $\alpha=\frac{m_j+\mu_j+\Delta+1}{2}$, $A=\frac{1}{s^2}$ we get

$$ {}_q l_2\left(f(t);s\right)=\frac{\left(1-q^2\right)^{\frac{\Delta}{2}}}{[2]s^{\Delta+1}\left(q^2;q^2\right)_\infty}\prod_{j=1}^{n}\left(\frac{a_j}{qs}\right)^{\mu_j} $$

$$ \cdot\sum_{m_j=0}^{\infty}\frac{\left(\frac{-a_jq}{s}\right)^{m_j}q^{m_j(m_j-1)}B_{m_j}\left(q^2\right){}_{q^2}\Gamma\left(\frac{m_j+\mu_j+\Delta+1}{2}\right)}{K\left(\frac{1}{s^2};\frac{m_j+\mu_j+\Delta+1}{2}\right)}. $$

4. Special Cases

1) Let $n=1$, $\mu_1=\mu$, $a_1=a$ in above theorems, respectively we have:

$$ {}_qL_2\left(t^{\Delta-1}J_{2\mu}^{(1)}\left(2\sqrt{at};q^2\right);s\right) = A_\Delta\left(\frac{a}{s}\right)^{\mu}\sum_{m=0}^{\infty}\left(\frac{-a}{s}\right)^{m}B_m\left(q^2\right)\Gamma_{q^2}\left(\frac{m+\mu+\Delta+1}{2}\right) \tag{29} $$

$$ {}_ql_2\left(t^{\Delta-1}J_{2\mu}^{(1)}\left(2\sqrt{at};q^2\right);s\right) = A_\Delta\left(\frac{a}{s}\right)^{\mu}\sum_{m=0}^{\infty}\frac{\left(\frac{-a}{s}\right)^{m}}{K\left(\frac{1}{s^2};\frac{m+\mu+\Delta+1}{2}\right)}B_m\left(q^2\right){}_{q^2}\Gamma\left(\frac{m+\mu+\Delta+1}{2}\right) \tag{30} $$

$$ {}_qL_2\left(t^{\Delta-1}J_{2\mu}^{(2)}\left(2\sqrt{at};q^2\right);s\right) = A_\Delta\left(\frac{a}{s}\right)^{\mu}\sum_{m=0}^{\infty}(-1)^m q^{2m(m+2\mu)}B_m\left(q^2\right)\Gamma_{q^2}\left(\frac{m+\mu+\Delta+1}{2}\right) \tag{31} $$

$$ {}_ql_2\left(t^{\Delta-1}J_{2\mu}^{(2)}\left(2\sqrt{at};q^2\right);s\right) = A_\Delta\left(\frac{a}{s}\right)^{\mu}\sum_{m=0}^{\infty}\frac{\left(\frac{-a}{s}\right)^{m}q^{2m(m+2\mu)}}{K\left(\frac{1}{s^2};\frac{m+\mu+\Delta+1}{2}\right)}B_m\left(q^2\right){}_{q^2}\Gamma\left(\frac{m+\mu+\Delta+1}{2}\right) \tag{32} $$

$$ {}_qL_2\left(t^{\Delta-1}J_{2\mu}^{(3)}\left(2\sqrt{aq^{-1}t};q^2\right);s\right) = A_\Delta\left(\frac{a}{qs}\right)^{\mu}\sum_{m=0}^{\infty}(-1)^m q^{m(m-1)}\left(\frac{aq}{s}\right)^{m}B_m\left(q^2\right)\Gamma_{q^2}\left(\frac{m+\mu+\Delta+1}{2}\right) \tag{33} $$

$$ {}_ql_2\left(t^{\Delta-1}J_{2\mu}^{(3)}\left(2\sqrt{aq^{-1}t};q^2\right);s\right) = A_\Delta\left(\frac{a}{s}\right)^{\mu}\sum_{m=0}^{\infty}\frac{\left(\frac{aq}{s}\right)^{m}q^{m(m-1)}}{K\left(\frac{1}{s^2};\frac{m+\mu+\Delta+1}{2}\right)}B_m\left(q^2\right){}_{q^2}\Gamma\left(\frac{m+\mu+\Delta+1}{2}\right) \tag{34} $$

2) Put $\Delta-1=\mu$ in part (29) above, then

$$ {}_qL_2\left(t^{\mu}J_{2\mu}^{(1)}\left(2\sqrt{at};q^2\right);s\right)=\frac{\left(1-q^2\right)^{\frac{\mu+1}{2}}}{[2]s^{\mu+2}\left(q^2;q^2\right)_\infty}\left(\frac{a}{s}\right)^{\mu} $$

$$\sum_{m=0}^{\infty}\left(\frac{a}{s}\right)^{m}\frac{\left(q^{2\mu+m+2};q^{2}\right)_{\infty}\left(1-q^{2}\right)^{\frac{m+\mu-1}{2}}}{\left(q^{2};q^{2}\right)_{m}}\Gamma_{q^{2}}\left(\frac{m+2\mu+2}{2}\right)$$

$$=\frac{\left(\frac{a}{s}\right)^{\mu}}{[2]s^{\mu+2}}\sum_{m=0}^{\infty}\frac{\left(\frac{-a}{s}\right)^{m}}{\left(q^{2};q^{2}\right)_{m}}=\frac{(a)^{\mu}}{[2]s^{2\mu+2}}e_{q^{2}}\left(\frac{-a}{s}\right).$$

3) Put $\mu=0$ we get

$$_{q}L_{2}\left(J_{0}^{(1)}\left(2\sqrt{at};q^{2}\right);s\right)=\frac{1}{[2]s^{2}}e_{q^{2}}\left(\frac{-a}{s}\right).$$

which is the same result cited by [7].

4) Put $\Delta-1$ in (33), then

$$_{q}L_{2}\left(t^{\mu}J_{2\mu}^{(3)}\left(2\sqrt{q^{-1}at}\right);s\right)=\frac{\left(1-q^{2}\right)^{\frac{\mu+1}{2}}}{[2]s^{\mu+2}\left(q^{2};q^{2}\right)_{\infty}}\left(\frac{a}{qs}\right)^{\mu}.$$

$$\sum_{m=0}^{\infty}(-1)^{m}\frac{q^{m(m-1)}\left(\frac{aq}{s}\right)^{m}\left(q^{2\mu+m+2};q^{2}\right)\left(1-q^{2}\right)^{\frac{m+\mu-1}{2}}\Gamma_{q^{2}}\left(\frac{m+2\mu+2}{2}\right)}{\left(q^{2};q^{2}\right)_{m}}$$

$$=\frac{\left(\frac{a}{q}\right)^{\mu}}{[2]s^{2\mu+2}}\sum_{m=0}^{\infty}(-1)^{m}\frac{\left(\frac{aq}{s}\right)^{m}q^{2m\frac{m-1}{2}}}{\left(q^{2};q^{2}\right)_{m}}=\frac{\left(\frac{a}{q}\right)^{\mu}}{[2]s^{2\mu+2}}E_{q^{2}}\left(\frac{aq}{s}\right).$$

5) Let $\mu=0$ and $a=0$ in (34), then

$$_{q}L_{2}\left(t^{\Delta-1};s\right)=\frac{\left(1-q^{2}\right)^{\frac{\Delta}{2}}}{[2]s^{\Delta+1}}\frac{1}{K\left(\frac{1}{s^{2}};\frac{\Delta+1}{2}\right)}\left(1-q^{2}\right)^{-\frac{1}{2}}\Gamma_{q^{2}}\left(\frac{\Delta+1}{2}\right)$$

replacing $\Delta-1$ by α, we get

$$_{q}L_{2}\left(t^{\alpha};s\right)=\frac{\left(1-q^{2}\right)^{\frac{\alpha}{2}}}{[2]s^{\alpha+2}}\frac{1}{K\left(\frac{1}{s^{2}};1+\frac{\alpha}{2}\right)}\Gamma_{q^{2}}\left(1+\frac{\alpha}{2}\right)$$

which is the same result in [8].

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Solution of Some Second Order Ordinary Differential Equations Using a Derived Algorithm

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Abstract

We emphasized explicitly on the derivation and implementation of a new numerical algorithm scheme which gave stable results that show the applicability of the method. In this paper, we aimed to solve some second order initial value problems of ordinary differential equations and compare the results with the theoretical solution. Using this method to solve some initial value problems of second order ordinary differential equations, we discovered that the results compared favorably with the theoretical solution which led to the conclusion that the new numerical algorithm scheme derived in the research is approximately correct and can be prescribed for any related ordinary differential equations.

Keywords

Numerical Scheme, Ordinary Differential Equation, Scheme Development

1. Introduction

Numerical methods are methods that are constructed through a given interval. The methods start with an initial point and then take a short step forward in time to find the next solution point. The process then continues with subsequent steps to map out the solution. There are two main numerical methods of solving initial value problems of ordinary differential equations. They are single step methods, also known as one step method and multistep methods. The single-step methods are the method that uses information about the solution at one point x_n, to advance it to the next point x_{n+1}. The single step methods have certain advantages which include, being self-starting and having the flexibility to change step size from one step to the next.

Various numerical methods have been developed for the solution of some initial value problems of ordinary differential equations. Some of the numerical analysts who have worked extensively on the development on numerical methods are: [1] [2] [3] [4] [5]. Development of a scheme for solving some initial value problem of ordinary differential equations with a particular basis function was carried out by [1] which was improved upon by [2] for solving related problems. [4] and [5] worked extensively in other to improve upon schemes developed by [1] and [2] and better methods were produced. The efficiency of all these contributed efforts in numerical analysis had been measured and tested for their stability, accuracy, convergence and consistency properties [6] [7] [8]. The accuracy properties of different methods are usually compared by considering the order of convergence as well as the truncation error coefficients of the various methods. Research has shown that for a method to be suitable for solving any sets of initial value problems (ivps) in ordinary differential equations (ODEs), it must have all the mentioned characteristics.

Recently [9] developed a scheme in which standard finite difference schemes were developed. Similarly, [4] also worked on some approximation techniques which were used to derive qualitatively stable non-standard finite difference schemes.

In this paper, a new one-step numerical method is developed with the above mentioned characteristics in mind to solve some initial value problems of ordinary differential equations which were based on the local representation of the theoretical solution to initial value problem of the form $y' = f(x, y); y(a) = \eta$. In the interval (x_n, x_{n+1}) by interpolating function $F(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + bxR_e(e^{kx+\mu})$, a_0, a_1, a_2, a_3 and b are real undetermined coefficients and k, μ are complex parameters. But in this paper, we shall be using the same assumptions but different interpolating functions such as:

$F(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + bxR_e(e^{kx+\mu})$, where a_0, a_1, a_2, a_3 and b are real undetermined coefficients and k, μ are complex parameters.

2. Methodogy

Considering an interpolating function:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + bxR_e(e^{kx+\mu}) \tag{1}$$

where a_0, a_1, a_2, a_3 and b are real undetermined coefficients and k, μ are complex parameters.

Since k and μ are complex parameters, then we have:

$$k = k_1 + ik_2 \tag{2}$$

Also, $\mu = i\theta$, where $i^2 = -1$, therefore putting this together with (2) in (1), we have the Interpolating function to be:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + bxe^{k_1}x\cos(k_2x + \sigma) \tag{3}$$

Let us define $R(x)$ and $\theta(x)$ as follows:

$$\left.\begin{aligned} R(x) &= xe^{k_1 x} \\ \theta(x) &= k_2 x + \sigma \end{aligned}\right\} \tag{4}$$

Putting (4) in (3), we have:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + bR(x)\cos\theta(x) \tag{5}$$

By assumption, y_n is a numerical estimate to the theoretical solution $y(x_n)$ and also $f_n = f(x_n, y_n)$. Let our mesh points (self length) be define as follows:

$$x_n = a + nh; n = 0,1,2,\cdots, a = 0, x_n = nh. \quad x_{n+1} = (n+1)h \tag{6}$$

Imposing the following constraints on the interpolating function (5), we have:

1) The interpolating function must coincide with the theoretical solution at $x = x_n$ and $x = x_{n+1}$. This required that:

$$f(x_{n+1}) = a_0 + a_1 x_{n+1} + a_2 x_{n+1}^2 + a_3 x_{n+1}^3 + bR(x_{n+1})\cos\theta(x_n). \tag{7}$$

That is, $f(x_n) = y(x_n)$ and

$$f(x_{n+1}) = a_0 + a_1 x_{n+1} + a_2 x_{n+1}^2 + a_3 x_{n+1}^3 + bR(x_{n+1})\cos\theta(x_{n+1}) \tag{8}$$

It implies that $f(x_{n+1}) = y(x_{n+1})$.

2) The first, second, third and fourth derivatives with respect to x of the interpolating function respectively coincide with the differential equation as well as its first, second, third and fourth derivatives with respect to x at x_n, *i.e.*

$$\left.\begin{aligned} F^1(x_n) &= f_n \\ F^2(x_n) &= f_n^1 \\ F^3(x_n) &= f_n^2 \\ F^4(x_n) &= f_n^3 \end{aligned}\right\} \tag{9}$$

From Equation (9) implies:

$$f'(x_n) = f_n + 2a_2 x_n + 3a_3 x_n^2 + \left[\cos\theta(x_n)\frac{\mathrm{d}}{\mathrm{d}x}\left(bR(x_n)\right) + bR(x_n)\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos\theta(x)\right)\right] \tag{10}$$

where

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(bR(x_n)\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(bxe^{k_1 x}\right) = e^{k_1 x}\cdot b + bx\cdot k_1 e^{k_1 x} = be^{k_1 x} + bk_1 xe^{k_{1x}} \\ &= be^{k_1 x} + bk_1 R(x_n) \end{aligned} \tag{11}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos\theta(x_n)\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos(k_2 x + \sigma)\right] = -k_2 \sin\theta(x_n) \tag{12}$$

Putting (11) & (12) in (10) we have:

$$\begin{aligned} f'(x_n) = f_n &= a_1 + 2a_2 x_n + 3a_3 x_n^2 \left[\cos\theta(x_n)\left\{be^{k_1 x} + bk_1 R(x_n)\right\}\right. \\ &\quad \left. + bR(x_n)\left(-k_2 \sin\theta(x_n)\right)\right] \\ &= a_1 + 2a_2 x_n + 3a_3 x_n^2 + \left[be^{k_1 x}\cos\theta(x_n) + bk_1 R(x_n)\cos\theta(x_n)\right. \\ &\quad \left. - bR(x_n)\left(-k_2 \sin\theta(x_n)\right)\right] \end{aligned}$$

$$\begin{aligned} f_n = a_1 + 2a_2 x_n + 3a_3 x_n^2 + b&\left[e^{k_1 x_n}\cos\theta(x_n) + k_1 R(x_n)\cos\theta(x_n)\right. \\ &\left. - k_2 R(x_n)\sin\theta(x_n)\right] \end{aligned} \tag{13}$$

That is, $F'(x_n) = f_n$

$$F''(x_n) = f_n' = 2a_2x_n + 6a_3x_n + b\Bigg[\left(e^{k_1x_n}\frac{\mathrm{d}}{\mathrm{d}x}\cos\theta(x_n) + \cos\theta(x_n)\frac{\mathrm{d}}{\mathrm{d}x}e^{k_1x_n}\right)$$
$$+\left(k_1R(x_n)\frac{\mathrm{d}}{\mathrm{d}x}\cos\theta(x_n) + \cos\theta(x_n)\frac{\mathrm{d}}{\mathrm{d}x}k_1R(x_n)\right) \tag{14}$$
$$-\left(R(x_n)\frac{\mathrm{d}}{\mathrm{d}x}\sin\theta(x_n) + \sin\theta(x_n)\frac{\mathrm{d}}{\mathrm{d}x}R(x_n)\right)\Bigg]$$

where

$$k_1R(x_n) = k_1\left[xe^{k_1x_n}\right] = k_1\left(x\frac{\mathrm{d}}{\mathrm{d}x}e^{k_1x_n} + e^{k_1x_n}\frac{\mathrm{d}}{\mathrm{d}x}x\right)$$
$$= k_1^2xe^{k_1x_n} + k_1e^{k_1x_n} = k_1^2R(x_n) + k_1e^{k_1x_n} \tag{15}$$

Since $R(x_n) = x_ne^{k_1}x_n = e^{k_1x_n} + k_1R(x_n)$.

Putting (15) in (14)

$$F''(x_n) = f_n' = 2a_2x_n + 6a_3x_n + b\Big[\Big\{e^{k_1x_n}\left(-\sin\theta(x_n)\right) + \cos\theta(x_n)\left(k_1e^{k_1x_n}\right)\Big\}$$
$$+k_1R(x_n)\left(-\sin\theta(x_n)\right) + \cos\theta(x_n)\left(k_1^2R(x_n) + k_1e^{k_1x_n}\right)$$
$$-\Big\{R(x_n)\cos\theta(x_n) + \sin\theta(x_n)\left(e^{k_1x_n} + k_1R(x_n)\right)\Big\}\Big]$$

$$F''(x_n) = 2a_2x_n + 6a_3x_n + b\Big[\Big\{-e^{k_1x_n}\sin\theta(x_n) + k_1e^{k_1x_n}\cos\theta(x_n)\Big\}$$
$$-k_1R(x_n)\sin\theta(x_n) + k_1^2R(x_n)\cos\theta(x_n) + k_1e^{k_1x_n}\cos\theta(x_n)$$
$$-R(x_n)\cos\theta(x_n) - e^{k_1x_n}\sin\theta(x_n) - k_1R(x_n)\sin\theta(x_n)\Big]$$

$$f_n' = 2a_2x_n + 6a_3x_n + b\Big\{-2e^{k_1x_n}\sin\theta(x_n) + 2k_1e^{k_1x_n}\cos\theta(x_n)$$
$$-2k_1R(x_n)\sin\theta(x_n) + k_1^2R(x_n)\cos\theta(x_n) - R(x_n)\cos\theta(x_n)\Big\} \tag{16}$$

$$F'''(x_n) = f_n'' = 6a_3 + b\Bigg[-\left\{2e^{k_1x_n}\frac{\mathrm{d}}{\mathrm{d}x}\sin\theta(x_n) + \sin\theta(x_n)\frac{\mathrm{d}}{\mathrm{d}x}2e^{k_1x_n}\right\}$$
$$+\left\{2k_1e^{k_1x_n}\frac{\mathrm{d}}{\mathrm{d}x}\cos\theta(x_n) + \cos\theta(x_n)\frac{\mathrm{d}}{\mathrm{d}x}2k_1e^{k_1x_n}\right\}$$
$$-\left\{2k_1R(x_n)\frac{\mathrm{d}}{\mathrm{d}x}\sin\theta(x_n) + \sin\theta(x_n)\frac{\mathrm{d}}{\mathrm{d}x}2k_1R(x_n)\right\}$$
$$+\left\{k_1^2R(x_n)\frac{\mathrm{d}}{\mathrm{d}x}\cos\theta(x_n) + \cos\theta(x_n)\frac{\mathrm{d}}{\mathrm{d}x}k_1^2R(x_n)\right\}$$
$$-\left\{R(x_n)\frac{\mathrm{d}}{\mathrm{d}x}\cos\theta(x_n) + \cos\theta(x_n)\frac{\mathrm{d}}{\mathrm{d}x}R(x_n)\right\}\Bigg]$$

$$F'''(x_n) = 6a_3 + b\Big[-\left(2e^{k_1x_n}\cos\theta(x_n) + 2k_1e^{k_1x_n}\sin\theta(x_n)\right)$$
$$+2k_1e^{k_1x_n}\left(-\sin\theta(x_n)\right) + 2k_1^2e^{k_1x_n}\cos\theta(x_n)$$
$$-2k_1R(x_n)\cos\theta(x_n) + \left[2k_1e^{k_1x_n} + 2k_1^2R(x_n)\right]\sin\theta(x_n)$$
$$+k_1^2R(x_n)\left(-\sin\theta(x_n)\right) + k_1^2e^{k_1x_n} + k_1^3R(x_n)\cos\theta(x_n)$$
$$-R(x_n)\left(-\sin\theta(x_n)\right) + \left(e^{k_1x_n} + k_1R(x_n)\right)\cos\theta(x_n)\Big]$$

$$F'''(x_n) = f_n'' = 6a_3 + b\Big[-3e^{k_1x_n}\cos\theta(x_n) - 6k_1e^{k_1x_n}\sin\theta(x_n) + 3k_1^2R(x_n)\sin\theta(x_n) - 2k_1R(x_n)\cos\theta(x_n) + k_1^3R(x_n)\cos\theta(x_n) + R(x_n)\sin\theta(x_n) - k_1e^{k_1x_n}\cos\theta(x_n)\Big] \tag{17}$$

$$\begin{aligned}
F'^{v}(x_n) &= f'''(x_n)\\
&= b\Big[-3\Big\{e^{k_1x_n}\frac{\mathrm{d}}{\mathrm{d}x}\cos\theta(x_n) + \cos\theta(x_n)\frac{\mathrm{d}}{\mathrm{d}x}e^{k_1x_n}\Big\}\\
&\quad -6\Big\{e^{k_1x_n}\frac{\mathrm{d}}{\mathrm{d}x}\sin\theta(x_n) + \sin\theta(x_n)\frac{\mathrm{d}}{\mathrm{d}x}e^{k_1x_n}\Big\}\\
&\quad +3k_1^2\Big\{e^{k_1x_n}\frac{\mathrm{d}}{\mathrm{d}x}\cos\theta(x_n) + \cos\theta(x_n)\frac{\mathrm{d}}{\mathrm{d}x}e^{k_1x_n}\Big\}\\
&\quad -3k_1^2\Big\{R(x_n)\frac{\mathrm{d}}{\mathrm{d}x}\sin\theta(x_n) + \sin\theta(x_n)\frac{\mathrm{d}}{\mathrm{d}x}R(x_n)\Big\}\\
&\quad -2k_1\Big\{R(x_n)\frac{\mathrm{d}}{\mathrm{d}x}\cos\theta(x_n) + \cos\theta(x_n)\frac{\mathrm{d}}{\mathrm{d}x}R(x_n)\Big\}\\
&\quad +k_1^3\Big\{R(x_n)\frac{\mathrm{d}}{\mathrm{d}x}\cos\theta(x_n) + \cos\theta\frac{\mathrm{d}}{\mathrm{d}x}R(x_n)\Big\}\\
&\quad +\Big\{R(x_n)\frac{\mathrm{d}}{\mathrm{d}x}\sin\theta(x_n) + \sin\theta(x_n)\frac{\mathrm{d}}{\mathrm{d}x}R(x_n)\Big\}\\
&\quad -k_1\Big\{e^{k_1x_n}\frac{\mathrm{d}}{\mathrm{d}x}\cos\theta(x_n) + \cos\theta(x_n)\frac{\mathrm{d}}{\mathrm{d}x}e^{k_1x_n}\Big\}
\end{aligned}$$

$$\begin{aligned}
F^{IV}(x_n) = b\Big[&4e^{k_1x_n}\sin\theta(x_n) - 11k_1e^{k_1x_n}\cos\theta(x_n) - 12k_1^2e^{k_1x_n}\sin\theta(x_n)\\
&+4k_1^3e^{k_1x_n}\cos\theta(x_n) - 5k_1^2R(x_n)\cos\theta(x_n) - 4k_1^3R(x_n)\sin\theta(x_n)\\
&+3k_1R(x_n)\sin\theta(x_n) + R(x_n)\cos\theta(x_n) + k_1e^{k_1x_n}\sin\theta(x_n)\\
&-k_1^2e^{k_1x_n}\cos\theta(x_n) + k_1^4R(x_n)\cos\theta(x_n)\Big]
\end{aligned} \tag{18}$$

$$b = \frac{f_n'''}{\begin{aligned}&e^{k_1x_n}\big\{4\sin\theta(x_n) - 11k_1\cos\theta(x_n) - 12k_1^2\sin\theta(x_n) + 4k_1^3\cos\theta(x_n)\\ &+k_1\sin\theta(x_n) - k_1^2\cos\theta(x_n)\big\} + R(x_n)\big\{-5k_1^2\cos\theta(x_n)\\ &-4k_1^3\sin\theta(x_n) + 3k_1\sin\theta(x_n) + \cos\theta(x_n) + k_1^4\cos\theta(x_n)\big\}\end{aligned}} \tag{19}$$

From (17), we have:

$$\begin{aligned}
a_3 = \frac{1}{6}\Big[f_n^2 - b\big\{&-3e^{k_1x_n}\cos\theta(x_n) - 6k_1e^{k_1x_n}\sin\theta(x_n) + 3k_1^2e^{k_1x_n}\cos\theta(x_n)\\
&-3k_1^2R(x_n)\sin\theta(x_n) - 2k_1R(x_n)\cos\theta(x_n) + k_1^3R(x_n)\cos\theta(x_n)\\
&+R(x_n)\sin\theta(x_n) - k_1e^{k_1x_n}\cos\theta(x_n)\big\}\Big]
\end{aligned} \tag{20}$$

Putting (19) in (20), we have:

$$a_3 = \frac{1}{6}\left[f_n^2 - \left(\frac{f_n^3}{\begin{array}{l} e^{k_1 x_n}\left\{4\sin\theta(x_n) - 11k_1\cos\theta(x_n) - 12k_1^2\sin\theta(x_n) + 4k_1^3\cos\theta(x_n)\right. \\ \left. + k_1\sin\theta(x_n) - k_1^2\cos\theta(x_n)\right\} + R(x_n)\left\{-5k_1^2\cos\theta(x_n)\right. \\ \left. -4k_1^3\sin\theta(x_n) + 3k_1\sin\theta(x_n) + \cos\theta(x_n) + k_1^4\cos\theta(x_n)\right\} \end{array}} \right) \right.$$
$$\times\left\{-3e^{k_1 x_n}\cos\theta(x_n) - 6k_1 e^{k_1 x_n}\sin\theta(x_n) + 3k_1^2 e^{k_1 x_n}\cos\theta(x_n) - 3k_1^2 R(x_n)\sin\theta(x_n)\right. \quad (21)$$
$$\left.\left. - 2k_1 R(x_n)\cos\theta(x_n) + k_1^3 R(x_n)\cos\theta(x_n) + R(x_n)\sin\theta(x_n) - k_1 e^{k_1 x_n}\cos\theta(x_n)\right\}\right]$$

Let:

$$\begin{aligned} v = & -3e^{k_1 x_n}\cos\theta(x_n) - 6k_1 e^{k_1 x_n}\sin\theta(x_n) + 3k_1^2 e^{k_1 x_n}\cos\theta(x_n) \\ & - 3k_1^2 R(x_n)\sin\theta(x_n) - 2k_1 R(x_n)\cos\theta(x_n) + k_1^3 R(x_n)\cos\theta(x_n) \\ & + R(x_n)\sin\theta(x_n) - k_1 e^{k_1 x_n}\cos\theta(x_n) \end{aligned} \quad (22)$$

Then, (21) becomes:

$$a_3 = \frac{1}{6}\left[f_n^2 - \left(\frac{f_n^3}{\begin{array}{l} e^{k_1 x_n}\left\{4\sin\theta(x_n) - 11k_1\cos\theta(x_n) - 12k_1^2\sin\theta(x_n) + 4k_1^3\cos\theta(x_n)\right. \\ \left. + k_1\sin\theta(x_n) - k_1^2\cos\theta(x_n)\right\} + R(x_n)\left\{-5k_1^2\cos\theta(x_n)\right. \\ \left. -4k_1^3\sin\theta(x_n) + 3k_1\sin\theta(x_n) + \cos\theta(x_n) + k_1^4\cos\theta(x_n)\right\} \end{array}} \right) v \right] \quad (23)$$

From (16), we have:

$$a_2 = \frac{1}{2}\left[f_n^1 - \left(f_n^2 - \left(\frac{f_n^3}{\begin{array}{l} e^{k_1 x_n}\left\{4\sin\theta(x_n) - 11k_1\cos\theta(x_n) - 12k_1^2\sin\theta(x_n) + 4k_1^3\cos\theta(x_n)\right. \\ \left. + k_1\sin\theta(x_n) - k_1^2\cos\theta(x_n)\right\} + R(x_n)\left\{-5k_1^2\cos\theta(x_n)\right. \\ \left. -4k_1^3\sin\theta(x_n) + 3k_1\sin\theta(x_n) + \cos\theta(x_n) + k_1^4\cos\theta(x_n)\right\} \end{array}} \right) v \right) x_n \right.$$
$$-b\left\{-2e^{k_1 x_n}\sin\theta(x_n) - 2k_1 e^{k_1 x_n}\cos\theta(x_n) - 2k_1 R(x_n)\sin\theta(x_n)\right. \quad (24)$$
$$\left.\left. + k_1^2 R(x_n)\cos\theta(x_n) - R(x_n)\cos\theta(x_n)\right\}\right]$$

Putting (19) and (23) in (24) we have:

$$a_2 = \frac{1}{2}\left[f_n^1 - \left(f_n^2 - \left(\frac{f_n^3}{\begin{array}{l} e^{k_1x_n}\left\{4\sin\theta(x_n) - 11k_1\cos\theta(x_n) - 12k_1^2\sin\theta(x_n) + 4k_1^3\cos\theta(x_n)\right. \\ \left. +k_1\sin\theta(x_n) - k_1^2\cos\theta(x_n)\right\} + R(x_n)\left\{-5k_1^2\cos\theta(x_n)\right. \\ \left. -4k_1^3\sin\theta(x_n) + 3k_1\sin\theta(x_n) + \cos\theta(x_n) + k_1^4\cos\theta(x_n)\right\} \end{array}} \right) v \right) x_n \right.$$
$$- \left(\frac{f_n^3}{\begin{array}{l} e^{k_1x_n}\left\{4\sin\theta(x_n) - 11k_1\cos\theta(x_n) - 12k_1^2\sin\theta(x_n) + 4k_1^3\cos\theta(x_n)\right. \\ \left. +k_1\sin\theta(x_n) - k_1^2\cos\theta(x_n)\right\} + R(x_n)\left\{-5k_1^2\cos\theta(x_n)\right. \\ \left. -4k_1^3\sin\theta(x_n) + 3k_1\sin\theta(x_n) + \cos\theta(x_n) + k_1^4\cos\theta(x_n)\right\} \end{array}} \right)$$
$$\times\left\{-2e^{k_1x_n}\sin\theta(x_n) - 2k_1e^{k_1x_n}\cos\theta(x_n) - 2k_1R(x_n)\sin\theta(x_n)\right.$$
$$\left.\left. + k_1^2R(x_n)\cos\theta(x_n) - R(x_n)\cos\theta(x_n)\right\}\right] \tag{25}$$

From (13) we have:

$$a_1 = f_n - 2a_2x_n - 3a_3x_n^2 - b\left[e^{k_1x_n}\cos\theta(x_n) + k_1R(x_n)\cos\theta(x_n) - R(x_n)\sin\theta(x_n)\right] \tag{26}$$

Putting (19), (23) and (25) in (26), we have:

$$a_1 = f_n - \left(\left[\frac{\begin{array}{l} -\left[f_n^2 - \left\{-3e^{k_1x_n}\cos\theta(x_n) - 6k_1e^{k_1x_n}\sin\theta(x_n) + 3k_1^2e^{k_1x_n}\cos\theta(x_n)\right.\right. \\ \left.\left. -3k_1^2R(x_n)\sin\theta(x_n) - 2k_1R(x_n)\cos\theta(x_n)\right\} f_n^3 \right] \end{array}}{\begin{array}{l} e^{k_1x_n}\left\{4\sin\theta(x_n) - 11k_1\cos\theta(x_n) - 12k_1^2\sin\theta(x_n) + 4k_1^3\cos\theta(x_n)\right. \\ \left. +k_1\sin\theta(x_n) - k_1^2\cos\theta(x_n)\right\} + R(x_n)\left\{-5k_1^2\cos\theta(x_n)\right. \\ \left. -4k_1^3\sin\theta(x_n) + 3k_1\sin\theta(x_n) + \cos\theta(x_n) + k_1^4\cos\theta(x_n)\right\} \end{array}} \right] x_n \right.$$
$$\left. - \frac{\begin{array}{l} \left\{-2e^{k_1x_n}\sin\theta(x_n) - 2k_1e^{k_1x_n}\cos\theta(x_n) - 2k_1R(x_n)\sin\theta(x_n)\right. \\ \left. +k_1^2R(x_n)\cos\theta(x_n) - R(x_n)\cos\theta(x_n)\right\} \end{array}}{\begin{array}{l} e^{k_1x_n}\left\{4\sin\theta(x_n) - 11k_1\cos\theta(x_n) - 12k_1^2\sin\theta(x_n) + 4k_1^3\cos\theta(x_n)\right. \\ \left. +k_1\sin\theta(x_n) - k_1^2\cos\theta(x_n)\right\} + R(x_n)\left\{-5k_1^2\cos\theta(x_n)\right. \\ \left. -4k_1^3\sin\theta(x_n) + 3k_1\sin\theta(x_n) + \cos\theta(x_n) + k_1^4\cos\theta(x_n)\right\} \end{array}} \right) x_n$$

$$-\frac{1}{2}\left[f_n^2-\frac{\begin{array}{l}\left\{-3e^{k_1x_n}\cos\theta(x_n)-6k_1e^{k_1x_n}\sin\theta(x_n)+3k_1^2e^{k_1x_n}\cos\theta(x_n)\right.\\ \left.-3k_1^2R(x_n)\sin\theta(x_n)-2k_1R(x_n)\cos\theta(x_n)\right\}\end{array}}{\begin{array}{l}e^{k_1x_n}\left\{4\sin\theta(x_n)-11k_1\cos\theta(x_n)-12k_1^2\sin\theta(x_n)+4k_1^3\cos\theta(x_n)\right.\\ \left.+k_1\sin\theta(x_n)-k_1^2\cos\theta(x_n)\right\}+R(x_n)\left\{-5k_1^2\cos\theta(x_n)\right.\\ \left.-4k_1^3\sin\theta(x_n)+3k_1\sin\theta(x_n)+\cos\theta(x_n)+k_1^4\cos\theta(x_n)\right\}\end{array}}\right]x_n^2$$

$$-\left(\frac{f_n^3}{\begin{array}{l}e^{k_1x_n}\left\{4\sin\theta(x_n)-11k_1\cos\theta(x_n)-12k_1^2\sin\theta(x_n)+4k_1^3\cos\theta(x_n)\right.\\ \left.+k_1\sin\theta(x_n)-k_1^2\cos\theta(x_n)\right\}+R(x_n)\left\{-5k_1^2\cos\theta(x_n)\right.\\ \left.-4k_1^3\sin\theta(x_n)+3k_1\sin\theta(x_n)+\cos\theta(x_n)+k_1^4\cos\theta(x_n)\right\}\end{array}}\right) \tag{27}$$
$$\times\left[e^{k_1x_n}\cos\theta(x_n)+k_1R(x_n)\cos\theta(x_n)-R(x_n)\sin\theta(x_n)\right]$$

For preservative of the scheme, then we can write the new scheme in a more compact form as:

$$\begin{aligned}y_{n+1}=y_n&+a_1h+a^2a_2(2n+1)+h^3a_3(3n^2+3n+1)\\&+bR_n\left[he^{k_1}h(\cos\theta_n\cos k_2h-\sin\theta_n\sin k_2h)-\cos\theta_n\right]\end{aligned}$$

Putting a_1, a_2 and a_3 as derived above, we arrived at a new scheme. But to test the scheme, we shall proceed to write a programme which will command the scheme to solve some firs order differential equations.

3. Implementation of the Scheem

PROBLEM 1. A spring with a mass of 2 kg has natural length m. A force of 5N is required to maintain it stretched to a length of m. If the spring is stretched to a length of m and then released with initial velocity 0, find the position of the mass at any time.

$$k(0.2)=25.6 \tag{1}$$

$$\text{So,}\quad k=\frac{25.6}{0.2}=128. \tag{2}$$

Using this value of the spring constant k, together with m = 2 then, we have

$$2\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+128x=0 \tag{3}$$

$$x(t)=c_1\cos 8t+c_2\sin 8t \tag{4}$$

$$x'(t)=-8c_1\sin 8t+8c_2\cos 8t \tag{5}$$

Since the initial velocity is given as $x'(0)=0$, we have $c_2=0$ and so the solution is

$$x(t)=\frac{1}{5}\cos 8t \tag{6}$$

The result compared favourably with the theoretical solutions.

PROBLEM 2:

Suppose that the spring of Problem 1 is immersed in a fluid with damping constant. Find the position of the mass at any time if it starts from the equilibrium Position and is given a push to start it with an initial velocity of m/s.

Mathematical Interpretation of the Problem

From Problem 1, the mass is $m=2$ and the spring constant is $k=128$, so the differential Equation (3) becomes

$$2\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+40\frac{\mathrm{d}x}{\mathrm{d}t}+128x=0 \tag{7}$$

or

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+20\frac{\mathrm{d}x}{\mathrm{d}t}+64x=0 \tag{8}$$

$$x(t)=c_1e^{-4t}+c_2e^{-16t} \tag{9}$$

We are given that $x(0)=0$, $c_1+c_2=0$. Differentiating, we get

$$x'(t)=-4c_1e^{-4t}+16c_2e^{-16t} \tag{10}$$

So

$$x'(0)=-4c_1+16c_2=0.6 \tag{11}$$

Since $c_2=-c_1$, this gives $12c_1=0.6$ or $c_1=0.05$. Therefore

$$x=0.05\left(e^{-4t}-e^{-16t}\right) \tag{12}$$

The result compares favourably with the theoretical solutions.

The result compares favourably with the theoretical solutions.

4. Conclusions

The procedure for development and implementation of a numerical algorithm has been examined in this work. The method employed a basis function for approximation. The scheme that formed the derived method was implemented to test the accuracy. The method gave a numerical algorithm capable of solving second order problems of ordinary differential equations. The new numerical algorithm scheme was employed to solve second order initial value problems of ordinary differential equations problems. The computations were carried out in computer codes. **Tables 1-4** represent the numerical results of the examples at various values of step size (H). From the tables, it could be concluded that a smaller value of H, gives a better result. **Figures 1-4** readily show the comparison of the newly derived method and the theoretical solution. Since the results obtained compare favorably with the theoretical, this shows the accuracy of the new method and so can be said to be approximately correct.

Table 1. Numerical results of problem 1 at $H = 0.01$.

$X(n)$	THEORITICAL SOLUTION	OLUBUNMI	TRUN
0.00000000	0.20000000	0.20000000	0.00000000
0.01000000	0.19936034	0.20000000	0.00063966
0.02000000	0.19744547	0.19999953	0.00255406
0.03000000	0.19426760	0.19626706	0.00199946
0.04000000	0.18984708	0.19087657	0.00102949
0.05000000	0.18421221	0.18522314	0.00101093
0.06000000	0.17739899	0,17885647	0.00145748
0.07000000	0.16945103	0.17057349	0.00112246
0.08000000	0.16041915	0.16179043	0.00137128
0.09000000	0.15036115	0.15114648	0.00078533
0.09999999	0.13934135	0.14011432	0.00077297
0.11000000	0.12743023	0.12878653	0.0013563
0.12000000	0.11470401	0.11534238	0.00063837
0.13000000	0.10124406	0.10815221	0.00690815
0.14000000	0.08713649	0.08937685	0.00224036
0.14999999	0.07247157	0.07387969	0.00140812
0.16000000	0.05734305	0.05876521	0.00142216
0.17000000	0.04184773	0.04497863	0.0031309
0.17999999	0.02608475	0.02765649	0.00157174
0.19000000	0.01015490	0.01217869	0.00202379
0.19999999	0.00583989	0.00592321	8.332E−05

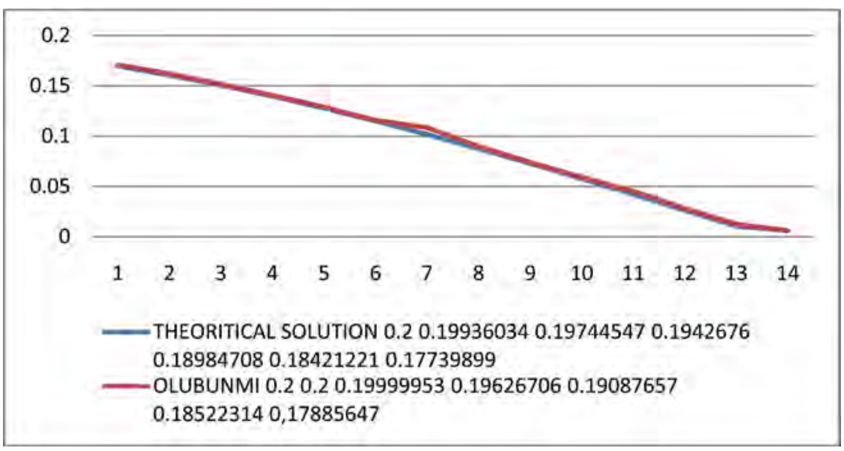

Figure 1. Graphical presentation of problem 1.

Table 2. Numerical results of problem 1 at $H = 0.001$.

$X(n)$	THEORETICAL SOLUTION	OLUBUNMI	TRUN
0.00000000	0.20000000	0.20000000	0.00000000
0.00100000	0.19999361	0.20000000	0.00000639
0.00200000	0.19997440	0.19998160	0.00000720
0.00300000	0.19994241	0.19995000	0.00000759
0.00400000	0.19989762	0.19990601	0.00000839
0.00500000	0.19984002	0.19984885	0.00000883
0.00600000	0.19976965	0.19977901	0.00000936
0.00700000	0.19968648	0.19969999	0.00001351
0.00800000	0.19959055	0.19960510	0.00001455
0.00900000	0.19948183	0.19949996	0.00001813
0.01000000	0.19936034	0.19937993	0.00001959
0.01100000	0.19922610	0.19924990	0.00002380
0.01200000	0.19907911	0.19910485	0.00002574
0.01300000	0.19891937	0.19894679	0.00002742
0.01400000	0.19874692	0.19877972	0.00003280
0.01500000	0.19856173	0.19859963	0.00003790
0.01600000	0.19836384	0.19840453	0.00004069
0.01700000	0.19815326	0.19819939	0.00004613
0.01800000	0.19792998	0.19797923	0.00004925
0.01900000	0.19769405	0.19774904	0.00005499
0.02000000	0.19744545	0.19750381	0.00005836

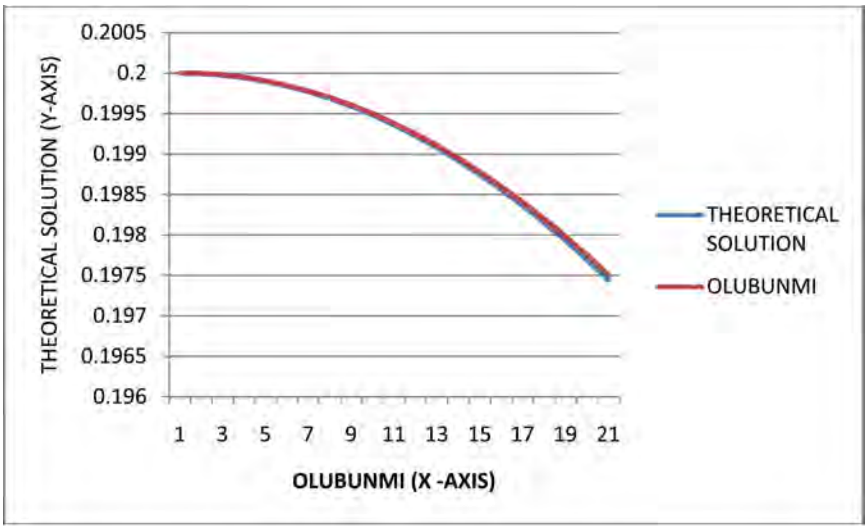

Figure 2. Graphical presentation of problem 1.

Table 3. Numerical results of problem 2 at $H = 0.001$.

$X(n)$	THEORETICAL SOLUTION	OLUBUNMI	TRUN
0.00000000	0.10000000	0.10000000	0.0000000
0.00100000	0.09900676	0.09900348	0.00000328
0.00200000	0.09802692	0.09807573	0.00004891
0.00300000	0.09706028	0.09713825	0.00007797
0.00400000	0.09610662	0.09618537	0.00007875
0.00500000	0.09516575	0.09527836	0.00011261
0.00600000	0.09423748	0.09434861	0.00011113
0.00700000	0.09332163	0.09343216	0.00011053
0.00800000	0.09241800	0.09252773	0.00010973
0.00900000	0.09152640	0.09173281	0.00020641
0.01000000	0.09064666	0.09081132	0.00016472
0.01100000	0.08977859	0.08999590	0.00021731
0.01200000	0.08892203	0.08899999	0.00022040
0.01300000	0.08807679	0.08835681	0.00028002
0.01400000	0.08724272	0.08752332	0.00028079
0.01500000	0.08641962	0.08673274	0.00031312
0.01600000	0.08560735	0.08570748	0.00030213
0.01700000	0.08480573	0.08413814	0.00043241
0.01800000	0.08401462	0.08427880	0.00051327
0.01900000	0.08323386	0.08382617	0.00059231
0.02000000	0.08246327	0.08209541	0.00063214

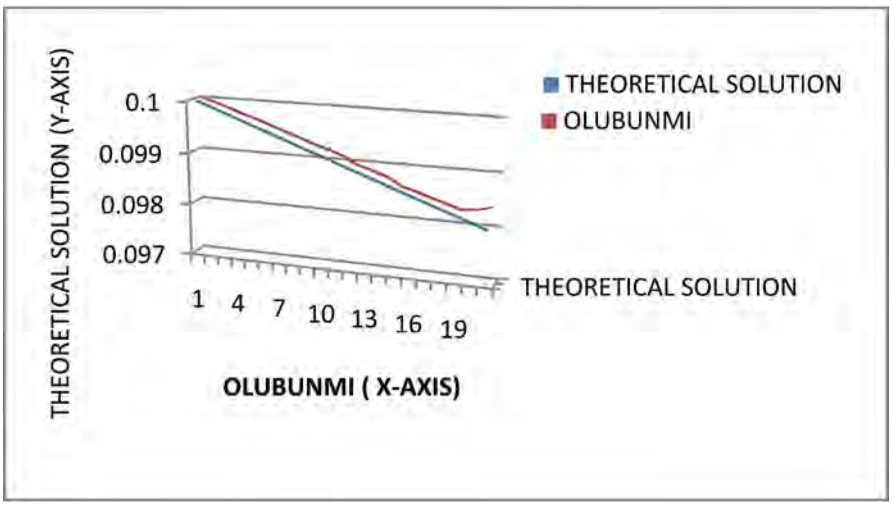

Figure 3. Graphical presentation of problem 2.

Table 4. Numerical results of problem 2 at $H = 0.0001$.

$X(n)$	THEORETICAL SOLUTION	OLUBUNMI	TRUN
0.00000000	0.10000000	0.10000000	0.00000000
0.00010000	0.09990007	0.09992720	0.00002713
0.00020000	0.09980027	0.09983454	0.00003427
0.00030000	0.09970061	0.09974292	0.00004231
0.00040000	0.09960109	0.09965531	0.00005422
0.00050000	0.09950170	0.09956093	0.00005923
0.00060000	0.09940244	0.09946347	0.00006123
0.00070000	0.09930332	0.09937088	0.00006676
0.00080000	0.09920434	0.09927559	0.00007125
0.00090000	0.09910548	0.09918759	0.00008211
0.00100000	0.09900676	0.09909777	0.00009101
0.00110000	0.09890819	0.09894247	0.00010113
0.00120000	0.09880973	0.09886919	0.00013274
0.00130000	0.09871142	0.09877561	0.00015777
0.00140000	0.09861323	0.09861727	0.00016238
0.00150000	0.09851518	0.09854087	0.00018621
0.00160000	0.09841727	0.09845110	0.00020000
0.00170000	0.09831949	0.09836652	0.00022135
0.00180000	0.09822183	0.09827469	0.00022927
0.00190000	0.09812431	0.09830139	0.00024221
0.00200000	0.09802692	0.09837561	0.00027725

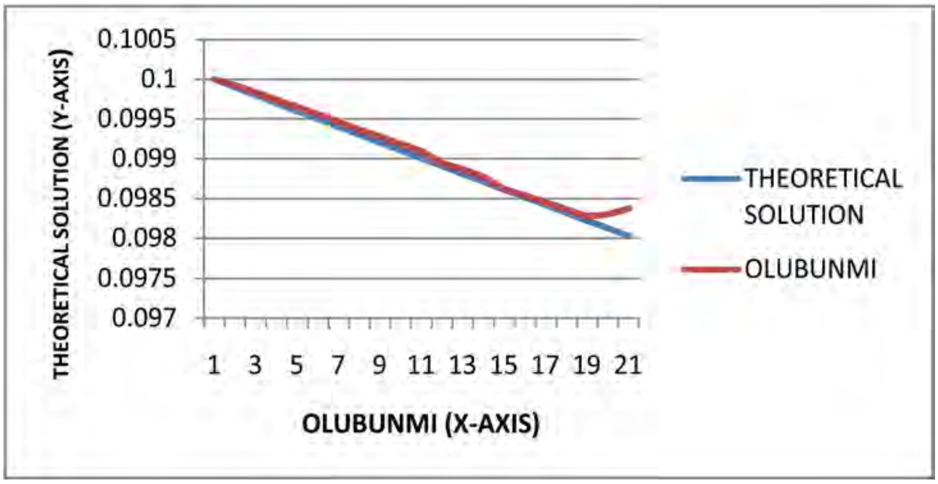

Figure 4. Graphical presentation of problem 2.

In our subsequent research, we shall pay more attention on the implementation of this new scheme to solve some higher order initial value problems of ordinary differential equation and also compare the results with the existing methods and thereafter we examine the characteristics properties such as the stability, convergence, accuracy and consistency of the scheme.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Behavior of a Scale Factor for Wiener Integrals of an Unbounded Function

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Abstract

The purpose of this paper is to investigate the behavior of a scale factor for Wiener integrals about the unbounded function $F(x)=\exp\left\{a\sum_{j=1}^{n}\int_0^T\alpha_j \mathrm{d}x\right\}$, where $\{\alpha_1,\alpha_2,\cdots,\alpha_n\}$ is an orthonormal set of elements in $L_2[0,T]$ on the Wiener space $C_0[0,T]$.

Keywords

Wiener Space, Wiener Integral, Analytic Wiener Integral, Analytic Feynman Integral, Scale Factor

1. Introduction

In [1], M. D. Brue introduced the functional transform on the Feynman integral (1972). In [2], R. H. Cameron wrote the paper about the translation pathology of a Wiener spac (1972). In [3] [4] [5], R. H. Cameron and W. T. Martin proved some theorems on the transformation and the translation and used the expression of the change of scale for Wiener integrals (1944, 1947). In [6] [7], R. H. Cameron and D. A. Storvick proved relationships between Wiener integrals and analytic Feynman integrals to prove the change of scale formula for Wiener integral on the Wiener space in 1987. In [8], M. D. Gaysinsky and M. S. Goldstein proved the Self-Adjointness of a Schrödinger Operator and Wiener Integrals (1992).

In [9], G. W. Johnson and M. L. Lapidus wrote the paper about the Feynman integral and Feynman's Operational Calculus (2000). In [10], G. W. Johnson and D. L. Skoug proved the scale-invariant measurability in Wiener Space (1979).

In [11] and [12], Y. S. Kim proved **a change of scale** formula for Wiener integrals about cylinder functions $f\left((h_1 x)^{\sim},\cdots,(h_n,x)^{\sim}\right)$ with $f\in L_p\left(R^n\right),1\le p\le\infty$ on the abstract Wiener space: the analytic Wiener integral exists for $f\in L_p\left(R^n\right),1\le p\le\infty$, and the analytic Feynman integral exists for $f\in L_1\left(R^n\right)$ (1998) and (2001). But the Feynman integral does not always exist for $1<p$.

In [13], Y. S. Kim investigates a behavior of a scale factor for the Wiener integral of a function $F(x)=\exp\left\{\int_0^T\theta(t,x(t))\mathrm{d}t\right\}$, where $\theta:[0,T]\times R\to\mathbf{C}$ is defined by $\theta(t,u)=\int_R\exp\{iuv\}\mathrm{d}\sigma_t(v)$ which is a Fourier-Stieltzes transform of a complex Borel measure $\sigma_t\in\mathbf{M}(R)$ and $\mathbf{M}(R)$ is a set of complex Borel measures defined on R.

In this paper, we investigate the behavior of a scale factor $\rho>0$ for the Wiener integral $\int_{C_0[0,T]}F(\rho x)\mathrm{d}m(x)$ which is defined on the Wiener space $C_0[0,T]$ about the unbounded function $F(x)=\exp\left\{a\sum_{j=1}^n\int_0^T\alpha_j\mathrm{d}x\right\}$ with $a>0$, where $\{\alpha_1,\alpha_2,\cdots,\alpha_n\}$ is an orthonormal set of elements in $L_2[0,T]$ on the Wiener space $C_0[0,T]$.

2. Definitions and Preliminaries

Let $C_0[0,T]$ denote the space of real-valued continuous functions x on $[0,T]$ such that $x(0)=0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let m denote a Wiener measure and $(C_0[0,T],\mathcal{M},m)$ be a Wiener measure space and we denote the Wiener integral of a function $F:C_0[0,T]\to\mathbf{C}$ by $\int_{C_0[0,T]}F(x)\mathrm{d}m(x)$.

A subset E of $C_0[0,T]$ is said to be scale-invariant measurable if $\rho E\in\mathcal{M}$ for each $\rho>0$, and a scale-invariant measurable set N is said to be scale-invariant null if $m(\rho N)=0$ for each $\rho>0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). If two functionals F and G are equal s-a.e., we write $F\approx G$. A function F defined on the scale invariant measurable set E is a scale invariant measurable function if $F(\alpha x)$ is a Wiener measurable function for all $\alpha>0$.

Throughout this paper, let $\mathbf{R}^n$ denote the n-dimensional Euclidean space and let $\mathbf{C},\mathbf{C}_+$, and $\mathbf{C}_+^{\sim}$ denote the set of complex numbers, the set of complex numbers with positive real part, and the set of non-zero complex numbers with nonnegative real part, respectively.

Definition 2.1. *Let F be a complex-valued measurable function on* $C_0[0,T]$ *such that the integral*

$$J(F;\lambda)=\int_{C_0[0,T]}F\left(\lambda^{-\frac{1}{2}}x\right)\mathrm{d}m(x) \tag{2.1}$$

exists for all real $\lambda>0$. If there exists a function $J^*(F;z)$ analytic on $\mathbf{C}_+$ such that $J^*(F;\lambda)=J(F;\lambda)$ for all real $\lambda>0$, then we define $J^*(F;z)$ to be the *analytic Wiener integral* of F over $C_0[0,T]$ with parameter z, and for each $z \in \mathbf{C}_+$, we write

$$I^{aw}(F;z) = J^*(F;z) \equiv \int_{C_0[0,T]}^{anw_z} F(x)\,\mathrm{d}m(x). \tag{2.2}$$

Let q be a non-zero real number and let F be a function defined on $C_0[0,T]$ whose analytic Wiener integral exists for each z in $\mathbf{C}_+$. If the following limit exists, then we call it the *analytic Feynman integral* of F over $C_0[0,T]$ with parameter q, and we write

$$I^{af}(F;q) = \lim_{z\to -iq} I^{aw}(F;z) \equiv \int_{C_0[0,T]}^{anw_q} F(x)\,\mathrm{d}m(x), \tag{2.3}$$

where z approaches $-iq$ through $\mathbf{C}_+$ and $i^2 = -1$.

Let $\{e_k\}_{k=1}^{n}$ be a complete orthonormal set and $e_k \in C[0,T] \cap B[0,T]$ for $k = 1, 2, \cdots, n$ and $\alpha \in L_2[0,T]$ and $x \in C_0[0,T]$. We define a Paley-Wiener-Zygmund integral (P.W.Z) of x with respect to α by

$$\int_0^T \alpha(t)\,\mathrm{d}x(t) \equiv \lim_{n\to\infty} \int_0^T \sum_{k=1}^{n} (\alpha, e_k) e_k(t)\,\mathrm{d}x(t).$$

Theorem 2.2 (Wiener Integration Formula). *Let $C_0[0,T]$ be a Wiener space. Then*

$$\begin{aligned} &\int_{C_0[0,T]} f\left(\int_0^T \alpha_1 \mathrm{d}x, \int_0^T \alpha_2 \mathrm{d}x, \cdots, \int_0^T \alpha_n \mathrm{d}x\right)\mathrm{d}m(x) \\ &= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbf{R}^n} f(\vec{u}) \exp\left\{-\frac{1}{2}\sum_{j=1}^{n} u_j^2\right\}\mathrm{d}\vec{u} \end{aligned} \tag{2.4}$$

where $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is an orthonormal set of elements in $L_2[0,T]$ and $f : \mathbf{R}^n \to \mathbf{C}$ is a Lebesgue measurable function and $\vec{u} = (u_1, u_2, \cdots, u_n)$ and $\mathrm{d}\vec{u} = \mathrm{d}u_1 \mathrm{d}u_2 \cdots \mathrm{d}u_n$ and $\int_0^T \alpha_j \mathrm{d}x$ is a Paley-Wiener-Zygmund integral for $1 \le j \le n$.

Remark. We will use several times the following well-known integration formula:

$$\int_{\mathbf{R}} \exp\left\{-au^2 + ibu\right\}\mathrm{d}u = \sqrt{\frac{\pi}{a}} \exp\left\{-\frac{b^2}{4a}\right\} \tag{2.5}$$

where a is a complex number with $Re\, a > 0$, b is a real number, and $i^2 = -1$.

3. Main Results

Define a function $F : C_0[0,T] \to \mathbf{C}$ on the Wiener space by

$$F(x) = \exp\left\{a \sum_{j=1}^{n} \int_0^T \alpha_j \mathrm{d}x\right\} \tag{3.1}$$

where $a > 0$ is a finite real number and $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is an orthonormal set of elements in $L_2[0,T]$.

Lemma 3.1. *For a finite real number $a > 0$, the unbounded cylinder function $F(x)$ in* (3.1) *is a Wiener integrable function.*

Proof. By the Wiener integration Formula (2.4), we have that for a finite real number $a>0$,

$$\begin{aligned}&\int_{C_0[0,T]} F(x)\,\mathrm{d}m(x)\\&=\left(\frac{1}{2\pi}\right)^{\frac{n}{2}}\cdot\int_{R^n}\exp\left\{a\sum_{j=1}^{n}u_j-\frac{1}{2}\sum_{j=1}^{n}u_j^2\right\}\mathrm{d}\vec{u}\\&=\left(\frac{1}{2\pi}\right)^{\frac{n}{2}}\cdot(2\pi)^{\frac{n}{2}}\cdot\exp\left\{+\frac{n}{2}a^2\right\}\\&=\exp\left\{+\frac{n}{2}a^2\right\}<\infty\end{aligned}\tag{3.2}$$

Remark. If we let $F(x)=f\left(\int_0^T\alpha_1\mathrm{d}x,\int_0^T\alpha_2\mathrm{d}x,\cdots,\int_0^T\alpha_n\mathrm{d}x\right)$ and $f:R^n\to\mathbf{C}$, then $f(\vec{u})=\exp\left\{a\sum_{j=1}^{n}u_j\right\}$ is unbounded for a finite real number $a>0$.

Lemma 3.2. *Let* $F:C_0[0,T]\to\mathbf{C}$ *be defined by* (3.1)*. For a finite real* $\rho>0$ *and a finite real* $a>0$,

$$\int_{C_0[0,T]} F(\rho x)\,\mathrm{d}m(x)=\exp\left\{+\frac{n}{2}a^2\rho^2\right\}\tag{3.3}$$

Proof. By the Wiener integration Formula (2.4), we have that

$$\begin{aligned}&\int_{C_0[0,T]} F(\rho x)\,\mathrm{d}m(x)\\&=\left(\frac{1}{2\pi}\right)^{\frac{n}{2}}\cdot\int_{R^n}\exp\left\{a\rho\sum_{j=1}^{n}u_j-\frac{1}{2}\sum_{j=1}^{n}u_j^2\right\}\mathrm{d}\vec{u}\\&=\left(\frac{1}{2\pi}\right)^{\frac{n}{2}}\cdot(2\pi)^{\frac{n}{2}}\cdot\exp\left\{+\frac{n}{2}a^2\rho^2\right\}\\&=\exp\left\{+\frac{n}{2}a^2\rho^2\right\}<\infty\end{aligned}\tag{3.4}$$

Lemma 3.3. *Let* $F:C_0[0,T]\to\mathbf{C}$ *be defined by* (3.1)*. For a finite real* $\rho>0$ *and a finte real* $a>0$,

$$\int_{C_0[0,T]} F(\rho x)\,\mathrm{d}m(x)=\left[\int_{C_0[0,T]} F(x)\,\mathrm{d}m(x)\right]^{\rho^2}\tag{3.5}$$

Proof. By the above Lemma, we have that

$$\begin{aligned}&\int_{C_0[0,T]} F(\rho x)\,\mathrm{d}m(x)\\&=\exp\left\{+\frac{n}{2}a^2\rho^2\right\}\\&=\left[\exp\left\{+\frac{n}{2}a^2\right\}\right]^{\rho^2}\\&=\left[\int_{C_0[0,T]} F(x)\,\mathrm{d}m(x)\right]^{\rho^2}\end{aligned}\tag{3.6}$$

Now we define a concept of **the scale factor for the Wiener integral** which was first defined in [13]:

Definition 3.4. *We define* **the scale factor for the Wiener integral** *by the* *real number* $\rho > 0$ *of the absolute value of the Wiener integral:*

$$G(\rho) = \left| \int_{C_0[0,T]} F(\rho x)\,\mathrm{d}m(x) \right| \tag{3.7}$$

where $G : R \to \mathbf{R}$ is a real valued function defined on *R*.

Property. <Behavior of a scale factor for the Wiener integral.>

We investigate the interesting behavior of the scale factor for the Wiener integral by analyzing the Wiener integral as followings: For real $\rho > 0$ and for a finite real number $a > 0$,

$$\int_{C_0[0,T]} F(\rho x)\,\mathrm{d}m(x) = \exp\left\{ +\frac{n}{2} a^2 \rho^2 \right\}. \tag{3.8}$$

Example. For the scale factor $\rho = \left\{ \cdots, \frac{1}{100}, \frac{1}{10}, 1, 10, 10^2, \cdots \right\}$, we can investigate the very interesting behavior of the Wiener integral:

$$\begin{aligned}
&\text{(a)}\ \int_{C_0[0,T]} F\left(\frac{1}{100}x\right)\mathrm{d}m(x) = \left[\int_{C_0[0,T]} F(x)\,\mathrm{d}m(x) \right]^{\frac{1}{10000}} \\
&\text{(b)}\ \int_{C_0[0,T]} F\left(\frac{1}{10}x\right)\mathrm{d}m(x) = \left[\int_{C_0[0,T]} F(x)\,\mathrm{d}m(x) \right]^{\frac{1}{100}} \\
&\text{(c)}\ \left[\int_{C_0[0,T]} F\left(\frac{1}{100}x\right)\mathrm{d}m(x) \right]^{100} = \left[\int_{C_0[0,T]} F\left(\frac{1}{10}x\right)\mathrm{d}m(x) \right] \\
&\text{(d)}\ \int_{C_0[0,T]} F(\rho x)\,\mathrm{d}m(x) = \left[\int_{C_0[0,T]} F(x)\,\mathrm{d}m(x) \right]^{\rho^2}
\end{aligned} \tag{3.9}$$

Remark. <Interpretation of a scale factor for Wiener integrals of an unbounded cylinder function.>

1) Whenever the scale factor $\rho > 1$ is increasing, the Wiener integral increases very rapidly. Whenever the scale factor $0 < \rho < 1$ is decreasing, the Wiener integral decreases very rapidly.

2) The function $G(\rho) = \left| \int_{C_0[0,T]} F(\rho x)\,\mathrm{d}m(x) \right|$ for $F(x)$ in (3.1) is an increasing function of a scale factor $\rho > 0$, because the exponential function $y = \mathrm{e}^{x^2}$ is an increasing function of $x \in R$.

3) Whenever the scale factor $\rho > 0$ is increasing and decreasing, the Wiener integral varies very rapidly.

4. Conclusions

What we have done in this research is that we investigate the very interesting behavior of the scale factor for the Wiener integral of an **unbounded** function.

From these results, we find an interesting property for the Wiener integral as a function of a scale factor which was first defined in [13].

Note that the function in [13] is bounded and the function of this paper is unbounded!

Finally, we introduce the motivation and the application of the Wiener integral and the Feynman integral and the relationship between the scale factor and the heat (or diffusion) equation:

Remark.

1) The solution of the heat (or diffusion) equation

$$\frac{\partial \psi}{\partial t} = -\frac{i}{h}\left[-\frac{h^2}{2m}\frac{\partial \psi^2}{\partial \xi^2} + V(\xi)\psi\right], \tag{3.10}$$

is that for a real $\lambda > 0$,

$$\psi_\lambda(t,\xi) = \int_{C_0^t} \exp\left\{-\frac{i}{h}\int_0^t V\left(\lambda^{-\frac{1}{2}}x(s)+\xi\right)\mathrm{d}s\right\}\cdot\psi\left(\lambda^{-\frac{1}{2}}x(s)+\xi\right)\mathrm{d}m(s) \tag{3.11}$$

where $\psi_\lambda(\cdot,\xi) = \phi(\xi)$ and $\phi \in L_2(R^d)$ and $\xi \in R^d$ and $x(\cdot)$ is a R^d-valued continuous function defined on $[0,t]$ such that $x(0) = 0$.

2) $H = -\Delta + V$ is the energy operator (or, Hamiltonian) and Δ is a Laplacian and $V : R^d \to R$ is a potential. This Formula (3.11) is called the Feynman-Kac formula. The application of the Feynman-Kac Formula (in various settings) has been given in the area: diffusion equations, the spectral theory of the schrödinger operator, quantum mechanics, statistical physics, for more details, see the paper [8] and the book [12].

3) If we let $\lambda = \rho^{-2}$, the solution of this heat (or diffusion) equation is

$$\psi_\rho(t,\xi) = \int_{C_0^t} \exp\left\{-\frac{i}{h}\int_0^t V(\rho x(s)+\xi)\mathrm{d}s\right\}\cdot\phi(\rho x(s)+\xi)\mathrm{d}m(s) \tag{3.12}$$

4) If we let $h = \frac{m}{i\lambda} = -im\rho^{-2}$, then

$$\psi_\rho(t,\xi) = \int_{C_0^t} \exp\left\{+m\rho^2\int_0^t V(\rho x(s)+\xi)\mathrm{d}s\right\}\cdot\phi(\rho x(s)+\xi)\mathrm{d}m(s) \tag{3.13}$$

is a solution of a heat (or diffusion) equation:

$$\frac{\partial \psi}{\partial t} = \frac{1}{m\rho^2}\left[\left(\frac{m^2\rho^2}{2m}\right)\frac{\partial \psi^2}{\partial \xi^2} + V(\xi)\psi\right]. \tag{3.14}$$

This equation is of the form:

$$\frac{\partial \psi}{\partial t} = \frac{1}{2}\frac{\partial \psi^2}{\partial \xi^2} + \frac{1}{m\rho^2}V(\xi)\psi. \tag{3.15}$$

5) If we let $F(x) = \exp\left\{-\frac{i}{h}\int_0^t V\left(\lambda^{-\frac{1}{2}}x(s)+\xi\right)\mathrm{d}s\cdot\phi\left(\lambda^{-\frac{1}{2}}x(s)+\xi\right)\right\}$, then we can express the solution of the heat (or diffusion) equation by the formula

$$\psi_\rho(t,\xi) = \int_{C_0^t} F(\rho x)\mathrm{d}m(x), \psi_\lambda(t,\xi) = \int_{C_0^t} F\left(\lambda^{-\frac{1}{2}}x\right)\mathrm{d}m(x) \tag{3.16}$$

6) By this motivation, we first define the scale factor of the Wiener integral by the real number $\rho > 0$ in the paper [13].

Remark. <Gratitude for the Refree> I am very grateful for the referee to comment in details.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Selection of Heteroscedastic Models: A Time Series Forecasting Approach

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Abstract

To overcome the weaknesses of in-sample model selection, this study adopted out-of-sample model selection approach for selecting models with improved forecasting accuracies and performances. Daily closing share prices were obtained from Diamond Bank and Fidelity Bank as listed in the Nigerian Stock Exchange spanning from January 3, 2006 to December 30, 2016. Thus, a total of 2713 observations were explored and were divided into two portions. The first which ranged from January 3, 2006 to November 24, 2016, comprising 2690 observations, was used for model formulation. The second portion which ranged from November 25, 2016 to December 30, 2016, consisting of 23 observations, was used for out-of-sample forecasting performance evaluation. Combined linear (ARIMA) and Nonlinear (GARCH-type) models were applied on the returns series with respect to normal and student-t distributions. The findings revealed that ARIMA (2,1,1)-EGARCH (1,1)-norm and ARIMA (1,1,0)-EGARCH (1,1)-norm models selected based on minimum predictive errors throughout-of-sample approach outperformed ARIMA (2,1,1)-GARCH (2,0)-std and ARIMA (1,1,0)-EGARCH (1,1)-std model chosen through in-sample approach. Therefore, it could be deduced that out-of-sample model selection approach was suitable for selecting models with improved forecasting accuracies and performances.

Keywords

ARIMA Model, GARCH-Type Model, Heteroscedasticity, Model Selection, Time Series Forecasting, Volatility

1. Introduction

Model selection is the act of choosing a model from a class of candidate models as a quest for a true model or best forecasting model or both (see also, [1], [2],

[3]). There are often several competing models that can be used for forecasting a particular time series. Consequently, selecting an appropriate forecasting model is considerably practical importance [4] [5]. Selecting the model that provides the best fit to historical data generally does not result in a forecasting method that produces the best forecasts of new data. Concentrating too much on the model that produces the best historical fit often leads to overfitting, or including too many parameters or terms. The best approach is to select the model that results in the smallest standard deviation or mean squared error of the one-step-ahead forecast errors when the model is applied to data set that was not used in fitting process [4]. There are two approaches to model selection in time series; the in-sample model selection and the out-of-sample model selection. The in-sample model selection is targeted at selecting a model for inference, which according to [1] is intended to identify the best model for the data and to provide a reliable characterization of the sources of uncertainty for scientific insight and interpretation. The in-sample model selection criteria include Akaike information criterion, AIC [6], Schwarz information criterion, SIC [7], and Hannan and Quinn information criteria, HQIC [8]. As captured in [9], AIC considered a discrepancy between the true model and a candidate, BIC approximated the posterior model probabilities in a Bayesian framework, and Hannan and Quinn proposed a related criterion which has a smaller penalty compared to BIC that yet permitted strong consistency property (for more details on information criteria, see [10] [11] [12] [13] [14]). However, the major drawbacks of in-sample model selection criteria are that, they are unstable and minimizing these criteria over a class of candidate models leads to a model selection procedure that is conservative or over-consistent in parameter settings [2] [9], and the inability to inform directly about the quality of the model [3]. On the other hand, out-of-sample model selection procedure is applied to achieve the best predictive performance, essentially at describing the characterization of future observations without necessarily considering the choice of true model, rather, the attention is shifted to choose a model with the smallest predictive errors [1] [2] [15] [16]. The out-of-sample forecast is accomplished when the data used for constructing the model are different from that used in forecasting evaluation. That is, the data is divided into two portions. The first portion is for model construction and the second is used for evaluating the forecasting performance with possibility of forecasting new future observations which can be checked against what is observed ([11] [16] [17]). Yet the choice of in-sample and out-of-sample model selection criteria is not without contention and such contention is well handled in [1] [15] [18] [19] [20].

With respect to heteroscedastic processes (or nonlinear time series), details regarding model selection are available in the studies of [21]-[27]. Meanwhile, in Nigeria, model selection in heteroscedastic processes are mainly based on in-sample criteria. For instance, the studies of [28]-[33] rely on the in-sample procedure to select the best fit model. Hence, this study seeks to improve on the work of [28] who applied the in-sample model selection criteria to choose best fitted heteroscedastic models by adopting out-of-sample forecasting approach in selecting heteroscedastic models that would best describe the accuracy and precision of future observations.

This work is further organized as follows: materials and methods are treated in Section 2, results and discussion covered in Section 3 and Section 4 takes care of conclusion.

2. Materials and Methods

2.1. Return

The return series R_t can be obtained given that P_t is the price of a unit share at time *t*, and P_{t-1} is the share price at time $t-1$.

$$R_t = \nabla \ln P_t = (1-B)\ln P_t = \ln P_t - \ln P_{t-1} \tag{1}$$

The R_t in Equation (1) is regarded as a transformed series of the share price, P_t meant to attain stationarity, that is, both mean and variance of the series are stable [29]. The letter B is the backshift operator.

2.2. Information Criteria

There are several information criteria available to determine the order, p, of an AR process and the order, q, of MA(q) process, all of them are likelihood based. The well-known Akaike information criterion (AIC), [6] is defined as

$$\text{AIC} = \frac{-2}{T}\ln(\text{likelihood}) + \frac{2}{T}x(\text{number of parameters}), \tag{2}$$

where the likelihood function is evaluated at the maximum likelihood estimates and T the sample size. For a Gaussian AR(p) model, AIC reduces to

$$\text{AIC}(P) = \ln\left(\hat{\sigma}_P^2\right) + \frac{2P}{T} \tag{3}$$

where $\hat{\sigma}_P^2$ is the maximum likelihood estimate of $\hat{\sigma}_a^2$, which is the variance of a_t, and T is the sample size. The first term of the AIC in Equation (6) measures the goodness-of-fit of the AR(p) model to the data whereas the second term is called the penalty function of the criterion because it penalizes a chosen model by the number of parameters used. Different penalty functions result in different information criteria.

The next commonly used criterion function is the Schwarz information criterion (SIC), [7]. For a Gaussian AR(p) model, the criterion is

$$\text{SIC}(P) = \ln\left(\hat{\sigma}_P^2\right) + \left(\frac{P\ln(T)}{T}\right) \tag{4}$$

Another commonly used criterion function is the Hannan Quinn information criterion (HQIC), [8]. For a Gaussian AR(p) model, the criterion is

$$\text{HQIC}(P) = \ln\left(\hat{\sigma}_P^2\right) + \frac{\ln\{\ln(T)\}}{T} \tag{5}$$

The penalty for each parameter used is 2 for AIC, ln(T) for SIC and ln{ln(T)}

for HQIC. These penalty functions help to ensure selection of parsimonious models and to avoid choosing models with too many parameters.

The AIC criterion asymptotically overestimates the order with positive probability, whereas the BIC and HQIC criteria estimate the order consistently under fairly general conditions ([11] [17]). Moreover, an in-sample model selection criterion is consistent if it chooses a true model when the true model is among those considered with probability approaching unity as the sample size becomes large, and if the true model is not among those considered, it selects the best approximation with probability approaching unity as sample size becomes larger [3]. The AIC is always considered inconsistent in that it does not penalize the inclusion of additional parameters. As such, relying on these criterion leads to overfitting. Meanwhile, the SIC and HQIC criteria are consistent in that it takes into account large size adjustment penalty. In contrast, consistency is not sufficiently informative. It turns out that the true model and any reasonable approximation to it are very complex. An asymptotically efficient model selection criterion chooses a sequence of models as the sample size get larger for which the one-step-ahead forecast error variances approach the one-step-ahead forecast error variance for the true model at least as fast as any other criterion [3]. The AIC is asymptotically efficient while SIC and HQIC are not. However, one major drawback of in-sample criteria is their inability to evaluate a candidate model's potential predictive performance.

2.3. Model Evaluation Criteria

It is tempting to evaluate performance on the basis of the fit of the forecasting or time series model to historical data [3]. The best way to evaluate a candidate model's predictive performance is to apply the out-of-sample forecast technique. This will provide a direct estimate of the one-step-ahead forecast error variance that guarantees an efficient model selection criterion. The methods of forecast evaluation based on forecast error include Mean Squared Error (MSE), Root Mean Squared Error (RMSE) and Mean Absolute Error (MAE). These criteria measure forecast accuracy. The forecast bias is measured by Mean Error (ME).

The measures are computed as follows:

$$\text{MSE} = \frac{1}{n}\sum_{i=1}^{n} e_i^2 \tag{6}$$

$$\text{RMSE} = \sqrt{\frac{1}{n}\sum_{i=1}^{n} e_i^2} \tag{7}$$

$$\text{MAE} = \frac{1}{n}\sum_{i=1}^{n} \left|e_i\right| \tag{8}$$

$$\text{ME} = \frac{1}{n}\sum_{i=1}^{n} \left(e_i\right) \tag{9}$$

where e_i is the forecast error and n is the number of forecast error. Also, it should be noted that in this work, the forecasts of the returns are used as proxies for the volatilities as they are not directly observable [34].

2.4. Autoregressive Integrated Moving Average (ARIMA) Model

[10] considered the extension of ARMA model to deal with homogenous non-stationary time series in which X_t, itself is non-stationary but its d^{th} difference is a stationary ARMA model. Denoting the d^{th} difference of X_t by

$$\varphi(B) = \phi(B)\nabla^d X_t = \theta(B)\varepsilon_t, \tag{10}$$

where $\varphi(B)$ is the nonstationary autoregressive operator such that d of the roots of $\varphi(B)=0$ are unity and the remainder lie outside the unit circle. $\phi(B)$ is a stationary autoregressive operator.

2.5. Heteroscedastic Models

Autoregressive Conditional Heteroscedastic (ARCH) Model: The first model that provides a systematic framework for modeling heteroscedasticity is the ARCH model of [35]. Specifically, an ARCH (q) model assumes that,

$$R_t = \mu_t + a_t, a_t = \sigma_t e_t,$$

$$\sigma_t^2 = \omega + \alpha_1 a_{t-1}^2 + \cdots + \alpha_q a_{t-q}^2, \tag{11}$$

where $[e_t]$ is a sequence of independent and identically distributed (i.i.d.) random variables with mean zero, that is $\mathrm{E}(e_t)=0$ and variance 1, that is $\mathrm{E}(e_t^2)=1$, $\omega>0$, and $\alpha_1,\cdots,\alpha_q \ge 0$ [36]. The coefficients α_i, for $i>0$, must satisfy some regularity conditions to ensure that the unconditional variance of a_t is finite.

Generalized Autoregressive Conditional Heteroscedastic (GARCH) Model: Although the ARCH model is simple, it often requires many parameters to adequately describe the volatility process of a share price return. Some alternative models must be sought. [37] proposed a useful extension known as the generalized ARCH (GARCH) model. For a return series, R_t, let $a_t = R_t - \mu_t$ be the innovation at time *t*. Then, a_t follows a GARCH(*q*, *p*) model if

$$a_t = \sigma_t e_t,$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i a_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2, \tag{12}$$

where again e_t is a sequence of i.i.d. random variance with mean, 0, and variance, 1, $\omega>0, \alpha_i \ge 0, \beta_j \ge 0$, and $\sum_{i=1}^{\max(p,q)} (\alpha_i + \beta_i) < 1$ (see [38]).

Here, it is understood that $\alpha_i = 0$, for $i>p$, and $\beta_i = 0$, for $i>q$. The latter constraint on $\alpha_i + \beta_i$ implies that the unconditional variance of a_t is finite, whereas its conditional variance σ_t^2, evolves over time.

Exponential Generalized Autoregressive Conditional Heteroscedastic (EGARCH) Model: The EGARCH model represents a major shift from ARCH and GARCH models [39]. Rather than modeling the variance directly, EGARCH models the natural logarithm of the variance, and so no parameter restrictions are required to ensure that the conditional variance is positive. The EGARCH(*q*, *p*) is defined as,

$$R_t = \mu_t + a_t, a_t = \sigma_t e_t,$$
$$\ln\sigma_t^2 = \omega + \sum_{i=1}^{q}\alpha_i\left|\frac{a_{t-i}}{\sqrt{\sigma_{t-i}^2}}\right| + \sum_{k=1}^{r}\gamma_k\left(\frac{a_{t-k}}{\sqrt{\sigma_{t-k}^2}}\right) + \sum_{j=1}^{p}\beta_j\ln\sigma_{t-j}^2, \tag{13}$$

where again, e_t is a sequence of i.i.d. random variance with mean, 0, and variance, 1, and γ_k is the asymmetric coefficient.

Glosten, Jagannathan and Runkle (GJR-GARCH) Model: The GJR-GARCH (*q*, *p*) model proposed by [40] is a variant, represented by

$$a_t = \sigma_t e_t,$$
$$\sigma_t^2 = \omega + \sum_{i=1}^{q}\alpha_i a_{t-i}^2 + \sum_{i=1}^{p}\gamma_i I_{t-i} a_{t-i}^2 + \sum_{j=1}^{p}\beta_j\sigma_{t-j}^2, \tag{14}$$

where I_{t-1} is an indicator for negative a_{t-i}, that is,

$$I_{t-1} = \begin{cases} 0 & \text{if } a_{t-i} < 0, \\ 1 & \text{if } a_{t-i} \ge 0, \end{cases}$$

and α_i, γ_i and β_j are nonnegative parameters satisfying conditions similar to those of GARCH models. Also the introduction of indicator parameter of leverage effect, I_{t-1} in the model accommodates the leverage effect, since it is supposed that the effect of a_{t-i}^2 on the conditional variance σ_t^2 is different accordingly to the sign of a_{t-i}.

2.6. Parametric Bootstrap

The parametric bootstrap is used in computing nonlinear forecasts given the fact that the model used in forecasting has been rigorously checked and is judged to be adequate for the series under study [39]. Let *T* be the forecast origin and *k* be the forecast horizon ($k > 0$). That is, we are at time index *T* and interested in forecasting R_{T+k}. The parametric bootstrap considered compute realizations $R_{T+1},\cdots,R_{T+k}$ sequentially by drawing a new innovation from the specific innovational distribution of the model, and computing R_{T+i} using the model, data, and previous forecasts $R_{T+1},\cdots,R_{T+i-1}$. This results in a realization for R_{T+k}. The procedure is repeated M times to obtain *M* realizations of R_{T+k} denoted by $\left\{R_{T+k}^{(j)}\right\}_{j=1}^{M}$. The point forecast of R_{T+k} is then the sample average of $R_{T+k}^{(j)}$.

Consequently, Forecasts of the ARCH model are obtained recursively. Let *T* be the starting date for forecasting, that is forecast origin. Let F_T be the information set available at time *T*. Then, the 1-step ahead forecast for conditional variance, σ_{T+1}^2 is

$$\sigma_T^2(1) = \hat{\omega} + \hat{\alpha}_1\hat{a}_T^2 + \cdots + \hat{\alpha}_p\hat{a}_{T+1-p}^2, \tag{15}$$

where $\hat{a}_T$ is the estimated residual. For the 2-step ahead forecast σ_{T+2}^2, we need a forecast of a_{T+1}^2. It is given by $\sigma_T^2(1)$. We therefore obtain

$$\sigma_T^2(2) = \hat{\omega} + \hat{\alpha}_1\sigma_T^2(1) + \hat{\alpha}_2\hat{a}_T^2 + \cdots + \hat{\alpha}_p\hat{a}_{T+2-p}^2. \tag{16}$$

The *k*-step ahead forecast for σ_{T+k}^2 is

$$\sigma_T^2(k) = \hat{\omega} + \hat{\alpha}_1\sigma_T^2(k-1) + \cdots + \hat{\alpha}_p\sigma_T^2(k-p), \tag{17}$$

with $\sigma_T^2(k-i)=\hat{a}_{T+k-i}^2$ if $k-i\le 0$.

Forecasts of the GARCH model are obtained recursively in a similar way as that of the ARCH model. Then, the 1-step ahead forecast for σ_{T+1}^2 is

$$\sigma_T^2(1)=\hat{\omega}+\hat{\alpha}_1\hat{a}_T^2+\hat{\beta}_1\hat{\sigma}_T^2, \tag{18}$$

since $a_T^2=\sigma_T^2e_T^2$, the GARCH (1,1) model can be rewritten as

$$\sigma_T^2=\omega+\alpha_1a_{T-1}^2+\beta_1\sigma_{T-1}^2=\omega+(\alpha_1+\beta_1)\sigma_{T-1}^2+\alpha_1\sigma_{T-1}^2\left(e_{T-1}^2-1\right),$$

so that, at time $T+2$, we have

$$\sigma_{T+2}^2=\omega+(\alpha_1+\beta_1)\sigma_{T+1}^2+\alpha_1\sigma_{T+1}^2\left(e_{T+1}^2-1\right),$$

with $E\left[\left(e_{T+1}^2-1\right)/F_T\right]=0$, we deduce the following 2-step ahead forecast for σ_{T+2}^2:

$$\sigma_T^2(2)=\hat{\omega}+\left(\hat{\alpha}_1+\hat{\beta}_1\right)\sigma_T^2(1).$$

Generally speaking, the k-step ahead forecast for σ_{T+k}^2 is

$$\sigma_T^2(k)=\hat{\omega}+\left(\hat{\alpha}_1+\hat{\beta}_1\right)\sigma_T^2(k-1), k>1. \tag{19}$$

One of the beauties of GARCH is that volatility forecasts for any horizon can be constructed from the estimated model. The estimated GARCH model is used to get forecasts of instantaneous forward volatilities, that is, the forecast for σ_{T+k}^2 made at time T and for every k step ahead.

For EGARCH model, assuming that the model parameters are known and the observations are standard Gaussian, for EGARCH (1,1) model, we have

$$\ln\sigma_T^2=(1-\alpha_1)\omega+\alpha_1\ln\sigma_{T-1}^2+g(\epsilon_{T-1}),$$

$$g(\epsilon_{T-1})=\theta\epsilon_{T-1}+\gamma\left(\left|\epsilon_{T-1}\right|-\sqrt{2/\pi}\right). \tag{20}$$

Taking exponentials, the model becomes

$$\sigma_T^2=\sigma_{T-1}^{2\alpha_1}\exp\left[(1-\alpha_1)\omega\right]\exp\left[g(\epsilon_{T-1})\right],$$

$$g(\epsilon_{T-1})=\theta\epsilon_{T-1}+\gamma\left(\left|\epsilon_{T-1}\right|-\sqrt{2/\pi}\right). \tag{21}$$

For the 1-step ahead forecast, σ_{T+1}^2 we have

$$\sigma_T^2(1)=\sigma_T^{2\alpha_1}\exp\left[(1-\alpha_1)\omega\right]\exp\left[g(\epsilon_T)\right]. \tag{22}$$

The 2-step-ahead forecast of σ_{T+2}^2 is given by

$$\sigma_T^2(2)=\hat{\sigma}_T^{2\alpha_1}(1)\exp\left[(1-\alpha_1)\omega\right]E_T\left\{\exp\left[g(\epsilon_T)\right]\right\},$$

where E_T denotes a conditional expectation taken at the time origin T with

$$E\left\{\exp\left[g(\epsilon_T)\right]\right\}=\exp\left(-\gamma\sqrt{2/\pi}\right)\left[\mathrm{e}^{(\theta+\gamma)^2/2}\Phi(\theta+\gamma)+\mathrm{e}^{(\theta-\gamma)^2/2}\Phi(\gamma-\theta)\right],$$

where $\Phi(x)$ is the cumulative density function of the standard normal distribution (see [39] for more details). Hence,

$$\begin{aligned}\hat{\sigma}_T^2(2)=&\hat{\sigma}_T^{2\alpha_1}(1)\exp\left[(1-\alpha_1)\omega-\gamma\sqrt{2/\pi}\right]\\&\times\left\{\exp\left[(\theta+\gamma)^2/2\right]\Phi(\theta+\gamma)+\exp\left[(\theta-\gamma)^2/2\right]\Phi(\gamma-\theta)\right\}\end{aligned}$$

Generally, the k-step-ahead forecast can be obtained as

$$\hat{\sigma}_T^2(k)=\hat{\sigma}_T^{2\alpha_1}(k-1)\exp\left[(1-\alpha_1)\omega-\gamma\sqrt{2/\pi}\right]$$
$$\times\left\{\exp\left[(\theta+\gamma)^2/2\right]\Phi(\theta+\gamma)+\exp\left[(\theta-\gamma)^2/2\right]\Phi(\gamma-\theta)\right\} \tag{23}$$

(See also, [34], [38]).

3. Results and Discussion

3.1. Plot Analysis

Figure 1 and Figure 2 are the share prices of Diamond and Fidelity Banks. Their movements appeared to fluctuate away from the common mean indicating the presence of stochastic nonstationarity.

Figure 3 and Figure 4 are the returns series of the respective banks and are found to cluster around the common mean signifying stationarity.

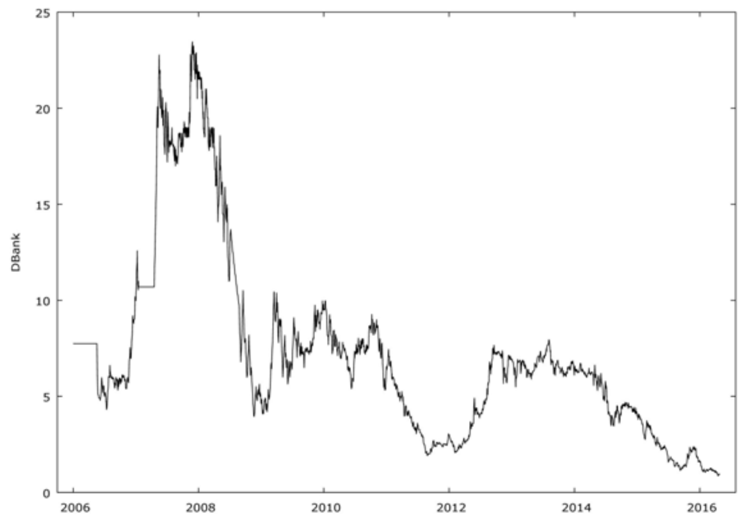

Figure 1. Share price series of diamond bank.

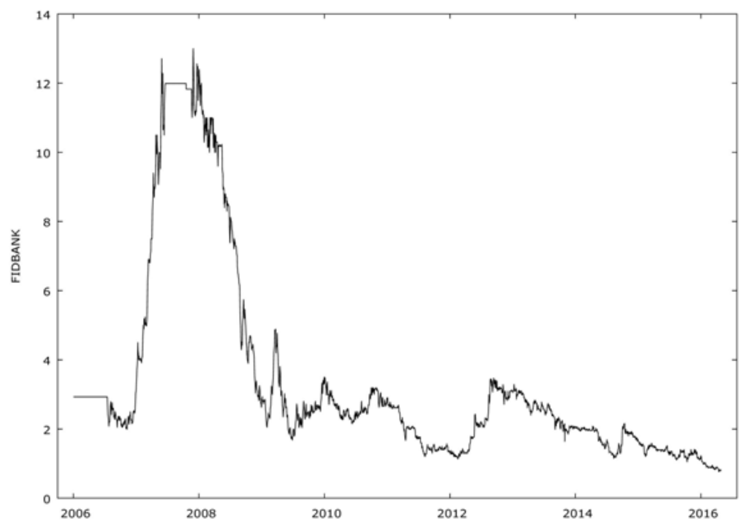

Figure 2. Share price series of fidelity bank.

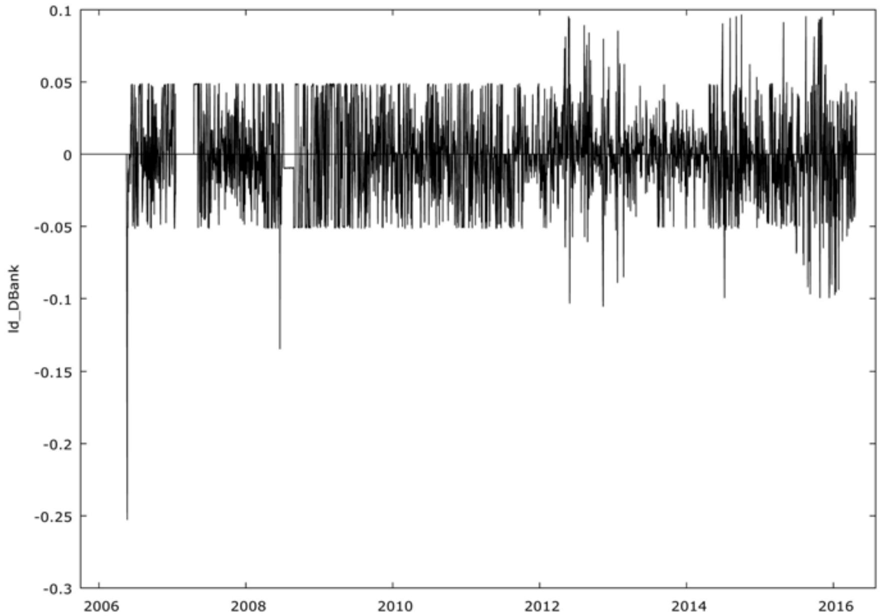

Figure 3. Return series of diamond bank.

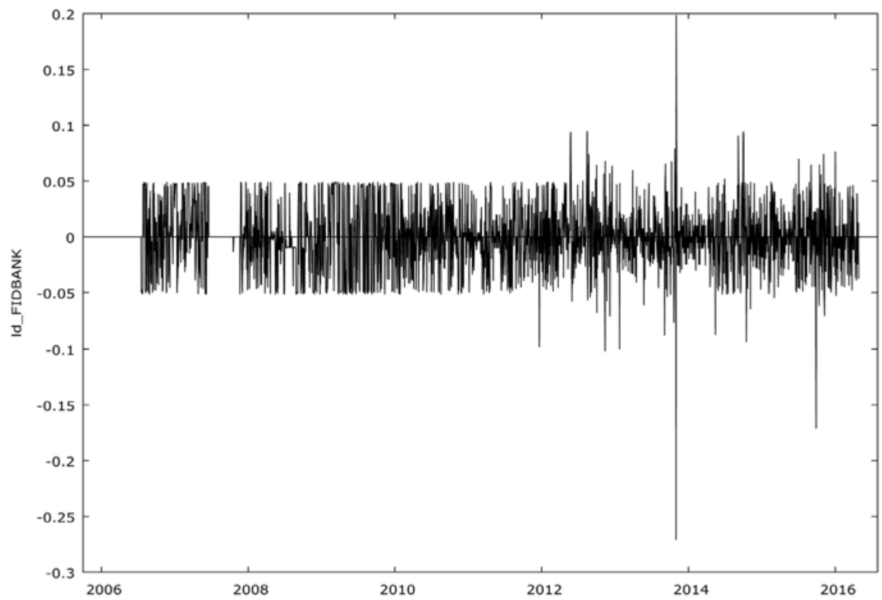

Figure 4. Return series of fidelity bank.

3.2. In-Sample Model Selection

Several models with respect to normal distribution (norm) and student-t distribution (std) such as ARIMA (2,1,1)-GARCH (1,0)-std, ARIMA (2,1,1)-GARCH (2,0)-std, ARIMA (2,1,1)-GARCH (1,1)-norm, ARIMA (2,1,1)-EGARCH (1,1)-norm and ARIMA (2,1,1)-EGARCH (1,1)-std were considered tentatively for the return series of Diamond Bank. ARIMA (2,1,1)-GARCH (2,0)-std was selected based on minimum information criteria (see **Table 1**). The model was found to be adequate given that the p-values corresponding to weighted Ljung-Box Q statistics at lags 1, 8 and 14 on standardized residuals, weighted

Table 1. Estimation of Heteroscedastic models of return series of diamond bank.

Model	Parameter	Estimate	s.e	t-ratio	p-value	Information Criteria AIC	BIC	HQIC
	μ	$-9.93e^{-4}$	$3.81e^{-4}$	−2.6037	0.0092			
	φ_1	0.6479	0.1134	5.7155	0.0000			
ARIMA (2,1,1)-GARCH (1,0)-std	φ_2	0.0115	0.0246	0.4676	0.6401	−4.3202	−4.3049	−4.3147
	θ_1	−0.7192	0.1107	−6.4961	0.0000			
	ω	$4.9e^{-4}$	$2.6e^{-5}$	18.9763	0.0000			
	α_1	0.5380	0.0581	9.2597	0.0000			
	μ	**0.0000**	**0.0000**	**−0.0179**	**0.9857**			
	φ_1	**−0.2748**	**0.1017**	**−2.7030**	**0.0069**			
	φ_2	**0.1899**	**0.0250**	**7.5938**	**0.0000**			
ARIMA (2,1,1)-GARCH (2,0)-std	θ_1	**0.2976**	**0.0988**	**3.0112**	**0.0026**	**−5.0430**	**−5.0255**	**−5.0367**
	ω	**0.0000**	**0.0000**	**0.0000**	**1.0000**			
	α_1	**0.5085**	**0.0215**	**23.6094**	**0.0000**			
	α_2	**0.4899**	**0.0216**	**22.6980**	**0.0000**			
	μ	$-1.89e^{-4}$	$4.6e^{-5}$	−4.1466	0.00003			
	φ_1	0.7177	0.1399	5.1280	0.0000			
	φ_2	0.0116	0.0248	0.4695	0.6387			
ARIMA (2,1,1)-GARCH (1,1)-norm	θ_1	−0.7663	0.1386	−5.5302	0.0000	−4.3997	−4.3843	−4.3941
	ω	$5.0e^{-6}$	0.0000	21.4307	0.0000			
	α_1	0.1499	0.0084	17.9265	0.0000			
	β_1	0.8491	0.0065	131.3783	0.0000			
	μ	$-1.325e^{-3}$	$4.67e^{-4}$	−2.8394	0.0045			
	φ_1	−0.6678	0.0235	−28.3624	0.0000			
	φ_2	−0.0247	0.0222	−1.1137	0.2654			
ARIMA (2,1,1)-EGARCH (1,1)-norm	θ_1	0.6243	0.0237	26.3269	0.0000	−4.3056	−4.2881	−4.2993
	ω	−1.8914	0.3467	−5.4553	0.0000			
	α_1	−0.0003	0.0199	−0.0137	0.9891			
	β_1	0.7326	0.0488	15.0204	0.0000			
	γ_1	0.3446	0.0484	7.12488	0.0000			
	μ	0.0000	$7.0e^{-6}$	−0.0005	0.9995			
	φ_1	−0.2876	0.0271	−10.6033	0.0000			
	φ_2	0.0023	0.0203	0.1135	0.9096			
ARIMA (2,1,1)-EGARCH (1,1)-std	θ_1	0.2356	0.0275	8.5685	0.0000	−4.4228	−4.4031	−4.4157
	ω	−0.8316	0.0198	−41.9417	0.0000			
	α_1	−0.0537	0.0284	−1.8871	0.0000			
	β_1	0.8820	0.0011	773.2336	0.0000			
	γ_1	0.9400	0.0378	24.8684	0.0000			

Ljung-Box Q statistics at lags 1, 5 and 9 on standardized squared residuals and weighted Lagrange Multiplier statistics at lags 3, 5 and 7 are all greater than 5% level of significance [see Table 2]. That is to say, the hypotheses of no autocorrelation and no remaining ARCH effect are not rejected.

Also, for Fidelity Bank, ARIMA (1,1,0)-GARCH (1,0)-norm, ARIMA (1,1,0)-GARCH (1,0)-std, ARIMA (1,1,0)-GARCH (1,1)-norm, ARIMA (1,1,0)-EGARCH (1,1)-norm and ARIMA (1,1,0)-EGARCH (1,1)-std were considered tentatively (Table 3). Based on smallest information criteria, ARIMA (1,1,0)-EGARCH (1,1)-std was chosen as the appropriate model. The selected model is adequate since all the p-values corresponding to weighted Ljung-Box Q statistics at lags 1, 2 and 5 on standardized residuals, weighted Ljung-Box Q statistics at lags 1, 5 and 9 on standardized squared residuals and weighted Lagrange Multiplier statistics at lags 3, 5 and 7 are greater than 5% level of significance [see Table 4]. That is to say, the null hypotheses of no autocorrelation and no ARCH effect are not rejected at 5% significance level.

3.3. Out-Of-Sample Forecasting Model Selection

Here, the out-of-sample forecast evaluation criteria; MAE, MSE and RMSE for each of the models are considered for the series of the banks. It was found that ARIMA (2,1,1)-EGARCH (1,1)-norm and ARIMA (1,1,0)-EGARCH (1,1)-norm possessed the smallest out-of-sample forecast evaluation criteria (see Table 5 and Table 6). Hence, the most appropriate for the return series of the respective banks.

Based on our findings, the in-sample model selection procedure favoured ARIMA (2,1,1)-GARCH (2,0)-std and ARIMA (1,1,0)-EGARCH (1,1)-std model while the out-of-sample model selection sufficed the choice of ARIMA (2,1,1)-EGARCH (1,1)-norm and ARIMA (1,1,0)-EGARCH (1,1)-norm models for the banks considered. Majorly, it is discovered that in each of the models selected through in-sample criteria are ill-conditioned. For instance, the constant term of the variance equation, ω of ARIMA (2,1,1)-GARCH (2,0)-std is zero which actually violates the constraint condition that requires $\omega > 0$. The implication is that, this model is not suitable for forecasting long-run variance as it would collapse at zero. Again, in EGARCH (1,1)-std, the stationarity condition which requires $\sum_j^p \beta_j < 1$, is violated. The implication is that, forecasting long-run variance using this model would not be realistic in that the variance

Table 2. Diagnostic checking for heteroscedastic models of return series of diamond bank.

Model	Standardized Residuals			Standardized Squared Residuals					
	Lag	Weighted LB	p-value	Lag	Weighted LB	p-value	Lag	Weighted ARCH-LM	p-value
ARIMA (2,1,1)-GARCH (2,0)-std	1	0.0001	0.9903	1	0.0004	0.9835	3	0.0004	0.9835
	8	0.0007	1.0000	5	0.0012	1.0000	5	0.0010	1.0000
	14	0.0011	1.0000	9	0.0021	1.0000	7	0.0015	1.0000

Table 3. Estimation of heteroscedastic models of return series of fidelity bank.

Model	Parameter	Estimate	s.e	t-ratio	p-value	Information Criteria		
						AIC	BIC	HQIC
ARIMA (1,1,0)-GARCH (1,0)-norm	μ	0.0181	$2.0e^{-6}$	7416.5857	0.0000	9.5841	9.5929	9.5873
	φ_1	−0.1188	$4.1e^{-5}$	−2892.9449	0.0000			
	ω	0.0000	0.0000	4.8845	0.0000			
	α_1	0.9755	0.0004	2331.4143	0.0000			
ARIMA (1,1,0)-GARCH (1,0)-std	μ	$-8.0e^{-4}$	0.0004	−1.9660	0.0493	−4.4395	−4.4286	−4.4356
	φ_1	−0.0671	0.0241	−2.7856	0.0053			
	ω	$4.0e^{-4}$	$2.3e^{-5}$	18.2910	0.0000			
	α_1	0.6093	0.0739	8.2413	0.0000			
ARIMA (1,1,0)-GARCH (1,1)-norm	μ	$-5.0e^{-4}$	$9.0e^{-5}$	−56.1884	0.0000	−4.5492	−4.5382	−4.5452
	φ_1	−0.0339	0.0225	−1.5108	0.1308			
	ω	$5.0e^{-6}$	0.0000	67.9874	0.0000			
	α_1	0.1528	0.0105	14.5331	0.0000			
	β_1	0.8462	0.0088	95.5935	0.0000			
ARIMA (1,1,0)-EGARCH (1,1)-norm	μ	0.0000	$2.0e^{-5}$	−0.0004	0.9997	−4.5912	−4.5781	−4.5865
	φ_1	−0.0363	0.0231	−1.5721	0.1159			
	ω	−0.6564	0.0065	−101.1563	0.0000			
	α_1	0.0155	0.0164	0.9474	0.3434			
	β_1	0.9063	0.0007	1341.0781	0.0000			
	γ_1	0.4184	0.0107	38.9643	0.0000			
ARIMA (1,1,0)-**EGARCH (1,1)-std**	μ	**$-4.0e^{-6}$**	**0.0000**	**−41.393**	**0.0000**	**−5.1507**	**−5.1354**	**−5.1452**
	φ_1	**−0.1572**	**0.0029**	**−56.078**	**0.0000**			
	ω	**0.0141**	**0.0003**	**49.642**	**0.0000**			
	α_1	**1.1589**	**0.0001**	**12452.741**	**0.0000**			
	β_1	**1.0000**	**$1.0e^{-5}$**	**84465.444**	**0.0000**			
	γ_1	**1.1730**	**0.0001**	**11481.44**	**0.0000**			

Table 4. Diagnostic checking for Heteroscedastic models of return series of fidelity bank.

Model	Standardized Residuals			Standardized Squared Residuals					
	Lag	Weighted LB	p-value	Lag	Weighted LB	p-value	Lag	Weighted ARCH-LM	p-value
ARIMA (1,1,0)-EGARCH (1,1)-std	1	0.0007	0.979	1	0.0010	0.9746	3	0.0010	0.9747
	2	0.0011	1.0000	5	0.0030	1.0000	5	0.0024	0.9999
	5	0.0065	1.0000	9	0.0051	1.0000	7	0.0036	1.0000

Table 5. Out-of-sample forecast evaluation criteria for diamond bank.

Evaluation Criteria	ARIMA (2,1,1)-GARCH (1,0)-std Model	ARIMA (2,1,1)-GARCH (2,0)-std Model	ARIMA (2,1,1)-GARCH (1,1)-norm Model	ARIMA (2,1,1)-EGARCH (1,1)-norm Model	ARIMA (2,1,1)-EGARCH (1,1)-std Model
MAE	0.019999	0.022278	0.020026	**0.019986**	0.020047
MSE	0.000629	0.000772	0.000634	**0.000628**	0.000636
RMSE	0.025084	0.027785	0.025179	**0.025078**	0.025218

Table 6. Out-of-sample forecast evaluation criteria for fidelity bank.

Evaluation Criteria	ARIMA (1,1,0)-GARCH (1,0)-norm Model	ARIMA (1,1,0)-GARCH (1,0)-std Model	ARIMA (1,1,0)-GARCH (1,1)-norm Model	**ARIMA (1,1,0)-EGARCH (1,1)-norm Model**	ARIMA (1,1,0)-EGARCH (1,1)-std Model
MAE	0.026079	0.021095	0.020999	**0.020938**	0.021193
MSE	0.000994	0.000680	0.000675	**0.000673**	0.000695
RMSE	0.001315	0.026084	0.025977	**0.025960**	0.026355

would converge at infinity. Moreover, the highly significance of the parameters of the models indicated that the models are over-fitted. Meanwhile, the models selected through out-of-sample criteria are characterized by non-significant parameters yet possessed smallest predictive errors and problem associated with over-fitting is overcome. In particular, this study showed that the study of [28] can be improved by adopting out-of-sample forecasting procedure. Furthermore, the study is in agreement with the works of [1], [2], [22] by supporting the choice of models based on smallest predictive errors.

4. Conclusion

In all, our study showed that out-of-sample model selection approach outperformed the in-sample counterpart in describing the characterization of future observations without necessarily considering the choice of true model. The major strength of this study is in utilizing the advantage of combining both ARIMA and GARCH-type models to achieve forecast accuracy. The weakness of this study is in adopting larger samples of training data against smaller sample sizes for forecast evaluation, which is suitable for achieving the best fitting models. However, this weakness could be overcome by adopting smaller sample sizes of data for model formulation and larger samples for forecast evaluation in future study.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Optimal Coordinated Search for a Discrete Random Walker

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Abstract

This paper presents the search technique for a lost target. A lost target is random walker on one of two intersected real lines, and the purpose is to detect the target as fast as possible. We have four searchers start from the point of intersection, they follow the so called Quasi-Coordinated search plan. The expected value of the first meeting time between one of the searchers and the target is investigated, also we show the existence of the optimal search strategy which minimizes this first meeting time.

Keywords

Random Walk, Coordinate Search Technique, Lost Targets, Expected Value, Optimal Search

1. Introduction

The search problem for a randomly moving target is very interesting because it may arise in many real world situations such as searching for lost persons on roads, the cancer cells in the human body and missing black box of a plane crash in the depth of the sea or ocean, also searching for a gold mine underground, Landmines and navy mines, a faulty unit in a large linear system such as electrical power lines, telephone lines, and mining system, and so on (see [1], [2], [3], [4] and [5]).

The aim of search, in many cases (see [6], and [7]) is to calculate the expected cost of detecting the target and is to obtain the search plan, which minimizes this expected cost. In the case of linear search for stationary or randomly moving targets many studies are made (see [8]-[26]).

The coordinated search method is one of the famous search methods which consider the searchers starting together from the origin and moving, seeking for a random walk target. Therefore, coordinated search technique is one of many techniques which studied previously on the line where the located targets have symmetric and unsymmetric distributions (see [27], [28], [29] and [30]), this technique has been illustrated on the circle with a known radius and the target equally likely to be anywhere on its circumference (see [31]), also this technique has been discussed in the plane when the located target has symmetric and asymmetric distribution (see [32] and [33]). There is obviously some similarity between this problem and the well known linear search problem.

In the present paper, we introduce the search problem for a random walk target motion on one of two intersected lines. This will happen by coordinating search between four searchers, all the searchers will start together at the same point of intersected their lines with zero as the starting and meeting point of the searchers. So that we may assume that two searchers always search to the right part and the other searchers search to the left part of intersected point. They return to zero after searching successively common distances until the target is found, we call this search as Quasi-Coordinated Linear Search Problem. We aim to minimize the expected value of the first meeting time between one of the searchers and the target. This paper is organized as follows. In Section 2 we formulate the problem and we give the conditions that make the expected value of the first meeting time between one of the searchers and the target which is finite. In Section 3 the existence of optimal search plan that minimizes the expected value of the first meeting time is presented. Finally, the paper concludes with a discussion of the results and directions for future research.

2. Problem Formulation

A target is assumed to move randomly on one of two intersected line according to a stochastic process $\{S(t), t \in I^+\}$, where I^+ is the set of non negative integers. Assume that $\{X_i\}_{i \ge 0}$ is a sequence of independent and identically distributed random variables such that for any $i \ge 1$: $P(X_i = 1) = p$ and $P(X_i = -1) = 1 - p = q, p, q > 0$. Thus, we have

$$S(t) = \sum_{i=1}^{t} X_i, t > 0 \text{ and } S(0) = 0 \tag{1}$$

as a Random Walk. Our aim is to calculate the expected value of the first meeting time between one of searcher and lost target, and investigate it, also we show the existence of a search plan which minimize this expected value.

In the present paper we take the search region to be the real lines. Assume that, we have four searchers S_1, S_2, S_3 and S_4 start together looking for the lost target from the intersected point $H_0 = 0$ on $L_1 \cap L_2$.The searchers coordinate their search to find the lost target, where each of the searchers S_1 and S_3 start search at H_0 and go to the right part of starting point as far as H_{11} and H_{21} respectively, and each of searchers S_2 and S_4 start search at H_0 and go the left part of the same lines as far as $-H_{11}$ and $-H_{21}$ respectively. Then, turn back to H_0 to tell the other searchers if the target is found or not.

Retrace the steps again to explore the right (left) part of H_{j1} $(-H_{j1})$ as far as H_{j2} $(-H_{j2})$, $j=1,2$ and so on, see Figure 1.

All the searchers S_1, S_2, S_3 and S_4 reach to $H_{11}, -H_{11}, H_{21}$ and $-H_{21}$ respectively in the same time G_1, then they come back to H_0 again in the same time G_2. If no one of the searchers do not find the lost target, then they begin their search from H_0 and reach to $H_{12}, -H_{12}, H_{22}$ and $-H_{22}$ in the time G_3, then they come back to H_0 again in the same time G_4 and so on. A search plan ϕ_r with speed V_r is a function $\phi_r : R^+ \to R$, $r=1,2,3,4$, such that:

$$\left|\phi_r(t_1) - \phi_r(t_2)\right| = V_r \left|t_1 - t_2\right|, \forall t_1, t_2 \in R^+ \tag{2}$$

where $\phi_r(0)=0, r=1,2,3,4$. Let the search plan be represented by $\phi_0 = (\phi_1, \phi_2, \phi_3, \phi_4), \phi_0 \in \Phi_0$ where Φ_0 is the set of all search plan.

We assume that Z_0 is a random variable represented the initial position of the target and valued in $2I$ (or $2I+1$) and independent with $S(t), t>0$. If $Z_0 = Z_1$ then the target moves on L_1 and if $Z_0 = Z_2$ the target moves on L_2 such that $P(Z_0 = Z_1) + P(Z_0 = Z_2) = 1$. There is a known probability measures v_j, such that $v_1 + v_2 = 1$ on $L_1 \cup L_2$, where v_1 is probability measure induced by the position of the target on L_1, while v_2 on L_2. The first meeting time is a random variable valued in I^+ defined as:

$$\tau_{\hat{\phi}_0} = \inf\left\{t, \bigcup_{r=1}^{2}\left|\phi_r(t)\right| = Z_1 + S(t) \text{ or } \bigcup_{r=3}^{4}\left|\phi_r(t)\right| = Z_2 + S(t)\right\}.$$

At the beginning of the search suppose that the lost target is existing on any integer point on L_1 but more than H_{11} or less than $-H_{11}$ or the lost target is existing on any integer point on L_2 but more than H_{21} or less than $-H_{21}$. Let $\tau_{\hat{\phi}_0}$ be the first meeting time between one of the searchers and the target. The main objective is to find the search plan such that $E\left(\tau_{\hat{\phi}_0}\right) < \infty$ and if $E\left(\tau_{\hat{\phi}_0^*}\right) < E\left(\tau_{\hat{\phi}_0}\right)$ where E terms to expectation value, then we call $\hat{\phi}_0^*$ is an optimal search plan. Given $n>0$, if x is:

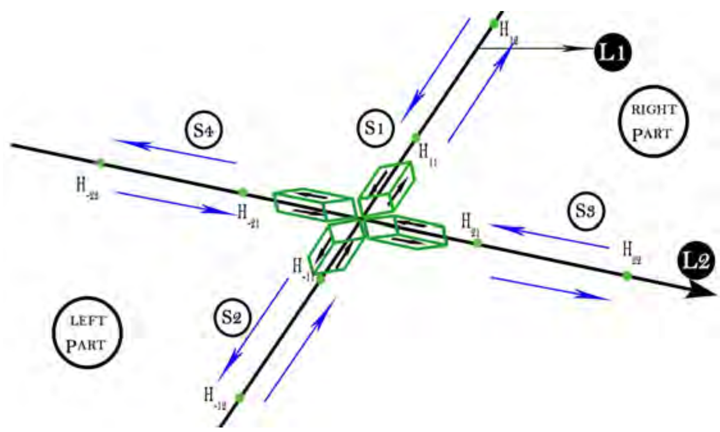

Figure 1. The searchers S_1 and S_2 start from the origin of L_1 after searching successively distances H_{11} and $-H_{11}$, respectively, they return to the origin (note the black arrow) and then they search the distances H_{12} and $-H_{12}$, respectively, they return to the origin (note the blue arrow) and so on also the same procedure for the searchers S_3 and S_4 on L_2.

$$0 \le k_1 \le \frac{n+x}{2} \le n$$

where k_1 is integer, then

$$P\left(S\left(n\right)=k_1\right)=\binom{n}{k_1}p^{k_1}q^{n-k_1} \tag{3}$$

Existence of a Finite Search Plan

Assuming that λ,ζ be positive integers such that: $\zeta>1, \lambda=k\theta$, where $k=1,2,\cdots$ and θ are positive integer numbers greater than one and $V=1$. We will shall define the following sequences $\{G_i\}_{i\ge1}$ and $\{H_{ji}\}_{i\ge1}, j=1,2$, for all the searchers $S_r, r=1,2,3,4$ on the line L_j, to obtain the distances which the searcher should do them as the functions of λ and ζ. In Figure 2 we can define

$$G_i=2^{\frac{1}{2}\left[1-(-1)^{i+1}\right]}\lambda\left(\zeta^{\frac{i}{2}+\frac{1}{4}-\frac{1}{4}(-1)^i}-1\right), \tag{4}$$

And

$$H_{ji}=G_{2i-1}=\frac{1}{2}G_{2i} \tag{5}$$

Also, we shall define the search paths as follows: for any $t\in I^+$, If $G_i \le t \le G_{i+1}$, $i=1,2,3,\cdots$, then

$$\phi_j\left(t\right)=\left(\frac{1}{2}H_{j\frac{i+1}{2}}\right)+\left(-1\right)^{i+1}\left(\frac{1}{2}H_{j\frac{i+1}{2}}\right)+\left(-1\right)^i\left(t-G_i\right) \tag{6}$$

and $\phi_k\left(t\right)=-\phi_{k+1}\left(t\right), k=1,3$. We define the notations $\varphi_j\left(t\right)=S\left(t\right)-t$ and $\tilde{\varphi}_j\left(t\right)=S\left(t\right)+t$ on $L_j, j=1,2$, respectively, $\tau_{\hat{\phi}_0}$ is the first meeting between one of the searcher and the target.

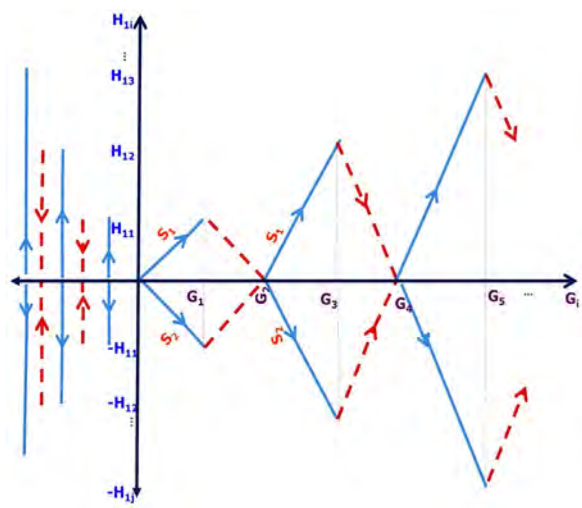

Figure 2. Plots the searched distances H_{ji} and times G_i on L_1.

Theorem 1. *If* $\phi_r\in\Phi_0, r=1,2,3,4$ is a search plan, then the expectation $E\left(\tau_{\hat{\phi}_0}\right)$ is finite if:

$$B_1(z_1)=\sum_{i=1}^{\infty}\left(\zeta^i-1\right)p\left(\tilde{\varphi}_1(G_{2i-1})<-z_1\right),$$

$$B_2(z_1)=\sum_{i=1}^{\infty}\left(\zeta^i-1\right)p\left(\varphi_1(G_{2i-1})>-z_1\right),$$

$$B_3(z_1)=\sum_{i=1}^{\infty}\left(\zeta^i(\zeta-2)+1\right)p\left(\tilde{\varphi}_1(G_{2i})<-z_1\right),$$

$$B_4(z_1)=\sum_{i=1}^{\infty}\left(\zeta^i(\zeta-2)+1\right)p\left(\varphi_1(G_{2i})>-z_1\right),$$

$$B_5(z_2)=\sum_{i=1}^{\infty}\left(\zeta^i-1\right)p\left(\tilde{\varphi}_2(G_{2i-1})<-z_2\right),$$

$$B_6(z_2)=\sum_{i=1}^{\infty}\left(\zeta^i-1\right)p\left(\varphi_2(G_{2i-1})>-z_2\right),$$

$$B_7(z_2)=\sum_{i=1}^{\infty}\left(\zeta^i(\zeta-2)+1\right)p\left(\tilde{\varphi}_2(G_{2i})<-z_2\right)$$

and

$$B_8(z_2)=\sum_{i=1}^{\infty}\left(\zeta^i(\zeta-2)+1\right)p\left(\varphi_2(G_{2i})>-z_2\right),$$

are finite.

Proof.

The hypothesis Z_1 and Z_2 are valued in $2I$ (or $2I+1$) and independent of $S(t), t>0$, if $Z_1>0$ then $Z_1+S(t)$ is greater than $\phi_1(t)$ until the first meeting between S_1 and the target on L_1 also if $Z_1<0$, then $Z_1+S(t)$ is smaller than $\phi_2(t)$ until the first meeting between S_2 and the target on L_1. The same thing for the second line by replacing Z_1 by Z_2 in the second line L_2 and $\phi_1(t), \phi_2(t)$ by $\phi_3(t), \phi_4(t)$ respectively.

Hence, for any $i>0$:

$$p\left(\tau_{\hat{\phi}_0}>t\right)\le\sum_{r=1}^{4}p\left(\tau_{\phi_r}>t\right)$$

hence,

$$E\left(\tau_{\hat{\phi}_0}\right)=\int_0^{\infty}p\left(\tau_{\hat{\phi}_0}>t\right)\mathrm{d}t$$

see ([5]),

$$\begin{aligned}
&=\sum_{i=0}^{\infty}\int_{G_i}^{G_{i+1}}p\left(\tau_{\hat{\phi}_0}>t\right)\mathrm{d}t\\
&\le\sum_{i=0}^{\infty}\int_{G_i}^{G_{i+1}}p\left(\tau_{\hat{\phi}_0}>G_i\right)\mathrm{d}t\\
&=\sum_{i=0}^{\infty}\left[\left(G_{i+1}-G_i\right)p\left(\tau_{\hat{\phi}_0}>G_i\right)\right]\\
&=\sum_{i=0}^{\infty}\left\{2^{\frac{1}{2}\left[1-(-1)^{i+2}\right]}\lambda\left(\zeta^{\frac{i+1}{2}+\frac{1}{4}-\frac{1}{4}(-1)^{i+1}}-1\right)\right.\\
&\qquad\left.-2^{\frac{1}{2}\left[1-(-1)^{i+1}\right]}\lambda\left(\zeta^{\frac{i}{2}+\frac{1}{4}-\frac{1}{4}(-1)^{i}}-1\right)\right\}p\left(\tau_{\hat{\phi}_0}>G_i\right)
\end{aligned}$$

$$\begin{aligned}
&= \lambda\Big\{\big((\zeta-2)+1\big)\Big[p\big(\tau_{\hat{\phi}_0}>0\big)+(\zeta-1)\,p\big(\tau_{\hat{\phi}_0}>G_1\big)\Big]\\
&\quad+\big(\zeta(\zeta-2)+1\big)\,p\big(\tau_{\hat{\phi}_0}>G_2\big)+\big(\zeta^2-1\big)\,p\big(\tau_{\hat{\phi}_0}>G_3\big)\\
&\quad+\big(\zeta^2(\zeta-2)+1\big)\,p\big(\tau_{\hat{\phi}_0}>G_4\big)+\big(\zeta^3-1\big)\,p\big(\tau_{\hat{\phi}_0}>G_5\big)\\
&\quad+\big(\zeta^3(\zeta-2)+1\big)\,p\big(\tau_{\hat{\phi}_0}>G_6\big)+\cdots\Big\}
\end{aligned}$$

To solve this equation we shall find the value of $p\big(\tau_{\hat{\phi}_0}>G_{2i-1}\big)$ and the value of $p\big(\tau_{\hat{\phi}_0}>G_{2i}\big), i\ge 1$ as the following

$$\begin{aligned}
p\big(\tau_{\hat{\phi}_0}>G_{2i-1}\big)\le&\int_{-\infty}^{0}p\big(Z_1+S(G_{2i-1})<-H_i\mid Z_i=z_1\big)v_1(\mathrm{d}z_1)\\
&+\int_{0}^{\infty}p\big(Z_1+S(G_{2i-1})>H_i\mid Z_i=z_1\big)v_1(\mathrm{d}z_1)\\
&+\int_{-\infty}^{0}p\big(Z_2+S(G_{2i-1})<-H_i\mid Z_2=z_2\big)v_2(\mathrm{d}z_2)\\
&+\int_{0}^{\infty}p\big(Z_2+S(G_{2i-1})>H_i\mid Z_2=z_2\big)v_2(\mathrm{d}z_2)
\end{aligned}$$

We get

$$\begin{aligned}
p\big(\tau_{\hat{\phi}_0}>G_{2i-1}\big)\le&\int_{-\infty}^{0}p\big(\tilde{\varphi}_1(G_{2i-1})<-z_1\big)v_1(\mathrm{d}z_1)+\int_{0}^{\infty}p\big(\varphi_1(G_{2i-1})>-z_1\big)v_1(\mathrm{d}z_1)\\
&+\int_{-\infty}^{0}p\big(\tilde{\varphi}_2(G_{2i-1})<-z_2\big)v_2(\mathrm{d}z_2)+\int_{0}^{\infty}p\big(\varphi_2(G_{2i-1})>-z_2\big)v_2(\mathrm{d}z_2)
\end{aligned}$$

also we get

$$\begin{aligned}
p\big(\tau_{\hat{\phi}_0}>G_{2i}\big)\le&\int_{-\infty}^{0}p\big(Z_1+S(G_{2i})<-2H_i\mid Z_1=z_1\big)v_1(\mathrm{d}z_1)\\
&+\int_{0}^{\infty}p\big(Z_1+S(G_{2i})>2H_i\mid Z_1=z_1\big)v_1(\mathrm{d}z_1)\\
&+\int_{-\infty}^{0}p\big(Z_2+S(G_{2i})<-2H_i\mid Z_2=z_2\big)v_2(\mathrm{d}z_2)\\
&+\int_{0}^{\infty}p\big(Z_2+S(G_{2i})>2H_i\mid Z_2=z_2\big)v_2(\mathrm{d}z_2)
\end{aligned}$$

We get

$$\begin{aligned}
p\big(\tau_{\hat{\phi}_0}>G_{2i}\big)\le&\int_{-\infty}^{0}p\big(\tilde{\varphi}_1(G_{2i})<-z_1\big)v_1(\mathrm{d}z_1)+\int_{0}^{\infty}p\big(\varphi_1(G_{2i})>-z_1\big)v_1(\mathrm{d}z_1)\\
&+\int_{-\infty}^{0}p\big(\tilde{\varphi}_2(G_{2i})<-z_2\big)v_2(\mathrm{d}z_2)+\int_{0}^{\infty}p\big(\varphi_2((G_{2i}))>-z_2\big)v_2(\mathrm{d}z_2)
\end{aligned}$$

Hence, we can get,

$$\begin{aligned}
E\big(\tau_{\hat{\phi}_0}\big)\le\lambda\Big\{&\big((\zeta-2)+1\big)p\big(\tau_{\hat{\phi}}>0\big)+(\zeta-1)\Big[\int_{-\infty}^{0}p\big(\tilde{\varphi}_1(G_1)<-z_1\big)v_1(\mathrm{d}z_1)\\
&+\int_{0}^{\infty}p\big(\varphi_1(G_1)>-z_1\big)v_1(\mathrm{d}z_1)\Big]+\big(\zeta(\zeta-2)+1\big)\Big[\int_{-\infty}^{0}p\big(\tilde{\varphi}_1(G_2)<-z_1\big)v_1(\mathrm{d}z_1)
\end{aligned}$$

$$
\begin{aligned}
&+\int_{0}^{\infty} p\left(\varphi_1\left(G_2\right)>-z_1\right) v_1\left(\mathrm{dz}_1\right)\Bigg]+\left(\zeta^2-1\right)\left[\int_{-\infty}^{0} p\left(\tilde{\varphi}_1\left(G_3\right)<-z_1\right) v_1\left(\mathrm{dz}_1\right)\right. \\
&+\int_{0}^{\infty} p\left(\varphi_1\left(G_3\right)>-z_1\right) v_1\left(\mathrm{dz}_1\right)\Bigg]+\left(\zeta^2(\zeta-2)+1\right)\left[\int_{-\infty}^{0} p\left(\tilde{\varphi}_1\left(G_4\right)<-z_1\right) v_1\left(\mathrm{dz}_1\right)\right. \\
&+\int_{0}^{\infty} p\left(\varphi_1\left(G_4\right)>-z_1\right) v_1\left(\mathrm{dz}_1\right)\Bigg]+\left(\zeta^3-1\right)\left[\int_{-\infty}^{0} p\left(\tilde{\varphi}_1\left(G_5\right)<-z_1\right) v_1\left(\mathrm{dz}_1\right)\right. \\
&+\int_{0}^{\infty} p\left(\varphi_1\left(G_5\right)>-z_1\right) v_1\left(\mathrm{dz}_1\right)\Bigg]+\left(\zeta^3(\zeta-2)+1\right)\left[\int_{-\infty}^{0} p\left(\tilde{\varphi}_6\left(G_2\right)<-z_1\right) v_1\left(\mathrm{dz}_1\right)\right. \\
&+\int_{0}^{\infty} p\left(\varphi_1\left(G_6\right)>-z_1\right) v_1\left(\mathrm{dz}_1\right)\Bigg]\Bigg\}+\lambda\Bigg\{((\zeta-2)+1) p\left(\tau_{\hat{\phi}}>0\right) \\
&+(\zeta-1)\left[\int_{-\infty}^{0} p\left(\tilde{\varphi}_2\left(G_1\right)<-z_2\right) v_2\left(\mathrm{dz}_2\right)+\int_{0}^{\infty} p\left(\varphi_2\left(G_1\right)>-z_2\right) v_2\left(\mathrm{dz}_2\right)\right] \\
&+(\zeta(\zeta-2)+1)\left[\int_{-\infty}^{0} p\left(\tilde{\varphi}_2\left(G_2\right)<-z_2\right) v_2\left(\mathrm{dz}_2\right)+\int_{0}^{\infty} p\left(\varphi_2\left(G_2\right)>-z_2\right) v_2\left(\mathrm{dz}_2\right)\right] \\
&+\left(\zeta^2-1\right)\left[\int_{-\infty}^{0} p\left(\tilde{\varphi}_2\left(G_3\right)<-z_2\right) v_2\left(\mathrm{dz}_2\right)+\int_{0}^{\infty} p\left(\varphi_3\left(G_2\right)>-z_2\right) v_2\left(\mathrm{dz}_2\right)\right] \\
&+\left(\zeta^2(\zeta-2)+1\right)\left[\int_{-\infty}^{0} p\left(\tilde{\varphi}_4\left(G_2\right)<-z_2\right) v_2\left(\mathrm{dz}_2\right)+\int_{0}^{\infty} p\left(\varphi_2\left(G_4\right)>-z_2\right) v_2\left(\mathrm{dz}_2\right)\right] \\
&+\left(\zeta^3-1\right)\left[\int_{-\infty}^{0} p\left(\tilde{\varphi}_5\left(G_2\right)<-z_2\right) v_2\left(\mathrm{dz}_2\right)+\int_{0}^{\infty} p\left(\varphi_5\left(G_2\right)>-z_2\right) v_2\left(\mathrm{dz}_2\right)\right] \\
&+\left(\zeta^3(\zeta-2)+1\right)\left[\int_{-\infty}^{0} p\left(\tilde{\varphi}_2\left(G_6\right)<-z_2\right) v_2\left(\mathrm{dz}_2\right)\right. \\
&+\int_{0}^{\infty} p\left(\varphi_6\left(G_2\right)>-z_2\right) v_2\left(\mathrm{dz}_2\right)\Bigg]+\cdots\Bigg\}
\end{aligned}
$$

hence,

$$
\begin{aligned}
E\left(\tau_{\hat{\phi}_0}\right) \leq \lambda &\Bigg\{((\zeta-2)+1) p\left(\tau_{\hat{\phi}_0}>0\right)+\left[\int_{-\infty}^{0} w_1\left(z_1\right) v_1\left(\mathrm{dz}_1\right)+\int_{0}^{\infty} w_2\left(z_1\right) v_1\left(\mathrm{dz}_1\right)\right] \\
&+\left[\int_{-\infty}^{0} w_3\left(z_1\right) v_1\left(\mathrm{dz}_1\right)+\int_{0}^{\infty} w_4\left(z_1\right) v_1\left(\mathrm{dz}_1\right)\right]\Bigg\} \\
&+\lambda\Bigg\{((\zeta-2)+1) p\left(\tau_{\hat{\phi}_0}>0\right)+\left[\int_{-\infty}^{0} w_5\left(z_2\right) v_2\left(\mathrm{dz}_2\right)+\int_{0}^{\infty} w_6\left(z_2\right) v_2\left(\mathrm{dz}_2\right)\right] \\
&+\left[\int_{-\infty}^{0} w_7\left(z_2\right) v_2\left(\mathrm{dz}_2\right)+\int_{0}^{\infty} w_8\left(z_2\right) v_2\left(\mathrm{dz}_2\right)\right]\Bigg\}
\end{aligned}
$$

where,

$$B_1\left(z_1\right)=\sum_{i=1}^{\infty}\left(\zeta^i-1\right) p\left(\tilde{\varphi}_1\left(G_{2i-1}\right)<-z_1\right),$$

$$B_2\left(z_1\right)=\sum_{i=1}^{\infty}(\zeta^i-1) p\left(\varphi_1\left(G_{2i-1}\right)>-z_1\right),$$

$$B_3(z_1)=\sum_{i=1}^{\infty}\left(\zeta^i(\zeta-2)+1\right)p\left(\tilde{\varphi}_1(G_{2i})<-z_1\right),$$

$$B_4(z_1)=\sum_{i=1}^{\infty}\left(\zeta^i(\zeta-2)+1\right)p\left(\varphi_1(G_{2i})>-z_1\right),$$

$$B_5(z_2)=\sum_{i=1}^{\infty}\left(\zeta^i-1\right)p\left(\tilde{\varphi}_2(G_{2i-1})<-z_2\right),$$

$$B_6(z_2)=\sum_{i=1}^{\infty}\left(\zeta^i-1\right)p\left(\varphi_2(G_{2i-1})>-z_2\right),$$

$$B_7(z_2)=\sum_{i=1}^{\infty}\left(\zeta^i(\zeta-2)+1\right)p\left(\tilde{\varphi}_2(G_{2i})<-z_2\right)$$

and

$$B_8(z_2)=\sum_{i=1}^{\infty}\left(\zeta^i(\zeta-2)+1\right)p\left(\varphi_2(G_{2i})>-z_2\right),$$

lemma 1.

For any ≥ 0, if $a_n\geq 0$ for $n>0$, and $a_{n+1}\leq a_n$, $\{d_n\}_{n\geq 0}$ be a strictly increasing sequence of integer numbers with $d_0=0$, then

$$\sum_{n=k}^{\infty}(d_{n+1}-d_n)a_{d_{n+1}}\leq\sum_{k=d_k}^{\infty}a_k\leq\sum_{n=k}^{\infty}(d_{n+1}-d_n)a_{d_n},$$

where $\sum_{n=k}^{\infty}(d_{n+1}-d_n)a_{d_{n+1}}$, $\sum_{k=d_k}^{\infty}a_k$ and $\sum_{n=k}^{\infty}(d_{n+1}-d_n)a_{d_n}$ are vectors see [32].

Theorem 2. For the two intersected lines the chosen search plan satisfies:

$$B_1(z_1)\leq B_9(|z_1|),\ B_2(z_1)\leq B_{10}(|z_1|),\ B_3(z_1)\leq B_{11}(|z_1|),$$

$$B_4(z_1)\leq B_{12}(|z_1|),\ B_5(z_2)\leq B_{13}(|z_2|),\ B_6(z_2)\leq B_{14}(|z_2|),$$

$$B_7(z_2)\leq B_{15}(|z_2|)\ \text{and}\ B_8(z_2)\leq B_{16}(|z_2|),$$

where $B_9(|z_1|)$, $B_{10}(|z_1|)$, $B_{11}(|z_1|)$, $B_{12}(|z_1|)$, $B_{13}(|z_2|)$, $B_{14}(|z_2|)$, $B_{15}(|z_2|)$ and $B_{16}(|z_2|)$ are linear functions.

Proof:

We shall prove the theorem for $B_2(z_1)$ and $B_6(z_2)$ since the other cases can be proved by similar ways.

$$B_2(z_1)=\sum_{i=1}^{\infty}\left(\zeta^i-1\right)p\left(\varphi_1(G_{2i-1})>-z_1\right)$$

and

$$B_6(z_2)=\sum_{i=1}^{\infty}\left(\zeta^i-1\right)p\left(\varphi_2(G_{2i-1})>-z_2\right)$$

Let us defined the following:

1) For $z_1\leq 0$, we have $B_2(z_1)\leq B_2(0)$

And for $z_2\leq 0$, we have $B_6(z_2)\leq B_6(0)$.

2) but for $z_1>0$, we have

$$B_2(z_1)=B_2(0)+\sum_{i=1}^{\infty}\left(\zeta^i-1\right)p\left(-z_1<\varphi_1(G_{2i-1})\leq 0\right),$$

$$B_2(z_1) - B_2(0) = \sum_{i=1}^{\infty}(\zeta^i - 1)\, p(-z_1 < \varphi_1(G_{2i-1}) \le 0).$$

and for $z_2 > 0$, we have

$$B_6(z_2) = B_6(0) + \sum_{i=1}^{\infty}(\zeta^i - 1)\, p(-z_2 < \varphi_2(G_{2i-1}) \le 0),$$

$$B_6(z_2) - B_2(0) = \sum_{i=1}^{\infty}(\zeta^i - 1)\, p(-z_2 < \varphi_2(G_{2i-1}) \le 0).$$

from theorem 2 see [13], we obtain:

$$B_2(0) = \sum_{i=0}^{\infty}(\zeta^i - 1)\, p(G_{2i-1} > 0) \le \sum_{i=1}^{\infty}(\zeta^i - 1)\, \varepsilon^{G_{2i-1}}, 0 < \varepsilon < 1$$

And

$$B_6(0) = \sum_{i=0}^{\infty}(\zeta^i - 1)\, p(G_{2i-1} > 0) \le \sum_{i=1}^{\infty}(\zeta^i - 1)\, \varepsilon^{G_{2i-1}}, 0 < \varepsilon < 1$$

Let us defined the following:

1)

$$V_1(n) = \frac{\varphi_1(n\theta_1)}{2} = \sum_{i=1}^{n} B_{1(i)}$$

where $\{B_{1i}\}$ is a sequence of independent identically distributed random variable.

$$V_2(n) = \frac{\varphi_2(n\theta_2)}{2} = \sum_{i=1}^{n} B_{2i}$$

where $\{B_{2i}\}$ is a sequence of independent identically distributed random variable.

2)

$$d_{1(n)} = \frac{G_{2n-1}}{\theta_1} = K(\zeta^n - 1)$$

$$d_{2(n)} = \frac{G_{2n-1}}{\theta_2} = K(\zeta^n - 1)$$

3) m_1 is an integer such that $d_{m1} = b_1|z_1| + b_2$, m_2 is an integer such that $d_{m2} = b_1|z_2| + b_2$.

4)

$$a_1(n) = \frac{n}{n+k}\, p(-z_1/2 < V_1(n) \le 0) = \sum_{i=0}^{(|z_1|/2)} p[-(i+1) < V_1(n) \le (-i)]$$

$$a_2(n) = \frac{n}{n+k}\, p(-z_2/2 < V_2(n) \le 0) = \sum_{i=0}^{(|z_2|/2)} p[-(i+1) < V_2(n) \le (-i)]$$

5)

$$\alpha_1 = \frac{\zeta}{(\zeta - 1)K},$$

$$\alpha_2 = \frac{\zeta}{(\zeta - 1)K}$$

and

6)

$$U_1(j, j+1) = \sum_{n=0}^{\infty} p\left[-(j+1) < V_1(n) \le (-j)\right],$$

$$U_2(j, j+1) = \sum_{n=0}^{\infty} p\left[-(j+1) < V_2(n) \le (-j)\right].$$

Then $U_1(j, j+1)$ and $U_2(j, j+1)$ satisfies the condition of the renewal equation see [33]. Thus, from lemma (1) we have $\alpha_1(n)$ and $\alpha_2(n)$ are non increasing if $n > d_{m1}$ and $n > d_{m2}$ see [4], consequently,

$$\begin{aligned} B_2(z_1) - B_2(0) &= \sum_{i=1}^{\infty} \left(\zeta^i - 1\right) p\left(-z_1 < \varphi_1(G_{2i-1}) \le 0\right) \\ &= \sum_{i=1}^{n_1} \left(\zeta^i - 1\right) p\left(-z_1 < \varphi_1(G_{2i-1}) \le 0\right) \\ &\quad + \sum_{i=n_1+1}^{\infty} \left(\zeta^i - 1\right) p\left(-z_1 < \varphi_1(G_{2i-1}) \le 0\right) \\ &= \sum_{n=1}^{n_1} \zeta^n a_1\left(d_{1(n)}\right) + \sum_{n=n_j+1}^{\infty} \zeta^n a_1\left(d_{1(n)}\right) \\ &\le \sum_{n=1}^{n_1} \zeta^n + \alpha_1 \sum_{n=n_j+1}^{\infty} \left(d_{1(n)} - d_{1(n-1)}\right) a_1\left(d_{1(n)}\right) \\ &\le \sum_{n=1}^{n_1} \zeta^n + \alpha_1 \sum_{n=d_{m_1}}^{\infty} a_1(n) \\ &\le \sum_{n=1}^{n_1} \zeta^n + \alpha_1 \sum_{n=d_{m_1}}^{\infty} \sum_{i=0}^{(|z_1|/2)} p\left[-(j+1) < V_1(n) \le (-j)\right] \\ &\le \sum_{n=1}^{n_1} \zeta^n + \alpha_1 \sum_{j=0}^{|z_1|/2} U_1(j, j+1) \end{aligned}$$

and

$$\begin{aligned} B_6(z_2) - B_6(0) &= \sum_{i=1}^{\infty} \left(\zeta^i - 1\right) p\left(-z_2 < \varphi_2(G_{2i-1}) \le 0\right) \\ &= \sum_{i=1}^{n_2} \left(\zeta^i - 1\right) p\left(-z_2 < \varphi_2(G_{2i-1}) \le 0\right) \\ &\quad + \sum_{i=n_2+1}^{\infty} \left(\zeta^i - 1\right) p\left(-z_2 < \varphi_2(G_{2i-1}) \le 0\right) \\ &= \sum_{n=1}^{n_2} \zeta^n a_2\left(d_{2(n)}\right) + \sum_{n=n_2+1}^{\infty} \zeta^n a_2\left(d_{2(n)}\right) \\ &\le \sum_{n=1}^{n_2} \zeta^n + \alpha_2 \sum_{n=n_2+1}^{\infty} \left(d_{2(n)} - d_{2(n-1)}\right) a_2\left(d_{2n}\right) \\ &\le \sum_{n=1}^{n_2} \zeta^n + \alpha_2 \sum_{n=d_{m_2}}^{\infty} a_2(n) \\ &\le \sum_{n=1}^{n_2} \zeta^n + \alpha_2 \sum_{n=d_{m_2}}^{\infty} \sum_{i=0}^{(|z_2|/2)} p\left[-(j+1) < V_2(n) \le (-j)\right] \\ &\le \sum_{n=1}^{n_2} \zeta^n + \alpha_2 \sum_{j=0}^{|z_2|/2} U_2(j, j+1) \end{aligned}$$

Such that for any line $U_1(j, j+1)$ and $U_2(j, j+1)$ satisfies the condition of the renewal equation see [26], we have $U_1(j, j+1)$ and $U_2(j, j+1)$ are bounded for all *j* by a constant so

$$B_2(z_1) \le B_2(0) + M_1 + M_2|z_1| = B_{10}(|z_1|),$$

then

$$B_6(z_1) \le B_6(0) + M_1 + M_2|z_2| = B_{14}(|z_2|).$$

The necessary and sufficient condition for the existence of a finite search plan is $E|Z_j| < \infty, j = 1,2$, that is sufficient from the consequence of Theorems 1,2 and the following Theorem 3.

Theorem. 3.

If there exists a finite search plan $\hat{\phi}_0 \in \hat{\Phi}_0$, then $E|Z_j|$ is finite, where Z_j is a random variable representing the initial position of the target on a line $L_j, j = 1,2$.

Proof

For $E(\tau_{\hat{\phi}_0}) < \infty$, we have $p(\tau_{\hat{\phi}_0} \text{ is finite}) = 1$, and so

$$\sum_{r=1}^{4} p(\tau_{\phi_r} \text{ is finite}) = 1$$

Therefore,

$$p(\tau_{\hat{\phi}_{01}} \text{ is finite}) + p(\tau_{\hat{\phi}_{02}} \text{ is finite}) = 1$$

where $\tau_{\hat{\phi}_{01}}, \tau_{\hat{\phi}_{02}}$ the first meeting time on the L_1 and L_2, respectively, then,

$$p(\tau_{\hat{\phi}_{01}} \text{ is finite}) = 1.$$

we conclude that:

$$p(\tau_{\hat{\phi}_{01}} \text{ is finite}) = 1 \text{ and } p(\tau_{\hat{\phi}_{02}} \text{ is finite}) = 0$$

$$\text{or } p(\tau_{\hat{\phi}_{01}} \text{ is finite}) = 0 \text{ and } p(\tau_{\hat{\phi}_{02}} \text{ is finite}) = 1$$

If $p(\tau_{\hat{\phi}_{01}} \text{ is finite}) = 1$ on the first line L_1, then $Z_1 = \phi(\tau_{\hat{\phi}_{01}}) - S(\tau_{\hat{\phi}_{01}})$ with probability one and hence

$$|Z_1| \le |\phi(\tau_{\hat{\phi}_{01}})| + |S(\tau_{\hat{\phi}_{01}})| \le \tau_{\hat{\phi}_{01}} + |S(\tau_{\hat{\phi}_{01}})|,$$

that leads to,

$$E|Z_1| \le E(\tau_{\hat{\phi}_{01}}) + E|S(\tau_{\hat{\phi}_{01}})|.$$

but $|S(\tau_{\hat{\phi}_{01}})| \le \tau_{\hat{\phi}_{01}}$, then $E|S(\tau_{\hat{\phi}_{01}})| \le E(\tau_{\hat{\phi}_{01}})$.

If $E(\tau_{\hat{\phi}_{01}}) < \infty$, then $E|S(\tau_{\hat{\phi}_{01}})| < \infty$, and $E|Z_1|$ is finite. On the second line L_2.

If $p\left(\tau_{\hat{\phi}_{02}} \text{ is finite}\right)=1$, then $Z_2=\phi\left(\tau_{\hat{\phi}_{02}}\right)-S\left(\tau_{\hat{\phi}_{02}}\right)$ with probability one, by the same way we can get $E\left|\left(\tau_{\hat{\phi}_{02}}\right)\right|<\infty$, and $E\left|Z_2\right|$ is finite.

3. Existence of an Optimal Search Plan

Definition 1.

Let $\hat{\phi}_{ji} \in \hat{\Phi}(t), j=1,2$ be two sequences of search plans, we say that $\hat{\phi}_{ji}$ converges to $\hat{\phi}_j$ as i tends to ∞ if and only if for any $t \in I^+$, converges to $\hat{\phi}_j$ uniformly on every compact subset see [13].

Theorem 1.4.

Let for any $t \in I^+$, and $S(t)$ be a process(one dimensional random walk). The mapping $\hat{\phi}=(\phi_1,\phi_2,\phi_3,\phi_4) \to E\left(\tau_{\hat{\phi}}\right) \in R^+ \to E\left(\tau_{\hat{\phi}}\right) \in R^+$ is lower semi-continuous on $\hat{\Phi}(t)$.

Proof

Let $I\left(\hat{\phi}_j,t\right)$ be the indicator function of the set $\left\{\tau_{\hat{\phi}_j} \geq t\right\}$, by the fatou-lebesque theorem see [5] we get:

$$E\left(\tau_{\hat{\phi}_j}\right)=E\left[\sum_{t=1}^{\infty} I\left(\hat{\phi}_j,t\right)\right]=E\left[\sum_{t=1}^{\infty} \liminf_{i\to\infty} I\left(\hat{\phi}_{ji},t\right)\right] \leq \liminf_{i\to\infty} E\left(\tau_{\hat{\phi}_j}\right)$$

for any sequence $\hat{\phi}_{ji} \to \hat{\phi}_j$ in $\hat{\Phi}(t)$, where $\hat{\Phi}(t)$ is sequentially compact see [29].Thus the mapping $\hat{\phi}_j \to E\left(\tau_{\hat{\phi}_j}\right)$ is lower semi continuous on $\hat{\Phi}(t)$, then this mapping attains its minimum.

4. Conclusions and Future Work

We illustrated that the quasi coordinated linear search technique for a random walk target on one of two intersected real lines has been presented, where the target initial position is given by a random variable. We introduced the proof of conditions that make the search plan which is finite in Theorem 1 based on the continuity of the search plan. In Theorems 2 we showed that the search plan is finite if the conditions, where

$$B_1(z_1) \leq B_9(|z_1|), B_2(z_1) \leq B_{10}(|z_1|), B_3(z_1) \leq B_{11}(|z_1|), B_4(z_1) \leq B_{12}(|z_1|),$$
$$B_5(z_2) \leq B_{13}(|z_2|), B_6(z_2) \leq B_{14}(|z_2|), B_7(z_2) \leq B_{15}(|z_2|) \text{ and } B_8(z_2) \leq B_{16}(|z_2|),$$

where $B_9(|z_1|), B_{10}(|z_1|), B_{11}(|z_1|), B_{12}(|z_1|), B_{13}(|z_2|), B_{14}(|z_2|), B_{15}(|z_2|)$ and $B_{16}(|z_2|)$ are linear functions. We use Theorem 3 to show that if there exists a finite search plan then the expected value of the target initial position $E\left|Z_j\right|$ is finite. It will also be interesting to see a direct consequence of Theorems 1, 2, and 3 satisfying the existence of a finite search plan if and only if $E\left|Z_j\right|$ is finite. We pointed to the existence of an optimal search plan in Theorem 4. The effectiveness of this model is illustrated using a real life application.

In future research, we have interesting search problems, study the coordinated search problem using multiple searchers, when the searchers star from any point on more lines rather than the origin.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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On the Coalitional Rationality and the Egalitarian Nonseparable Contribution

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Abstract

In earlier works we introduced the Inverse Problem, relative to the Shapley Value, then relative to Semivalues. In the explicit representation of the Inverse Set, the solution set of the Inverse Problem, we built a family of games, called the almost null family, in which we determined more recently a game where the Shapley Value and the Egalitarian Allocations are colalitional rational. The Egalitarian Nonseparable Contribution is another value for cooperative transferable utilities games (TU games), showing how to allocate fairly the win of the grand coalition, in case that this has been formed. In the present paper, we solve the similar problem for this new value: given a nonnegative vector representing the Egalitarian Nonseparable Contribution of a TU game, find out a game in which the Egalitarian Nonseparable Contribution is kept the same, but it is colalitional rational. The new game will belong to the family of almost null games in the Inverse Set, relative to the Shapley Value, and it is proved that the threshold of coalitional rationality will be higher than the one for the Shapley Value. The needed previous results are shown in the introduction, the second section is devoted to the main results, while in the last section are discussed remarks and connected problems. Some numerical examples are illustrating the procedure of finding the new game.

Keywords

Shapley Value, Egalitarian Nonseparable Contribution, Inverse Set, Family of Almost Null Games, Coalitional Rationality

1. Introduction

In [1], we introduced a new problem, connected to the Shapley Value, that was called the Inverse Problem, relative to the Shapley Value: Let $L \in R_{+}^{n}$ be the Shapley Value of a given TU game. It is well known that the set of cooperative

TU games (N,v), with $SH(N,v)=L$, is a vector space. In this vector space, called the Inverse Set, we defined a basis, called a potential basis, such that any element of the vector space may be written as

$$v=\sum_{S\subset N,|S|\le n-2} c_S W_S + c_N\left(W_N+\sum_{i\in N} W_{N-\{i\}}\right)-\sum_{i\in N} L_i W_{N-\{i\}}, \tag{1}$$

where the basis is $W=\left\{W_T\in R^n : T\subseteq N, T\neq\varnothing\right\}$, expressed in terms of the Shapley Value weights, as

$$W_T(T)=\frac{1}{p_t^t}, W_T(S)=\sum_{l=0}^{l=s-t}\frac{(-1)^l\binom{s-t}{l}}{p_{t+l}^{t+l}}, \forall S\supseteq T, T\subseteq N, \tag{2}$$

This set of games is called the Inverse Set, relative to the vector L. More recently, we introduced, in [2], a new problem connected to the Inverse Problem, relative to the Shapley value: to find out, in the Inverse Set of a TU game relative to the Shapley Value, a game in which this value is colalitional rational. The main idea in solving this problem was to look for the solution in what we called the almost null family of the Inverse Set, defined by the formula

$$w=c_N\left(w_N+\sum_{i\in N} w_{N-\{i\}}\right)-\sum_{i\in N} L_i w_{N-\{i\}}, \tag{3}$$

in which c_N is a parameter, in fact, the potential of the game. The scalar form of this family of games is

$$\begin{aligned} &w(N-\{i\})=(n-1)(c_N-L_i), \forall i\in N,\\ &w(N)=\sum_{i\in N} L_i, w(T)=0, \forall T\subset N, |T|\le n-2. \end{aligned} \tag{4}$$

The coalitional rationality conditions that give the appurtenance of the Shapley Value to the CORE of these games, are

$$\begin{aligned} &c_N\le\frac{1}{n-1}\left[w(N)+(n-2)L_i\right], \forall i\in N,\\ &\text{or } c_N\le\frac{1}{n-1}\left[v(N)+(n-2)Min_i L_i\right]=\alpha. \end{aligned} \tag{5}$$

Now, for any value of the parameter satisfying (5), substituted in the above scalar form (4), we get a solution of the last problem. Of course, no computation is needed, in case that for the given game the Shapley Value is already colalitional rational. Obviously, there is an infinite set of solutions, corresponding to the infinite possible choices of the parameter, belonging to the interval $[0,\alpha]$. This last number will be called the threshold of coalitional rationality.

A similar problem may be considered relative to the Banzhaf Value (see [3]).

Another new problem was discussed in the very recent paper [4]: Let us take another efficient value, called the Egalitarian Allocation and try to solve the similar problem: **If the value is not colalitional rational, find out in the Inverse Set, relative to the Shapley Value, a new game in which the value is kept the same, but it is colalitional rational.** Recall that the Egalitarian Allocation is defined by

$$EA_i(N,v)=\frac{v(N)}{n},\forall i\in N. \tag{6}$$

As shown by (6), this value depends only on the worth of the grand coalition, so that the main idea is that of trying to find a solution also in the family of the almost null game of the Inverse Set, relative to the Shapley value, where this value is kept unchanged. Therefore, we have tried again to find it in the family of the almost null games of the Inverse Set, relative to the Shapley Value, where the coalitional rationality conditions, providing a new threshold for coalitional rationality, where imposed (see [4]).

In this paper, we discuss the similar problem for another value, the Egalitarian Nonseparable Contribution, a value introduced in [5], and defined by

$$\begin{aligned}&ENSC_i(N,v)\\&=v(N)-v(N-\{i\})+\frac{1}{n}\left\{v(N)-\sum_{j\in N}\left[v(N)-v(N-\{j\})\right]\right\},\forall i\in N.\end{aligned} \tag{7}$$

The same basic idea from [4] will be used, that is a solution will be found in the family of almost null games in the Inverse Set, relative to the Shapley Value. The difference is that now we should show that the new game has the same Egalitarian Nonseparable Contribution like before and the threshold of coalitional rationality is given by a new formula which allows a comparison with the other thresholds of the two values considered in the previous works.

The interesting fact in answering the question why should we use the new value is provided by the nice interpretation of the ENSC: in a first stage, we allocate to each player his marginal contribution to the grand coalition, and if the win of the grand coalition is not exhausted, the reminder will be shared equally. Namely, if the total allocation is smaller than the win of the grand coalition, the difference will be shared equally; otherwise, each player will return an equal share of the difference.

Recall that in [4] we used the scalar form (4), of the games in the family of almost null games, as well as the definition (6) for the Egalitarian Allocation, to express the coalitional rationality conditions. We obtained

$$(n-1)\frac{w(N)}{n}\geq(n-1)(c_N-L_i),\forall i\in N,\quad\text{or}\quad c_N\leq\frac{v(N)}{n}+Min_iL_i=\beta. \tag{8}$$

The last number was the threshold of coalitional rationality for the Egalitarian Allocation, and in [4] we proved also the inequality $\alpha\geq\beta$. The same steps will be used in the case of Egalitarian Nonseparable Contributions. But, we have to compute the representation of the ENSC for games in the almost null family and to prove that the ENSC will be the same as in the initial game. This will be done in the next section, and remarks derived from examples will be in the last section.

2. The ENSC Value for the Games in the Almost Null Family

We compute the terms of formula (7) for the ENSC value by using (4); we get the sum of marginal contributions

$$\sum_{i\in N}\left[w(N)-w(N-\{i\})\right]=(2n-1)w(N)-n(n-1)c_N, \tag{9}$$

then, the average of leftover, after subtracting the initial allocations, namely

$$\frac{1}{n}\left\{w(N)-\sum_{i\in N}\left[w(N)-w(N-\{i\})\right]\right\}=(n-1)c_N-\frac{2(n-1)}{n}w(N). \tag{10}$$

In this way, from (4), (7), and (10), we obtain the components for the ENSC value

$$ENSC_i(N,w)=(n-1)L_i-\frac{n-2}{n}w(N),\forall i\in N. \tag{11}$$

Do not forget that in (11) we have the component of the Shapley Value. Now, on the one hand, from (11), it is easy to check the efficiency of the value in the new game; on the other hand, formula (11) shows the result that in the ENSC does not occur the value of the parameter. It follows that whatever would be the choice of the parameter, the ENSC has the same value. Of course, this includes the value which was providing the ENSC for the initially given game. Hence, any choice for the parameter should only satisfy in the new game the coalitional rationality conditions. Further, we shall impose, by means of Formulas (4), the coalitional rationality conditions

$$w(N)-ENSC_i(N,w)\geq w(N-\{i\}),\forall i\in N, \tag{12}$$

or, in another form

$$c_N\leq\frac{2}{n}v(N)=\gamma. \tag{13}$$

We proved the following result:

Theorem: The family of almost null games in the Inverse Set, relative to the Shapley Value, is providing a family of TU games in which the ENSC value will be unchanged and coalitional rational, if the parameter satisfies the inequality (13), providing a new coalitional rationality threshold.

Note that beside the coalitional thresholds given by Formulas (5) and (8), we have also a new threshold, for ENSC, offered by formula (13).

Example 1: Consider the three-person game

$$v(1)=v(2)=v(3)=0,v(1,2)=22,v(1,3)=v(2,3)=18,v(1,2,3)=25. \tag{14}$$

First, compute the Shapley Value and the Egalitarian Allocation of this game, by using the Shapley formula and the definition (6) given in the first section:

$$SH(N,v)=(9,9,7),EA(N,v)=\left(\frac{25}{3},\frac{25}{3},\frac{25}{3}\right). \tag{15}$$

The thresholds for coalitional rationality for them are $\alpha=16$ and $\beta=\frac{46}{3}$, hence we can get solutions of our problem, in the case of both values, for games corresponding to values of the parameter in the interval $\left[0,\frac{46}{3}\right]$. The number γ necessary for getting a solution for the ENSC value is $\gamma=\frac{50}{3}$; hence the values of the parameter providing solutions of our problems for all three values are those in the interval $\left[0, \frac{46}{3}\right]$. Now, a solution may be obtained by taking the maximal value in this interval, which provides the game

$$\begin{aligned} &w(1)=w(2)=w(3)=0, w(1,2)=\frac{50}{3}, \\ &w(1,3)=w(2,3)=\frac{38}{3}, w(1,2,3)=25. \end{aligned} \tag{16}$$

For our game (16), beside the same Shapley Value and Egalitarian Value, both coalitional rational, we obtain the same ENSC value as in the initially given game, namely the coalitional rational value

$$ENSC(N,v)=\left(\frac{29}{3}, \frac{29}{3}, \frac{17}{3}\right), \tag{17}$$

This was not coalitional rational in the given game, as it did not belong to the CORE. An interesting remark is that the value of the threshold of coalitional rationality for the Egalitarian Nonseparable Contribution is higher than the one for the Shapley Value and the Egalitarian Allocation, and we may wonder whether, or not, this is a general situation. This will provide a second main result after the above theorem.

To see that, first we should compare the numbers

$$\alpha=\frac{1}{n-1}\left[v(N)+(n-2)Min_i L_i\right], \beta=\frac{1}{n}v(N)+Min_i L_i, \gamma=\frac{2}{n}v(N), \tag{18}$$

that decide the coalitional rationality in the family of almost null games, in the Inverse Set, relative to the Shapley Value. Taking into account that in such games we have

$$w(N)=\sum_{i\in N} L_i \geq nMin_i L_i, \tag{19}$$

we can compute the differences

$$\begin{aligned} &\gamma-\alpha=\left(\frac{2}{n}-\frac{1}{n-1}\right)w(N)-\frac{n-2}{n-1}Min_i L_i=\frac{n-2}{n-1}\left(\frac{1}{n}w(N)-Min_i L_i\right)\geq 0, \\ &\gamma-\beta=\frac{1}{n}w(N)-Min_i L_i \geq 0, \end{aligned} \tag{20}$$

and conclude the result:

Theorem: In the family of almost null games from the Inverse Set, relative to the Shapley Value, we have:

1) The thresholds for coalitional rationality (18) satisfy the inequalities $\gamma \geq \alpha \geq \beta$,

2) A game in which the Shapley Value, the Egaltarian Allocation and the Egalitarian Nonseparable Contribution are all coalitional rational can be obtained by taking $c_N \in [0, \beta]$.

Example 2. An interesting situation occurs in case of the game

$$v(1)=v(2)=v(3)=0, v(1,2)=v(1,3)=v(2,3)=v(1,2,3)=1 \tag{21}$$

where the CORE is empty, as a constant sum game, and we have

$$\alpha=\beta=\gamma=\frac{2}{3}, SH(N,v)=EA(N,v)=ENSC(N,v)=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right), \tag{22}$$

and one of the solutions of our problem for all three values is the game obtained for the maximal value shown by the previous theorem:

$$w(1)=w(2)=w(3)=0, w(1,2)=w(1,3)=w(2,3)=\frac{2}{3}, w(1,2,3)=1. \tag{23}$$

Finally, we could check to see that the Egalitarian Nonseparable Contribution will be unchanged for the games in the family of almost null games in the Inverse Set, relative to the Shapley Value.

3. Discussion and Remarks

In the examples 1 and 2, we have shown cases of games in which the thresholds for coalitional rationality are satisfying the inequalities proved in the recent work [4] and in the last theorem. In example 1, all hold with strict inequality signs, while in example 2, all hold with equal signs. In [4], an example was given, where the parameter was chosen between the two smallest values. Now, let us see that for the ENSC it is possible to have a case where the parameter is chosen at the maximal value, while the other two values are not colalitional rational.

Example 3: Return to the game of example 1, given by Formulas (14). The Shapley Value and the Egalitarian Value are given by Formulas (15), while the ENSC is given by formula (17). We have the values of thresholds

$$\gamma=\frac{50}{3}\geq\alpha=16\geq\beta=\frac{46}{3}, \tag{24}$$

that is, the inequalities are satisfied with strict signs. Let us use the Formulas (4) and choose the parameter equal to the maximal value of the threshold for the ENSC, that is $c_N=\frac{50}{3}$, to compute the new game in the family of the almost null games in the Inverse Set relative to the Shapley Value. We obtain the game

$$\begin{gathered} w(1)=w(2)=w(3)=0, w(1,2)=\frac{58}{3}, \\ w(1,3)=w(2,3)=\frac{46}{3}, w(1,2,3)=25. \end{gathered} \tag{25}$$

We can compute and check that the three values are unchanged, and also check the coalitional rationality. For the Shapley Value and the Egalitarian Allocations, the inequalities $x_1+x_2\geq w(1,2)=\frac{58}{3}$, from the definition of the CORE, do not hold, while all the others hold, hence these two values are not colalitional rational in the game (25). On the other side, the ENSC satisfies all conditions of coalitional rationality. This provides an illustration of the above statement.

A good question is whether or not, there are other efficient values that generate subfamilies of the family of almost null games in the Inverse Set, relative to the Shapley Value, in which these values are colalitional rational. This may be the topic for future research.

Note also that the Egalitarian Nonseparable Contribution, beside the efficiency, has the property of possessing the coalitional rationality inside the Inverse Set, relative to another value, the Shapley Value. This was also true for the Egalitarian Allocations, but it was not that obvious like in the ENSC case.

We may also check that whatever value satisfying the condition of the above theorem was chosen, the ENSC value is the same and equal to the initially computed ENSC for the given game, shown above in (17). This is shown in connection with formula (11), but we may check it by taking any other value of the parameter. For example, if we take $c_N = \frac{41}{3}$, that is below the common threshold for coalitional rationality, and use the Formulas (4) and (7), then we obtain the new game

$$\begin{aligned} &w(1) = w(2) = w(3) = 0, w(1,2) = w(1,3) = \frac{28}{3}, \\ &w(2,3) = \frac{40}{3}, w(1,2,3) = 25, \end{aligned} \tag{26}$$

and if we compute the ENSC, we shall get the same result like in (17). The similar result will be obtained for any other choice. The new game is different, but the ENSC is the same as initially, and it is colalitional rational.

Acknowledgements

The present work is a natural continuation of the paper [4], just published in AJOR, (2018), vol. 8. Note that the ENSC has been introduced in a chapter of the book [5].

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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