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Disintegration of Group Representations on Direct Integrals of Banach Spaces

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Abstract

In this paper, let G be a Polish locally compact group acting on a Polish space X with a G-invariant probability measures $\sum_j \mu^j$. Factorize the integral with respect to $\sum_j \mu^j$ in terms of the integrals with respect to the ergodic measures on X, and showed that $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right),(0\le\epsilon<\infty)$ are G-equivariantly isometric ally lattice isomorphic to an $L^{(1+\epsilon)}$-direct integral of the spaces $L^{(1+\epsilon)}\left(X,\lambda^j\right)$, where λ^j ranges over the ergodic measures on X. This yields a disintegration of the canonical representation of G as isometric lattice auto orphisms of $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$ as an $L^{(1+\epsilon)}$-direct integral of order indecomposable representations. If $\left(X',\sum_j \mu'^j\right)$ are probability space, and, for some $0\le\epsilon<\infty$, G acts in a strongly continuous manner on $L^{(1+\epsilon)}\left(X',\sum_j \mu'^j\right)$ as isometric lattice auto orphisms that leave the constants fixed, then G acts on $L^{(1+\epsilon)}\left(X',\sum_j \mu'^j\right)$ in a similar fashion for all $0\le\epsilon<\infty$. Moreover, there exists an alternative model in which these representations originate from a continuous action of G on a compact Hausdorff space. If $\left(X',\sum_j \mu'^j\right)$ are separable, the representation of G on $L^{(1+\epsilon)}\left(X',\sum_j \mu'^j\right)$ can then be disintegrated into order indecomposable representations. The notions of $L^{(1+\epsilon)}$-direct integrals of Banach spaces and representation is developed for extend those in the literature.

Keywords

Positive Representation L^p-Space, Order Indecomposable Representation, Direct Integral of Banach Lattice

1. Introduction and Overview

There is unitary group representation. Apart from an intrinsic interest and mathematical relevance, the wish to understand such representations originates from quantum theory, where the unitary representations of the symmetry group of a physical system have a natural role. However, whereas symmetry yields a unitary representation of the pertinent symmetry group, there is also a family of canonical representations on Banach lattices in [1]. The rotation group of $\mathbb{R}^3$ acts on the 2-sphere in a measures-preserving fashion, yielding a canonical unitary representation on $L^2\left(S^2, d\sigma^j\right)$, but there are, in fact, canonical strongly continuous representations as isometric lattice automorphisms of the Banach lattice $L^{(1+\epsilon)}\left(S^2, d\sigma^j\right)$ for all $0 \le \epsilon < \infty$.

Likewise, for all $0 \le \epsilon < \infty$, the motion group of $\mathbb{R}^{(1+\epsilon)}$ acts in a strongly continuous fashion as isometric lattice automorphisms on the Banach lattice $L^{(1+\epsilon)}\left(\mathbb{R}^{(1+\epsilon)}, dx\right)$. Representations of groups as isometric lattice automorphisms of Banach lattices are quite common. In spite of this, not much is known about such representation or, for that matter, about the related positive representation of ordered Banach algebra and Banach lattice algebra in Banach lattice; the material in [2] [3] [4] [5] is a modest start at best. Nevertheless, it seems quite natural to investigate such representations. Moreover, given the long-term success, in a Hilbert space context, of the passage from single operator theory to groups and algebras and their representations—a development that was initially also stimulated and guided by the wish to understand unitary group representations—it seems promising to develop a similar theory for representations in Banach lattices.

One of the highlights in abstract representation theory in Hilbert spaces is the insight that every strongly continuous unitary representation of a separable locally compact Hausdorff group on a separable Hilbert space can be disintegrated into irreducible unitary representations. This follows from a similar theorem for C^*-algebras and the standard relation between the unitary representations of a group and the non-degenerate representations of its group C^*-algebra; every representation is thus built from irreducible ones. Is something analogous possible for strongly continuous actions of a locally compact Hausdorff group as isometric lattice automorphisms of Banach lattices? This seems a natural guiding question when studying representations in an ordered context. It is still very far from having been answered in general, and presumably one will have to restrict oneself to a class of suitable Banach lattices. After all, the unitary theory works particularly well in just one space, namely ℓ^2, and it seems doubtful that there can be a uniform answer for the existing diversity of Banach lattices.

What, exactly, should "irreducible" mean in an ordered context? When searching for the parallel with unitary representations it is actually more convenient to think of irreducible unitary representations as indecomposable unitary representations, which happens to be the same notion, and look for the analogue of the latter. Given a representation of a group G as lattice automorphisms of two vector lattices E_1 and E_2, there is a natural representation of G as lattice automorphisms of the vector lattice $E = E_1 \oplus E_2$. If a representation of G as lattice automorphisms of a given vector lattice E is not such an order direct sum of two non-trivial subrepresentations, then one will want to call it for order indecomposable. Actually, if $E = E_1 \oplus E_2$ is an order direct sum of vector lattice, then more is true than one would perhaps expect. E_1 and E_2 are automatically projection bands, and they are each other's disjoint complement; this is a special case of [[6], Theorem 11.3].

Coming from the other side, if a projection band in E is invariant under a group of lattice automorphisms, then so is its disjoint complement, and hence there is a corresponding decomposition of the representation into two sub-representations as lattice automorphisms. All in all, we have the following natural definition.

Definition 1.1. Let E be a vector lattice, and let ρ be a homomorphism from G into the group of lattice automorphisms of E. Then the representation ρ is order indecomposable if {0} and E are the only G-invariant projection bands in E.

Note that G acts on E as lattice automorphisms precisely when it acts as positive sequence of operators; hence one can also refer to such a representation as a positive representation of G on E.

It is a non-trivial fact that an order indecomposable positive representation of a finite group on a Dedekind complete vector lattice is finite dimensional; this follows from [[7, Theorem 3.14]. It is also possible to show that every finite dimensional positive representation of a finite group on an Archimedean vector lattice is an order direct sum of order indecomposable positive representations, where the latter can be classified [[7], Theorem 4.10 and Corollary 4.11]. This answers the question about disintegrating finite dimensional positive representations of finite groups. The matter is still open for infinite dimensional positive representations of finite groups.

For positive representations of an abstract group G on a normalized Banach sequence space E, it is true that the representation is order direct sum of order indecomposable positive representations; see [[8], Theorem 5.7]. If the group has compact image in the strong sequence of operators topology, and E has order continuous norm (this includes the spaces $\ell^{(1+\epsilon)}$ for $0 \le \epsilon < \infty$), then these order indecomposable positive representations are all finite dimensional. This is an analogue of the well-known theorem for unitary representations of compact Hausdorff groups.

This first main step—omit the necessary conditions for the sake of clarity—consists of a disintegration into order indecomposable representations of the representations of a locally compact Hausdorff group G as isometric lattice automorphisms of $L^{(1+\epsilon)}$-spaces, as canonically associated with an action of G on a Bore probability space $\left(X,\sum_j \mu^j\right)$ with invariant measure $\sum_j \mu^j$. Such a representation is order indecomposable precisely when $\sum_j \mu^j$ is ergodic. One might therefore hope that, somehow, a disintegration of $\sum_j \mu^j$ into ergodic measures λ^j will yield a disintegration of the canonical positive representation on $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$ in terms of the order indecomposable canonical representations on $L^{(1+\epsilon)}\left(X,\lambda^j\right)$ for ergodic λ^j. This can be done, and Theorem 4.9 clarifies what is here: in a G-equivariant fashion, the Banach lattice $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$ is an $L^{(1+\epsilon)}$-direct integral of the Banach lattices $L^{(1+\epsilon)}\left(X,\lambda^j\right)$ for ergodic λ^j, where the $L^{(1+\epsilon)}$-direct integral is with respect to a Borel probability measure on the set of ergodic measures. Apart from the framework of direct integrals of Banach spaces as such, which could also have representation theoretical applications in other contexts, the principal ingredient for the proof of this result is a factorization of the integral over X with respect to $\sum_j \mu^j$ in terms of those with respect to the ergodic measures; see Theorem 4.5. In spite of its aesthetic appeal, not aware of a reference for the pertinent formula in this Tonelli-Fubini-type theorem, which itself is based on the aforementioned disintegration of $\sum_j \mu^j$ into ergodic measures.

Aside, let us briefly mention that there is no uniqueness statement concerning the isomorphism classes occurring in the disintegration Theorem 4.9. Given the subtleties necessary in the study of Type I groups and C^*-algebras in the Hilbert space context, it does not seem to be realistic to strive for such a result at this moment.

The second main step consists of removing the hypothesis that the given representation of G on $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$ originate from an action on the underlying probability space $\left(X,\sum_j \mu^j\right)$. Under mild conditions, it can be shown that an action of G on $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$ as isometric lattice automorphisms that leave the constants fixed, can be transferred to another model where there is such an underlying action; see Theorem 5.14. Then back in the ergodic theoretical context, and combination with the result from the first main step yields a disintegration result for these representations into order indecomposable representations as well. The pertinent Theorem 5.15 should be thought of as an ordered relative of the general unitary disintegration result. The key transfer Theorem 5.14 for this step is strongly inspired by the material in [9].

It seems that, for practical purposes, the main results have a rather broad range of validity; we will make a few technical remarks to support this statement.

One of the re-occurring hypotheses is that a space is Polish. For a locally compact Hausdorff space, being Polish is equivalent to be second countable; see

[[10], Theorem 5.3].

Thus all Lie groups are Polish see [[11], Section 1.3]), and, more generally, so are all differentiable manifolds. Therefore, the factorization Theorem 4.5 and the disintegration Theorem 4.9—for which the underlying Polish space X need not even be locally compact—are applicable to all actions of Lie groups on differentiable manifolds. In a similar vein, note that it follows from the combination of [[12], Vol. I, Exercise 1.12.102] and [[12], Vol. II, Example 6.5.2] that the measure space $\left(X, \sum_j \mu^j\right)$ is always separable whenever X is a separable metric space and $\sum_j \mu^j$ is a Borel probability measure on X. Therefore, the disintegration Theorem 5.15, where this separ ability is assumed, covers several commonly occurring situations as well.

This paper is organized as follows:

In Sec 2, introduce some terminology and notation, and establish a few preliminary results on order in decomposability and strong continuity of canonical representations of groups on $L^{(1+\epsilon)}$-spaces.

The first part of Sect.3 is concerned with an extension of part of the theory of direct integrals of Banach spaces and Banach lattices. The measurable families of norms figuring in [13] are not sufficient for the context, where a measurable family of semi-norms occurs naturally. Moreover, the measures need not be complete. Generalize the theory accordingly. After that, $L^{(1+\epsilon)}$-direct integrals of representations are introduced, and possible perspectives in representation theory are briefly discussed. The usual direct integrals of representations on separable Hilbert spaces are shown to be special cases of the general formalism.

Section 4 contains the results of the first main step, *i.e.* the factorization Theorem 4.5 and the disintegration Theorem 4.9 in the case of an action on the underlying measure space. As a worked example, give a concrete disintegration of the representations of the unit circle on the $L^{(1+\epsilon)}$-spaces of the closed unit disk, as these are canonically associated with the action of the circle on this disk as rotations.

Section 5 is concerned with disintegrating representations when there is no action on an underlying measure space. Its main result, the disintegration Theorem 5.15, is the ordered relative of the general unitary disintegration.

Section 6 contains some remarks on the current status of the theory and on possible further developments. Even though this paper was motivated by a representation theoretical question in an ordered context, the interpretation of the main results as answers to this question is almost just an afterthought. The reader can find definitions and terminology concerning vector lattices, but, if so desired, the limited number of occurrences of this terminology in the sequel that stand beyond the notions of a vector lattice and a lattice homomorphism can also safely be ignored. Then be read from a primarily ergodic theoretical, functional analytical, or general representation theoretical perspective.

2. Preliminaries

In this section, fix terminology and notation, and establish a few preliminary results on group representations.

2.1. Terminology and Notation

All vector spaces, except the Hilbert spaces, are over the real numbers. This is no essential restriction, as the results extend to complex $L^{(1+\epsilon)}$-spaces and (in Sect. 3) complex Backspaces and Banach lattices in an obvious manner, but this convention reduces the necessary terminology and size of the proofs.

Topological spaces are not assumed to be Hausdorff. A topological space is called locally compact if every point has an open neighborhood with compact closure.

If X is a topological space, then $C_c(X)$ and $C_b(X)$ denote the continuous functions on X that have compact support and that are bounded, respectively.

Topological groups are groups for which inversion is continuous and multiplications continuous in two variables simultaneously. They are not assumed to be Hausdorff or locally compact.

A topological dynamical system is a pair (G, X), where the topological group G acts as homeomorphisms on the topological space X such that the map $(g_j, x) \mapsto (g_j)_x$ are continuous from $G \times X$ to X. The system is called Polish if both G and X are Polish.

A measure on a σ^j-algebra are σ^j-additive and takes values in $[0,\infty]$. It is not assumed to be σ^j-finite. If X is a topological space, then a Borel measure is a measure on the Borel σ^j-algebra of X, without any further assumptions.

For $\left(X, \sum_j \mu^j\right)$ a measure space and $0 \le \epsilon < \infty$, $\mathcal{L}^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ denotes the semi-normed space of all $(1+\epsilon)$-integrable extended functions $f_j : X \to \mathbb{R} \cup \{-\infty, \infty\}$, and $L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ denotes the Banach lattice of all equivalence classes of extended functions $f_j \in \mathcal{L}^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$, under $\sum_j \mu^j$-almost everywhere equality. Often work with an extended functions f_j that is an element of $\mathcal{L}^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ for different measures $\sum_j \mu^j$ on X, and consider the equivalence classes of f_j in $L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ for these $\sum_j \mu^j$. It is essential to keep a clear distinction between these objects, so do not identify functions that are equal almost everywhere, and, when $(1+\epsilon)$ is fixed, denote the equivalence class in $L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ of an element $f_j \in L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ by $\sum_j [f_j]_{\mu^j}$.

In the same vein, if V is a vector space, ω^j is an index, and $\sum_j \|\cdot\|_{\omega^j}$ are semi-norms on V, then denote the equivalence class of $x \in V$ in $V/\ker\left(\|\cdot\|_{\omega^j}\right)$ by $\sum_j [x]_{\omega^j}$.

If Y is a subset of X, then $\mathbf{1}_Y$ is the characteristic function of Y on X.

If B_j are normed space, then $\mathcal{L}(B_j)$ denotes the bounded linear sequence of operators on B_j.

2.2. Preliminaries on Group Representations

Suppose that the abstract group G acts as measure preserving transformations on the measure spaces $\left(X,\sum_j \mu^j\right)$. Then say that $\sum_j \mu^j$ are G-invariant measure. In this case, for every $0 \le \epsilon < \infty$, $\sum_j \rho_{\mu^j}\left(g_j\right)\left[f_j\right]_{\sum_j \mu^j} := \sum_j\left[x \mapsto f_j\left(g_j^{-1}x\right)\right]_{\sum_j \mu^j}$ are well-defined representation of G as isometric lattice isomorphism of $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$. Refer to this representation as the canonical representation on $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$; in the literature this is also called a Koopmans representation.

A measurable subset Y of X is $\sum_j \mu^j$-essentially G-invariant if $\sum_j \mu^j\left(g_j Y \,\Delta\, Y\right)=0$ for all m, where $Y \,\Delta\, g_j Y := \left(Y \cup g_j Y\right)\setminus\left(Y \cap g_j Y\right)$ are the symmetric difference of Y and $g_j Y$. An ergodic measure on X is a G-invariant measures $\sum_j \mu^j$ such that $\sum_j \mu^j(Y)=0$.

Or $\sum_j \mu^j(Y)=1$ for all $\sum_j \mu^j$-essentially G-invariant measurable subset Y of X.

Investigate the relationship between the ergodicity of the measure μ^j and the order in decomposability of $\rho_{\sum_j \mu^j} : G \to \mathcal{L}\left(L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)\right)$. This is essential for the representation theoretical interpretation of the disintegration results, but not for these results as such, so that a primarily ergodic theoretical or functional analytic interest can skip the next two results. The following lemma.

Lemma 2.1. Let $\left(X,\sum_j \mu^j\right)$ be a σ^j-finite measure space, and let $0 \le \epsilon < \infty$. If $Y \subseteq X$ is measurable, let

$$\sum_j \left(B_j\right)_Y = \left\{\sum_j \left[f_j\right]_{\mu^j} \in L^{1+\epsilon}\left(X,\sum_j \mu^j\right) : f_j(y)=0 \text{ for } \mu^j\text{-almost all } y \in Y\right\}.$$

Then $\left(B_j\right)_Y$ are projection band in $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$, and all projection bands in $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$ are of this form. If Y_1 and Y_2 are measurable subsets of X, then

$$\left(B_j\right)_{Y_1} = \left(B_j\right)_2 \text{ if and only if } \sum_j \mu^j\left(Y_1 \Delta Y_2\right)=0.$$

Recall that the measure algebra $\left(A_j\right)_{\sum_j \mu^j}$ of $\left(X,\sum_j \mu^j\right)$ consists of the equivalence classes $[Y]_{\sum_j \mu^j}$ of measurable subsets Y of X, where Y_1 and Y_2 are equivalent when $\sum_j \mu^j\left(Y_1 \Delta Y_2\right)=0$. Lemma 2.1 shows that there is a bisection between the elements of $\left(A_j\right)_{\sum_j \mu^j}$ and the projection bands in $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$, where an element of $\sum_j [Y]_{\sum_j \mu^j}$ of the measure algebra corresponds to the well-defined bands $\sum_j \left(B_j\right)_{[Y]_{\sum_j \mu^j}} := \sum_j \left(B_j\right)_Y$.

If an abstract group G' acts as positive sequence of operators on $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$, then it permutes the projection bands in $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$. If, as is the case for the group G, this positive action originates canonically from an action as measure preserving transformations on $\left(X,\sum_j \mu^j\right)$, then G also acts canonically on $\left(A_j\right)_{\sum_j \mu^j}$: for $g_j \in G$ and $[Y] \in \left(A_j\right)_{\sum_j \mu^j}$, the action $\sum_j g_j [Y]_{\sum_j \mu^j} := \sum_j \left[g_j Y\right]_{\sum_j \mu^j}$ is well-defined (see, e.g. by Marcel de Jeu, J Rozendaal [1]). These two actions are compatible with the maps $\sum_j [Y]_{\sum_j \mu^j} \mapsto \sum_j B_j [Y]_{\sum_j \mu^j}$. This is the content of part (1) of the next result, and it is exploited in parts (2), (3), and (4).

Proposition 2.2. Let G be an abstract group, acting as measure preserving transformations on a σ^j-finite measure space $\left(X, \sum_j \mu^j\right)$, and let $0 \le \epsilon < \infty$.

(1) If $\sum_j [Y]_{\sum_j \mu^j} \in \sum_j \left(A_j\right)_{\sum_j \mu^j}$, and $\left(B_j\right)_{[Y]_{\sum_j \mu^j}}$ is the corresponding projection b and in $L^{(1+\epsilon)}\left(X, \sum_j \sum_j \mu^j\right)$, then

$$\sum_j \rho_{\sum_j \mu^j}\left(g_j\right)\left(B_j\right)_{[Y]_{\mu^j}} = \sum_j \left(B_j\right)_{g_j [Y]_{\sum_j \mu^j}} \left(g_j \in G\right).$$

(2) for $g_j \in G$, the projection bands in $L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ that are fixed by g_j correspond.

To the fixed points of g_j in $\left(A_j\right)_{\sum_j \mu^j}$.

(3) The G-invariant projection bands in $L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ correspond to the fixed points of G in $\left(A_j\right)_{\sum_j \mu^j}$.

(4) The canonical representation $\rho_{\sum_j \mu^j} : G \to \mathcal{L}\left(L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)\right)$ of G as isometric lattice automorphisms on $L^{(1+\epsilon)}\left(X, \mu^j\right)$ is order indecomposable if and only if $\sum_j \mu^j$ are ergodic.

Proof. As for (1), let $\sum_j \left[f_j\right]_{\sum_j \mu^j} \in \sum_j \left(B_j\right)[Y]_{\sum_j \mu^j}$, so that $\sum_j \mu^j\left(\left\{Y \cap \operatorname{supp} f_j\right\}\right) = 0$. By the invariance of $\sum_j \mu^j$, have $\sum_j \mu^j\left(\left\{g_j Y \cap g_j \operatorname{supp} f_j\right\}\right) = 0$. Since $g_j \operatorname{supp} f_j = \operatorname{supp} g_j f_j$, see that $\sum_j \mu^j\left(\left\{g_j Y \cap \operatorname{supp} g_j f_j\right\}\right) = 0$, *i.e.* $\sum_j \rho_{\sum_j \mu^j} \in \left(B_j\right)_{g_j Y} = \sum_j \left(B_j\right)_{[g_j Y]_{\sum_j \mu^j}} = \sum_j \left(B_j\right)_{g_j [Y]_{\sum_j \mu^j}}$. Hence $\sum_j \rho_{\mu^j}\left(g_j\right)\left(B_j\right)_{[Y]_{\sum_j \mu^j}} \subseteq \sum_j \left(B_j\right)_{g_j [Y]_{\sum_j \mu^j}}$.

Then also

$$\begin{aligned} \sum_j \rho_{\sum_j \mu^j} g_j^{-1}\left(B_j\right)_{g_j [Y]_{\sum_j \mu^j}} &= \sum_j \rho_{\mu^j} g_j^{-1}\left(B_j\right)_{[g_j Y]_{\sum_j \mu^j}} \\ &\subseteq \sum_j \left(B_j\right)_{g_j^{-1}[g_j Y]_{\sum_j \mu^j}} = \sum_j \left(B_j\right)_{[Y]_{\sum_j \mu^j}} \end{aligned}$$

, so that $\sum_j \left(B_j\right)_{g_j [Y]_{\sum_j \mu^j}} \subseteq \sum_j \rho_{\sum_j \mu^j}\left(g_j\right)\left(B_j\right)_{[Y]_{\sum_j \mu^j}}$.

The parts (2) and (3) are immediate from (1).

As for (4), we know from (2) that $\rho_{\sum_j \mu^j}$ are order indecomposable if and only if $\sum_j [\varnothing]_{\sum_j \mu^j}$ and $\sum_j [X]_{\sum_j \mu^j}$ are the only points of $\left(A_j\right)_{\sum_j \mu^j}$ that are fixed by the G-action. The latter condition is equivalent to the ergodicity of $\sum_j \mu^j$.

As a further preliminary, investigate the strong continuity of canonical representations of topological groups on spaces of continuous functions with compact support and on $L^{(1+\epsilon)}$-spaces, the latter being the principal point of interest.

The matter is usually considered in the context of a locally compact Hausdorff group and a locally compact Hausdorff space in [14], but more can be said.

The results clarify natural questions concerning the context, and, in view of possible future study of canonical group actions on $L^{(1+\epsilon)}$-spaces, this seems a natural moment to collect a few basic facts in a sharp formulation.

A reference for the following result would be desirable, but not aware of one for the statement in this generality. The left and right uniform continuity of compactly supported continuous functions on a locally compact Hausdorff group are special cases.

Lemma 2.3. Let (G,X) be a topological dynamical system. Then the canonical representation ρ of G as isometric lattice automorphisms of $(C_c(X),\|\cdot\|_\infty)$ is strongly continuous.

Proof. It is sufficient to prove that $g_j \mapsto \rho(g_j)f_j$ are continuous at for all $f_j \in C_c(X)$.

Let $\epsilon > 0$. For all $x \in X$, there exist a symmetric open neighborhood U_x of e in G and an open neighborhood V_x of x in X such that $\sum_j \left|f_j(g_j^{-1}y) - f_j(x)\right| < \epsilon/2$ for all $g_j \in U_x$ and $y \in V_x$. Let $\bigcup_{i=1}^n V_{x_i}$ be a finite cover of sup, f_j and put $U = \bigcap_{i=1}^n U_{x_i}$.

If $x \in \operatorname{supp} f_j$, say $x \in V_{x_{i_0}}$, and $g_j \in U \subseteq U_{x_{i_0}}$, then $\sum_j \left|f_j(g_j^{-1}x) - f_j(x)\right| \le \sum_j \left|f_j(g_j^{-1}x) - f_j(x_{i_0})\right| + \sum_j \left|f_j(x) - f_j(x_{i_0})\right| < \epsilon/2 + \epsilon/2 = \epsilon$. Since U is symmetric, also have $\sum_j \left|f_j(g_j x) - f_j(x)\right| < \epsilon$ for all $x \in \operatorname{supp} f_j$ and $g_j \in U$. Therefore, if $g_j \in U$ and $x \in X$ are such that $g_j^{-1}x \in \operatorname{supp} f_j$, Have $\sum_j \left|f_j(g_j^{-1}x) - f_j(x)\right| = \sum_j \left|f_j(g_j(g_j^{-1}x)) - f_j(g_j^{-1}x)\right| < \epsilon$.

Have shown that, for $g_j \in U$, $\sum_j \left|f_j(g_j^{-1}x) - f_j(x)\right| < \epsilon$ whenever $x \in \operatorname{supp} f_j$ or $g_j^{-1}x \in \operatorname{supp} f_j$. Since $\sum_j \left|f_j(g_j^{-1}x) - f_j(x)\right| = 0$ for all remaining x, are done.

Proposition 2.4. Let (G,X) be a topological dynamical system, and assume that G is locally compact. Let $\sum_j \mu^j$ be a Borel measure on X that is finite on compact subsets of X. Then, for $0 \le \epsilon < \infty$, the canonical representation $\rho_{\sum_j \mu^j}$ of possibly unbounded lattice automorphisms of $\left(C_c(X),\|\cdot\|_{(1+\epsilon)}\right)$ is strongly continuous.

If $\sum_j \mu^j$ are G-invariant, then the canonical representation $\rho_{\sum_j \mu^j}$ of G as isometric lattice Automorphisms of the closure of $C_c(X)$ in $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$ are strongly continuous.

Proof. Let $f_j \in C_c(X), (g_j)_0 \in G$, and $\epsilon > 0$ be given. Choose an open neighbourhood V of e in G with compact closure. Then $(g_j)_0 \overline{V} \operatorname{supp} f_j$ are compact, hence has finite measure. By Lemma 2.3, there exist an open neighbourhood U of e in G such that

$\sum_j \mu^j\left(\left(g_j\right)_0 \bar{V}\operatorname{supp} f_j\right)\sum_j\left\|\rho_{\sum_j\mu^j}\left(g_j\right)f_j-\rho_{\sum_j\mu^j}\left(\left(g_j\right)_0\right)\right\|_\infty^{(1+\epsilon)}<\epsilon^{(1+\epsilon)}$ for all $g_j\in\left(g_j\right)_0 U$. Assume that $U\subseteq V$. Then, for $g_j\in\left(g_j\right)_0 U$,

$$\sum_j\left\|\rho_{\sum_j\mu^j}\left(g_j\right)f_j-\rho_{\sum_j\mu^j}\left(\left(g_j\right)_0\right)\right\|_{(1+\epsilon)}^{(1+\epsilon)}$$

$$=\int_X\sum_j\left|\left(\rho_{\sum_j\mu^j}\left(g_j\right)f_j\right)(x)-\left(\rho_{\sum_j\mu^j}\left(g_j\right)_0 f_j\right)(x)\right|^{(1+\epsilon)}\mathrm{d}\sum_j\mu^j(x)$$

$$=\sum_j\int_{g_j\operatorname{supp} f_j\cup\left(g_j\right)_0\operatorname{supp} f_j}\left|\left(\rho_{\sum_j\mu^j}\left(g_j\right)f_j\right)(x)-\left(\rho_{\sum_j\mu^j}\left(\left(g_j\right)_0\right)f_j\right)(x)\right|^{(1+\epsilon)}\mathrm{d}\sum_j\mu^j(x)$$

$$\leq\int_{\left(g_j\right)_0\bar{V}\cdot\operatorname{supp} f_j}\left|\sum_j\left(\rho_{\sum_j\mu^j}\left(g_j\right)f_j\right)(x)-\left(\rho_{\sum_j\mu^j}\left(\left(g_j\right)_0\right)f_j\right)(x)\right|^{(1+\epsilon)}\mathrm{d}\sum_j\mu^j(x)$$

$$\leq\sum_j\int_{\left(g_j\right)_0\bar{V}\cdot\operatorname{supp} f_j}\left\|\rho_{\sum_j\mu^j}\left(g_j\right)f_j-\rho_{\sum_j\mu^j}\left(\left(g_j\right)_0\right)f_j\right\|_\infty^{(1+\epsilon)}\mathrm{d}\sum_j\mu^j(x)\leq\epsilon^{(1+\epsilon)}.$$

The final statement follows from a 3ϵ-argument.

Proposition 2.4 points at the heart of the matter: under a mild condition on the G-invariant Borel measure $\sum_j\mu^j$, the natural representation subspace for G in $L^{(1+\epsilon)}\left(X,\sum_j\mu^j\right)$ is the closure of $C_c(X)$. In some cases, this closure equals $L^{(1+\epsilon)}\left(X,\sum_j\mu^j\right)$, and include this well-known result for the sake of completeness. For this, recall that a Borel measure μ^j on a locally compact Hausdorff space is said to be regular if $\sum_j\mu^j(K)<\infty$ for all compact subsets K of X, $\sum_j\mu^j(Y)=\inf\left\{\mu^j(V):Y\subseteq V,V\text{ open}\right\}$ for all Borel subsets Y of X,

And $\sum_j\mu^j(V)=\sup\left\{\sum_j\mu^j(K):K\subseteq V,K\text{ compact}\right\}$ for all open subsets V of X.

For such a measure, $C_c(X)$ is dense in $L^{(1+\epsilon)}\left(X,\sum_j\mu^j\right)$; in [15]. Combination with Proposition 2.4 gives the following, generalizing the well-known fact that the left and right regular representations of a locally compact Hausdorff group G on $L^{(1+\epsilon)}(G)$ are strongly continuous for $0\leq\epsilon<\infty$.

Corollary 2.5. Let (G,X) be a topological dynamical system, and assume that G is locally compact and that X is a locally compact Hausdorff space. Let $\sum_j\mu^j$ bea G-invariant regular Borel measure on X. Then, for $0\leq\epsilon<\infty$, the canonical representation $\rho_{\sum_j\mu^j}$ of G as isometric lattice automorphisms of $L^{\sum_j\mu^j}\left(X,\sum_j\mu^j\right)$ are strongly continuous.

Although [[14], p. 68]—where it is assumed that G is Hausdorff—mentions that the above result holds, and it is likewise stated—for locally compact second countable Hausdorff G and X—without proof on [[16], p. 875].

If G and X are not both locally compact, the proof of the strong continuity in Corollary 2.5 breaks down. However, there is an alternative context where a similar result can still be established along similar lines.

Lemma 2.6. Let X be a metric space, and let $\sum_j\mu^j$ be a Borel probability measure on *X*. Then $C_b(X)$ is dense in $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$ for $0 \le \epsilon < \infty$.

Proof. It is sufficient to approximate the characteristic function $\mathbf{1}_Y$ of an arbitrary Borel subset *Y* of *X* by elements of $C_b(X)$. Since know that, for every Borel subset *Y* of *X*, $\sum_j \mu^j(Y) = \inf\left\{\sum_j \mu^j(U) : Y \subseteq U, U \text{ open}\right\}$, it is sufficient to approximate $\mathbf{1}_U$ for an arbitrary open subset *U* of *X*. Assume that $U \ne X$. In that case, let $(f_j)_n(x) = \min\left(1, nd\left(x, U^c\right)\right)(n = 1,2,\cdots)$. Then $(f_j)_n \in C_b(X)$ and $0 \le (f_j)_n \uparrow \mathbf{1}_U$, so that $\sum_j \left\|(f_j)_n - \mathbf{1}_U\right\|_{(1+\epsilon)} \to 0$ as $n \to \infty$ by the dominated convergence theorem.

Proposition 2.7. Let *G* be a first countable group, acting as Borel measurable transformations on a metric space *X* with a *G*-invariant Borel probability measure $\sum_j \mu^j$.

Suppose that, for all $x \in X$, the map $g_j \mapsto g_j x$ is continuous from *G* to *X*. Then, for $0 \le \epsilon < \infty$, the canonical representation $\rho_{\sum_j \mu^j}$ of *G* as isometric lattice automorphisms of $L^{(1+\epsilon)}\left(X, \mu^j\right)$ are strongly continuous.

Proof. In view of Lemma 2.6 and a 3ϵ-argument, it is sufficient to prove that the map $g_j \mapsto \rho_{\sum_j \mu^j}(g_j) f_j$ are continuous for all $f_j \in C_b(X)$. Since *G* is first countable, continuity at a point $g_j \in G$ is the same as sequential continuity at g_j. If $(g_j)_n \to g_j$ as $n \to \infty$, then $(g_j)_n f_j \to g_j f_j$ point wise as $n \to \infty$, by the continuity assumption on the *G*-action and the continuity of f_j. The dominated convergence theorem then yields that

$$\sum_j \left\|\rho_{\sum_j \mu^j}\left((g_j)_n\right) f_j - \rho_{\sum_j \mu^j}(g_j) f_j\right\|_{(1+\epsilon)} \to 0 \text{ as } n \to \infty.$$

Corollary 2.8. Let (G, X) be a Polish topological dynamical system, and suppose that $\sum_j \mu^j$ are *G*-invariant Borel probability measure on *X*. Then, for each $0 \le \epsilon < \infty$, the canonical representation $\rho_{\sum_j \mu^j}$ of *G* as isometric lattice automorphisms of $L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ are strongly continuous.

Remark 2.9. Every Borel probability measure on a Polish space is regular; (see e.g. [1]). However, since local compactness of *G* and *X* are not assumed in Corollary 2.8, Corollary 2.5 is still not applicable here.

3. $L^{(1+\epsilon)}$-Direct Integrals of Banach Spaces and Representations

This section provides the framework for the disintegration Theorems 4.9 and 5.15. Start by defining $L^{(1+\epsilon)}$-direct integrals of Banach spaces and Banach lattices in the spirit of [[13], Sections 6.1 and 7.2]. The idea roughly, to begin with a "core" vector space *V* that is supplied with a family of norms $\sum_j \|\cdot\|_{\omega^j}$, depending on the points ω^j of a measure space $\left(\Omega, \nu^j\right)$. If $\left\{(B_j)_{\omega^j}\right\}_{\omega^j \in \Omega}$ are the corresponding family of Banach space completions of *V*, then one can consider sections from to $\coprod_{\omega^j \in \Omega} (B_j)_{\omega^j}$. There is a natural notion of measurable section s^j, and the $\sum_j (B_j)_{\omega^j}$ are "glued together" by restricting attention to measurable sections and identifying measurable sections that are ν^j-almost everywhere equal. For any Kothespace *E* associated with $\left(\Omega, \nu^j\right)$, one can then require, for a measurable section s^j, that the functions $\sum_j \omega^j \mapsto \sum_j \left\|s^j\left(\omega^j\right)\right\|$

be in E. If E satisfies appropriate additional properties, then the equivalence classes of such sections form a Banach space, which is called the E-direct integral of the families $\left\{\left(B_j\right)_{\omega^j}\right\}_{\omega^j \in \Omega}$.

In Sect.3.1, this program is carried the for $E = L^{(1+\epsilon)}\left(\Omega, \nu^j\right), \left(0 \le \epsilon < \infty\right)$, but with two noticeable modifications as compared to [13]. The first is that the family of norms figuring is replaced with a family of semi-norms. The need for this comes up quite naturally in the context, and it seems to the authors that this may also be the case elsewhere. The second difference is that the measure ν^j need not be complete. Completeness of measures is the standing assumption, but the measure apply the formalism need not be complete One has to be extra cautious then, and particularly in a vector-valued context; the proof of Proposition 3.2 may serve as evidence for this. As a consequence of this choice of presentation, we are also able to give a precise discussion of the relation with the Bochner integral and with the usual theory of direct integrals of separable Hilbert spaces and of decomposable sequence of operators, proving that these are particular cases of the general theory.

In Sect. 3.2, define decomposable sequence of operators and the $L^{(1+\epsilon)}$-direct integral of a decomposable family of representations of a group G, which is a representation of G on the $L^{(1+\epsilon)}$-direct integral of Banach spaces from Sect.3.1. One way to obtain such a decomposable family of representations is when it originates from one common "core" representation $\tilde{\rho}$ of G on the 'core' vector space V. Even though it is all fairly natural, not aware of previous similar work in the context of (dis)integrating representations.

As shown in Sect.3.3, the framework in Sect. 3.2 includes the usual theory of direct integrals of separable Hilbert spaces.

Finally, in Section 3.4, sketch a perspective that a more or less obvious extension of the formalism could have in representation theory.

3.1. $L^{(1+\epsilon)}$-Direct Integrals of Banach Spaces

Define $L^{(1+\epsilon)}$-direct integrals of a suitable family of Banach spaces. These are Banach spaces that generalize the Bochner $L^{(1+\epsilon)}$-spaces and the direct integrals of separable Hilbert spaces. Let $\left(\Omega, \nu^j\right)$ be a measure space, and let V be a vector space. For clarity, let us recall that the measures need not be finite (or even σ^j-finite) or complete. Say that a collection $\left\{\|\cdot\|_{\omega^j}\right\}_{\omega^j \in \Omega}$ are measurable family of semi-norms on V if $\sum_j \|\cdot\|_{\omega^j}$ is a semi-norm on V for all $\omega^j \in \Omega$, and $\omega^j \mapsto \|x\|_{\omega^j}$ are measurable function on Ω for all $x \in V$. For later use, let us record that this is the same as requiring that the (identical) functions $\sum_j \omega^j \mapsto \sum_j \|x\|_{\omega^j}$ are measurable function on for all $x \in V$, where $\sum_j \left\|[x]_{\omega^j}\right\|_{\omega^j}$ are the value of the induced norm $\sum_j \|\cdot\|_{\omega^j}$ on $V / \ker\left(\|\cdot\|_{\omega^j}\right)$ at the equivalence class $\sum_j [x]_{\omega^j}$ of x in $V / \ker\left(\sum_j \|\cdot\|_{\omega^j}\right)$.

Let $\left\{\left(B_j\right)_{\omega^j}\right\}_{\omega^j \in \Omega}$ be collections of Banach spaces and suppose that

$\left\{\left(B_j\right)_{\omega^j}\right\}_{\omega^j\in\Omega}$ are measurable family of semi-norms on V such that, for all $\omega^j\in\Omega$, $\left(B_j\right)_{\omega^j}$ are the Banach space completion of $V/\ker\left(\|\cdot\|_{\omega^j}\right)$ with respect to the induced norm $\sum_j\|\cdot\|_{\omega^j}$ on $V/\ker\left(\|\cdot\|_{\omega^j}\right)$. Then say that $\left\{\left(B_j\right)_{\omega^j}\right\}_{\omega^j\in\Omega}$ are measurable family of Banach spaces over $\left(\Omega,\nu^j,V\right)$. For conciseness, usually do not explicitly mention the specific measurable family of semi- series norms $\left\{\left(B_j\right)_{\omega^j}\right\}_{\omega^j\in\Omega}$ on V that gives rise to $\left\{\left(B_j\right)_{\omega^j}\right\}_{\omega^j\in\Omega}$, as this family will generally be clear from the context.

Analogously, suppose that V is a vector lattice and that $\left\{\left(B_j\right)_{\omega^j}\right\}_{\omega^j\in\Omega}$ are measurable family of lattice semi-norms on V such that, for all $\omega^j\in\Omega$, $\left(B_j\right)_{\omega^j}$ are the Banachlattice completion of $V/\ker\left(\|\cdot\|_{\omega^j}\right)$ with respect to the induced lattice series norm $\|\cdot\|_{\omega^j}$ and the induced ordering on $V/\ker\left(\|\cdot\|_{\omega^j}\right)$. Then say that a families $\left\{\left(B_j\right)_{\omega^j}\right\}_{\omega^j\in\Omega}$ of Banach lattices is a measurable family of Banach lattices over $\left(\Omega,\nu^j,V\right)$. When using this terminology, tacitly assume that V is a vector lattice, and that the $\|\cdot\|_{\omega^j}$ are lattice semi-norms.

Let $\left\{\left(B_j\right)_{\omega^j}\right\}_{\omega^j\in\Omega}$ be a measurable family of Banach spaces over $\left(\Omega,\nu^j,V\right)$. Say that a map $S:\to\coprod_{\omega^j\in\Omega}\left(B_j\right)_{\omega^j}$ is a section of $\left\{\left(B_j\right)_{\omega^j}\right\}_{\omega^j\in\Omega}$ if $\sum_j s^j\left(\omega^j\right)\in\left(B_j\right)_{\omega^j}$ for all $\omega^j\in\Omega$.

A simple section is a section s^j for which there exist $n\in N$, $x_1,\cdots,x_n\in V$, and measurable subsets $\left(A_j\right)_1,\cdots,\left(A_j\right)_n$ of Ω such that $s^j\left(\omega^j\right)=\left[\sum_{k=1}^n\mathbf{1}_k\left(\omega^j\right)x_k\right]_{\omega^j}$ for all $\omega^j\in\Omega$. Choosing the $\left(A_j\right)_k$ disjoint, have $\sum_j\left\|s^j\left(\omega^j\right)\right\|_{\omega^j}=\sum_{k=1}^n\sum_j\mathbf{1}_{\left(A_j\right)_k}\left(\omega^j\right)\left\|\left[x_k\right]_{\omega^j}\right\|_{\omega^j}$, so that the function $\omega^j\mapsto\left\|s^j\left(\omega^j\right)\right\|_{\omega^j}$ on Ω is measurable for all simple section s^j.

A sectionsof $\left\{\left(B_j\right)_{\omega^j}\right\}_{\omega^j\in\Omega}$ is said to be measurable if there exists a sequence $\left(s_k^j\right)_{k=1}^\infty$ of simple sections such that, for all $\omega^j\in\Omega$, $s_k^j\left(\omega^j\right)\to s^j\left(\omega^j\right)$ in $\left(B_j\right)_{\omega^j}$ as $k\to\infty$. Then clearly $\sum_j\left\|s^j=\left(s^j\right)_k\left(\omega^j\right)\right\|_{\omega^j}\to\sum_j\left\|s^j\left(\omega^j\right)\right\|_{\omega^j}$ for all $\omega^j\in\Omega$ as $k\to\infty$, and hence, as a consequence of the measurability of the functions $\sum_j\omega^j\mapsto\sum_j\left\|\left(s^j\right)_k\left(\omega^j\right)\right\|_{\omega^j}$ on, the functions $\sum_j\omega^j\mapsto\sum_j\left\|s^j\left(\omega^j\right)\right\|_{\omega^j}$ are measurable function on Ω for all measurable section s^j. The measurable sections form a vector space, and denote the section that maps every $\omega^j\in\Omega$ to the zero element of $\left(B_j\right)_{\omega^j}$ simply by 0. Also note that, if A_j are measurable subset of Ω and s^j is a simple section, then $\mathbf{1}_{A_j}s$ is again a simple section. It follows easily from this that the measurable sections area module over the measurable functions on Ω under point wise operations.

Define the direct integral $\sum_j\int_\Omega^\oplus\left(B_j\right)_{\omega^j}\mathrm{d}\nu^j\left(\omega^j\right)$ of $\left\{\left(B_j\right)_{\omega^j}\right\}_{\omega^j\in\Omega}$ with respect to ν^j to be the space of all equivalence classes $\left[s^j\right]\nu^j$ of measurable sections s^j of $\left\{\left(B_j\right)_{\omega^j}\right\}_{\omega^j\in\Omega}$, where two measurable sections are equivalent if they agree ν^j-almost everywhere on. Say that the $\sum_j (B_j)_{\omega^j}$ are the fibers of $\sum_j \int_\Omega^\oplus (B_j)_{\omega^j} \,\mathrm{d}\nu^j(\omega^j)$, and introduce a vector space structure on $\int_\Omega^\oplus \sum_j (B_j)_{\omega^j} \mathrm{d}\nu^j(\omega^j)$ in the usual representative-independent way.

If $\{(B_j)_{\omega^j}\}_{\omega^j\in\Omega}$ are measurable family of Banach lattices over (Ω,ν^j,V), then, in addition, can meaningfully define a natural partial ordering on $\int_\Omega^\oplus \sum_j (B_j)_{\omega^j} \mathrm{d}\nu^j(\omega^j)$ by $\sum_j [s^j]_{\nu^j} \geq \sum_j [(1+\epsilon)]_{\nu^j} \Leftrightarrow s^j(\omega^j) \geq (1+\epsilon)(\omega^j)$ for ν^j-almost all $\omega^j \in \Omega$ for $\sum_j [s^j]_{\nu^j}, \sum_j [(1+\epsilon)]_{\nu^j} \in \int_\Omega^\oplus \sum_j (B_j)_{\omega^j} \mathrm{d}\nu^j(\omega^j)$. Then $\int_\Omega^\oplus \sum_j (B_j)_{\omega^j} \mathrm{d}\nu^j(\omega^j)$ are ordered vector space. In fact, it is a vector lattice. For the latter statement, note that the point wise supermom and infimum of two measurable sections are measurable again, as a consequence of the continuity of the lattice operations in all $(B_j)_{\omega^j}$ and the fact that the point wise supremum and infimum of two simple sections are simple sections again. It is then easily verified that, for $\sum_j [s^j]_{\nu^j}, \sum_j [(1+\epsilon)]_{\nu^j} \in \int_\Omega^\oplus \sum_j (B_j)_{\omega^j} \mathrm{d}\nu^j(\omega^j)$, $\sum_j [s^j]_{\nu^j} \vee \sum_j [(1+\epsilon)]_{\nu^j}$ exists in $\int_\Omega^\oplus \sum_j (B_j)_{\omega^j} \mathrm{d}\nu^j(\omega^j)$, and that, in fact, $\sum_j [s^j]_{\nu^j} \vee \sum_j [(1+\epsilon)]_{\nu^j} = \sum_j [s^j \vee (1+\epsilon)]_{\nu^j}$, where $\sum_j [s^j \vee (1+\epsilon)]_{\nu^j} \in \int_\Omega^\oplus \sum_j (B_j)_{\omega^j} \mathrm{d}\nu^j(\omega^j)$ are defined by $\sum_j (s^j \vee (1+\epsilon))(\omega^j) := \sum_j s^j(\omega^j) \vee (1+\epsilon)(\omega^j)\,(\omega^j \in \Omega)$. The expression for the infimum is similar.

For $(0 \leq \epsilon < \infty)$, let the $L^{(1+\epsilon)}$-direct integral $\left(\int_\Omega^\oplus \sum_j \mathrm{d}\nu^j(\omega^j)\right)_{L^{(1+\epsilon)}}$ of $\{(B_j)_{\omega^j}\}_{\omega^j\in\Omega}$ with respect to ν^j be the subset of $\int_\Omega^\oplus \sum_j (B_j)_{\omega^j} \mathrm{d}\nu^j(\omega^j)$ consisting of those $\sum_j [s^j] \in \int_\Omega^\oplus \sum_j (B_j)_{\omega^j} \mathrm{d}\nu^j(\omega^j)$ such that the functions $\sum_j \omega^j \mapsto \left\|\sum_j s^j(\omega^j)\right\|_{\omega^j}$, which know to be measurable, is in $\mathcal{L}^{(1+\epsilon)}(\Omega,\nu^j)$.

This criterion is evidently independent of the particular representatives of $\sum_j [s^j]\nu^j$, and call $\sum_j [s^j]_{\nu^j}$ and its representatives $(1+\epsilon)$-integrable (with respect to ν^j). It follows from the triangle inequality for all $\sum_j \|\cdot\|_{\omega^j}$ that $\left(\int_\Omega^\oplus \sum_j (B_j)_{\omega^j} \mathrm{d}\nu^j(\omega^j)\right)_{L^{(1+\epsilon)}}$ are subspace of $\int_\Omega^\oplus \sum_j (B_j)_{\omega^j} \mathrm{d}\nu^j(\omega^j)$ and that

$$\sum_j \left\|[s^j]_{\nu^j}\right\|_{(1+\epsilon)} := \left(\int_\Omega \sum_j \left\|s^j(\omega^j)\right\|_{\omega^j}^{(1+\epsilon)} \mathrm{d}\nu^j\right)^{\frac{1}{(1+\epsilon)}} \left([s^j]_{\nu^j} \in \left(\int_\Omega^\oplus \sum_j (B_j)_{\omega^j} \mathrm{d}\nu^j(\omega^j)\right)_{L^{(1+\epsilon)}}\right) \tag{1}$$

defines series norms $\sum_j [s^j] \mapsto \sum_j \left\|[s^j]_{\nu^j}\right\|_{(1+\epsilon)}$ on $\left(\int_\Omega^\oplus \sum_j (B_j)_{\omega^j} \mathrm{d}\nu^j(\omega^j)\right)_{L^{(1+\epsilon)}}$.

If V is a vector lattice and $\{(B_j)_{\omega^j}\}_{\omega^j\in\Omega}$ is measurable family of Banach lattices over (Ω,ν^j,V), then it is easily verified that $\left(\int_\Omega^\oplus \sum_j (B_j)_{\omega^j} \mathrm{d}\nu^j(\omega^j)\right)_{L^{(1+\epsilon)}}$ are vector sub lattice of $\int_\Omega^\oplus \sum_j (B_j)_{\omega^j} \mathrm{d}\nu^j(\omega^j)$, and that (1) supplies it with a lattice norm.

The $L^{(1+\epsilon)}$-direct integrals of Banach spaces, as defined above, are, in fact, Banach spaces. To show this, use that the equivalence classes of the $(1+\epsilon)$-integrable simple sections are dense. This density, which is also a key ingredient of the proof of the disintegration Theorem 4.9, is established in the following stronger result, based on a familiar truncation argument as in e.g. [[17], proof of Proposition 2.16].

Lemma 3.1. Let (Ω,ν^j) be a measure space, let V be a vector space, and let $(0 \le \epsilon < \infty)$. Let $\{(B_j)_{\omega^j}\}_{\omega^j \in \Omega}$ be measurable families of Banach spaces over (Ω,ν^j,V), and let $\sum_j [s^j]_{\nu^j} \in \left(\int_\Omega^\oplus \sum_j (B_j)_{\omega^j} \mathrm{d}\nu^j(\omega^j)\right)_{L^{(1+\epsilon)}}$. Then, for all $\epsilon > 0$, there exists a sequence $\left((s^j)_k\right)_{k=1}^\infty$ of $(1+\epsilon)$-integrable simple sections such that $\sum_j \left\|(s^j)_k(\omega^j)\right\|_{\omega^j} \le (1+\epsilon)\left\|\sum_j s^j(\omega^j)\right\|_{\omega^j}$ for all $k \in \mathbb{N}$ and $\omega^j \in \Omega$, $(s^j)_k(\omega^j) \to s^j(\omega^j)$ in $(B_j)_{\omega^j}$ as $k \to \infty$ for all $\omega^j \in \Omega$, and $\sum_j \left\|[s^j]_{\nu^j} - [(s^j)_k]_{\nu^j}\right\|_{(1+\epsilon)} \to 0$ as $k \to \infty$.

If $\{(B_j)_{\omega^j}\}_{\omega^j \in \Omega}$ are measurable family of Banach lattices over (Ω,ν^j,V) and $\sum_j [s^j]_{\nu^j} \ge 0$, then the sequence $\sum_j \left((s^j)_k\right)_{k=1}^\infty$ can be chosen such that, in addition, $\sum_j \left[(s^j)_k\right]_{\nu^j} \ge 0$ for all $k \in N$.

Proof. Let $\sum_j [s^j]_{\nu^j} \in \left(\int_\Omega^\oplus \sum_j (B_j)_{\omega^j} \mathrm{d}\nu^j(\omega^j)\right)_{L^{(1+\epsilon)}}$. Then there exists a sequence $\left((\acute{s}^j)_k\right)_{k=1}^\infty$ of simple sections such that, for all $\omega^j \in \Omega$, $(\acute{s}^j)_k(\omega^j) \to s^j(\omega^j)$ in $(B_j)_{\omega^j}$ as $k \to \infty$. For $k \in \mathbb{N}$, let $\sum_j (A_j)_k := \left\{\omega^j \in \Omega : \sum_j \left\|(\acute{s}^j)_k(\omega^j)\right\|_{\omega^j} \le (1+\epsilon)\left\|\sum_j s^j(\omega^j)\right\|_{\omega^j}\right\}$. Then $(A_j)_k$ are measurable subset of, hence the sections $(s^j)_k$, defined by $\sum_j (s^j)_k := \sum_j \mathbf{1}_{(A_j)_k}(\acute{s}^j)_k$, is simple again. Furthermore, $\sum_j \left\|(s^j)_k(\omega^j)\right\|_{\omega^j} \le (1+\epsilon)\left\|\sum_j s^j(\omega^j)\right\|_{\omega^j}$ for all k and all $\omega^j \in \Omega$, so that each $(s^j)_k$ is $(1+\epsilon)$-integrable with respect to ν^j. For all $\omega^j \in \Omega$, $\sum_j (s^j)_k(\omega^j) \to \sum_j s^j(\omega^j)$ in $(B_j)_{\omega^j}$ as $k \to \infty$. It then follows from the dominated convergence theorem that $\sum_j \left[(s^j)_k\right]_{\nu^j} \to \sum_j [s^j]_{\nu^j}$ in $\left(\int_\Omega^\oplus \sum_j (B_j)_{\omega^j} \mathrm{d}\nu^j(\omega^j)\right)_{L^{(1+\epsilon)}}$.

For the second statement, suppose that $\sum_j [s^j]_{\nu^j} \ge 0$, and let $\epsilon > 0$. Choose a sequence $\left((\acute{s}^j)_k\right)_{k=1}^\infty$ of simple sections with the three properties in the first part of the lemma.

There exists a measurable subsets A_j of Ω such that $\sum_j \nu^j(A_j)=0$ and $\sum_j s^j(\omega^j)\geq 0$ for all $\omega^j \in A_j^c$. Then the sequences $\left(\left(s^j\right)_k\right)_{k=1}^{\infty}$, given by $\sum_j \left(s^j\right)_k\left(\omega^j\right) := \sum_j \mathbf{1}_{A_j^c}\left(\omega^j\right)\left(\acute{s}^j\right)_k\left(\omega^j\right)^+ + \sum_j \mathbf{1}_{A_j}\left(\omega^j\right)\left(\acute{s}^j\right)_k\left(\omega^j\right)$ for $k \in N$ and $\omega^j \in \Omega$, is as desired.

Establish the completeness of $L^{(1+\epsilon)}$-direct integrals of Banach spaces.

Proposition 3.2. Let $\left(\Omega,\nu^j\right)$ be a measure space, let V be a vector space, and let $\left(0\leq\epsilon<\infty\right)$. If $\left\{\left(B_j\right)_{\omega^j}\right\}_{\omega^j\in\Omega}$ are measurable family of Banach spaces over $\left(\Omega,\nu^j,V\right)$, then $\left(\int_{\Omega}^{\oplus}\sum_j\left(B_j\right)_{\omega^j}\mathrm{d}\nu^j\left(\omega^j\right)\right)_{L^{(1+\epsilon)}}$ is a Banach space. If $\left\{\left(B_j\right)_{\omega^j}\right\}_{\omega^j\in\Omega}$ are measurable family of Banach lattices over $\left(\Omega,\nu^j,V\right)$, then $\left(\int_{\Omega}^{\oplus}\sum_j\left(B_j\right)_{\omega^j}\mathrm{d}\nu^j\left(\omega^j\right)\right)_{L^{(1+\epsilon)}}$ are Banach lattice.

Proof. Let $\left(\sum_j\left[\left(s^j\right)_k\right]_{\nu^j}\right)_{k=1}^{\infty}$ be a sequence in $\left(\int_{\Omega}^{\oplus}\sum_j\left(B_j\right)_{\omega^j}\mathrm{d}\nu^j\left(\omega^j\right)\right)_{L^{(1+\epsilon)}}$ such that $\sum_{k=1}^{\infty}\left\|\sum_j\left[\left(s^j\right)_k\right]_{\nu^j}\right\|_{(1+\epsilon)}<\infty$, one shows that there exists measurable subsets A_j of such that $\sum_j \nu^j\left(A_j^c\right)=0$ and $\sum_j s^j\left(\omega^j\right) := \lim_{n\to\infty}\sum_{k=1}^{n}\sum_j \mathbf{1}_{A_j}\left(\omega^j\right)\left(s^j\right)_k\left(\omega^j\right)$ exists for all $\omega^j\in\Omega$. If one knew s^j to be a measurable section, then the conclusion of the standard proof would show that the series $\sum_{k=1}^{\infty}\sum_j\left[\left(s^j\right)_k\right]_{\nu^j}$ converges to s^j. The point wise limit of a sequence of scalar-valued measurable functions is measurable, and, more generally in the context of the Bochner integral, the limit of a sequence of strongly measurable functions is strongly measurable in [[18], Theorem E.9]. In the context, however, have no such result. Fortunately, the following easily verified fact saves the day: If X is a normed space and Y is a dense subspace with the property that every absolutely convergent series with terms from Y converges in X, then X is a Banach space. With this and Lemma 3.1 in mind, see that it is sufficient to prove convergence of the series when the $\left(s^j\right)_k$ are simple sections. In that case, s^j is the pointwise limit of simple sections, hence is measurable by definition.

Remark 3.3. (1) If V is a Banach space with norm $\|\cdot\|$, and if take $\sum_j\|\cdot\|_{\omega^j}=\|\cdot\|$ for all $\omega^j\in\Omega$, then all $\sum_j\left(B_j\right)_{\omega^j}$ equal V. Claim that, in this case, $\left(\int_{\Omega}^{\oplus}\sum_j\left(B_j\right)_{\omega^j}\mathrm{d}\nu^j\left(\omega^j\right)\right)_{L^{(1+\epsilon)}}$ are the Bochner space $L^{(1+\epsilon)}\left(\Omega,V,\nu^j\right)$ as it is defined for an arbitrary measure are defined, starting from the strongly measurable functions, in the same canonical fashion as $\left(\int_{\Omega}^{\oplus}\sum_j\left(B_j\right)_{\omega^j}\mathrm{d}\nu^j\right)_{L^{(1+\epsilon)}}$ are defined, starting from the measurable sections, both spaces coincide.

(2) Although it is usually not observed as such, the direct integrals of separable Hilbert spaces as they are defined in the literature are Bochner L^2-spaces.

3.2. Decomposable Operators and $L^{(1+\epsilon)}$-Direct Integrals of Representations

Define decomposable sequence of operators, and, subsequently, a decomposable representation of a group on an $L^{(1+\epsilon)}$-direct integral of a measurable family of Banach spaces, that can be called the $L^{(1+\epsilon)}$-direct integral of the fiber wise representations.

Both are a natural generalization of the corresponding notion in the context of the usual direct integral of separable Hilbert spaces.

Let (Ω,ν^j) be a measure space, let V be a vector space, and let $\left\{(B_j)_{\omega^j}\right\}_{\omega^j\in\Omega}$ be a measurable family of Banach spaces over (Ω,ν^j,V), originating from the measurable family of semi-series norms $\left\{\|\cdot\|_{\omega^j}\right\}_{\omega^j\in\Omega}$ on V. A decomposable the sequence of operators T^j on $\left\{(B_j)_{\omega^j}\right\}_{\omega^j\in\Omega}$ are maps $\sum_j \omega^j \mapsto \sum_j (T^j)_{\omega^j} \in \mathcal{L}\left((B_j)_{\omega^j}\right)(\omega^j\in\Omega)$ such that, for each measurable section s^j, the section $T^j s$, defined by $\sum_j\left((T^j)s^j\right)(\omega^j) := \sum_j (T^j)_{\omega^j}\left(s^j(\omega^j)\right)$, are measurable again, and such that the possibly non-measurable, functions $\sum_j \omega^j \to \sum_j \left\|(T^j)_\omega\right\|_{\omega^j}$ are ν^j-essentially bounded. Then, for $0\le\epsilon<\infty$, T^j induces a bounded sequence of operators $\sum_j (T^j)_{(1+\epsilon)}$ (also denoted by $\left(\int_\Omega^\oplus \sum_j (T^j)_{\omega^j} \mathrm{d}\nu^j(\omega^j)\right)_{L^{(1+\epsilon)}}$) on $\left(\int_\Omega^\oplus \sum_j (B_j)_{\omega^j} \mathrm{d}\nu^j(\omega^j)\right)_{L^{(1+\epsilon)}}$: for $\sum_j [s^j]_{\nu^j} \in \left(\int_\Omega^\oplus \sum_j (B_j)_{\omega^j} \mathrm{d}\nu^j(\omega^j)\right)_{L^{(1+\epsilon)}}$, let $\sum_j (T^j)_{(1+\epsilon)}[s^j]_{\nu^j} := \sum_j [T^j s^j]_{\nu^j}$.

If the $(B_j)_{\omega^j}$ are Banach lattices and ν^j-almost all $T^j\omega^j$ are positive sequence of operators, then $\sum_j (T^j)_{(1+\epsilon)}$ are appositive sequence of operators. If ν^j-almost all $\sum_j (T^j)_{\omega^j}$ are lattice homomorphisms, then $\sum_j (T^j)_{(1+\epsilon)}$ are a lattice homomorphism.

Let G be an abstract group. A decomposable representation ρ of G on $\left\{(B_j)_{\omega^j}\right\}_{\omega^j\in\Omega}$ are family $\left\{\rho_{\omega^j}\right\}_{\omega^j\in\Omega}$, where ρ_{ω^j} are representation of G on $(B_j)_{\omega^j}(\omega^j\in\Omega)$, such that, for all $g_j\in G$, the map $\sum_j \omega^j \to \sum_j \rho_{\omega^j}(g_j)$ are decomposable the sequence of operators on $\left\{(B_j)\right\}_{\omega^j\in\Omega}$; denote this decomposable the sequence of operators by $\rho(g_j)$. Then, for $(0\le\epsilon<\infty)$, the map ρ induces representations $\rho_{(1+\epsilon)}$ of G as bounded the sequence of operators on $\left(\int_\Omega^\oplus \sum_j (B_j)_{\omega^j} \mathrm{d}\nu^j(\omega^j)\right)_{L^{(1+\epsilon)}}$, defined by

$$\sum_j \rho_{(1+\epsilon)}(g_j) = \sum_j \left(\rho(g_j)\right)(1+\epsilon) = \left(\int_\Omega^\oplus \sum_j \rho_{\omega^j}(g_j) \mathrm{d}\nu^j(\omega^j)\right)_{L^{(1+\epsilon)}}$$

$(g_j\in G)$. If the $(B_j)_{\omega^j}$ is Banach lattices and ν^j-almost all ρ_{ω^j} are positive representations, then $\rho_{(1+\epsilon)}$ is a positive representation.

Call $\rho_{(1+\epsilon)}$ the $L^{(1+\epsilon)}$-direct integral of the representations $\left\{\rho_{\omega^j}\right\}_{\omega^j\in\Omega}$ with respect to ν^j, and write $\rho_{(1+\epsilon)} = \left(\int_\Omega \sum_j \rho_{\omega^j} \mathrm{d}\nu^j(\omega^j)\right)_{L^{(1+\epsilon)}}$.

If G is a topological group, it is easy to write down various conditions for the decomposable representations $\left\{\rho_{\omega^j}\right\}_{\omega^j\in\Omega}$ of G on $\left\{(B_j)_{\omega^j}\right\}_{\omega^j\in\Omega}$ that are sufficient to ensure the strong continuity of $\rho_{(1+\epsilon)}$, together with that of all

$\rho_{\omega^j}\left(\omega^j \in \Omega\right)$. A crude and $(1+\epsilon)$ independent one is e.g. that there exists a constant M such that $\left\|\sum_j \rho_{\omega^j}\left(g_j\right)\right\|_{\omega^j} \leq M$ for all $g_j \in G$ and $\omega^j \in \Omega$, and that, for each $x \in V$ and $\epsilon > 0$, there exists a neighbourhood $U_{x,\epsilon}$ of e in G such that $\left\|\sum_j \rho_{\omega^j}\left(g_j\right)[x]_{\omega^j} - [x]_{\omega^j}\right\| < \epsilon$ for all $g_j \in U_{x,\epsilon}$ and $\omega^j \in \Omega$. Indeed, for all $\omega^j \in \Omega$, this certainly implies that, for all $x \in V$, the map $\sum_j g_j \to \sum_j \rho_{\omega^j}\left(g_j\right)[x]_{\omega^j}$ are continuous at *e*. By density, the uniform boundedness of the $\rho_{\omega^j}\left(g_j\right)$ then implies that, for all $b \in \left(B_j\right)_{\omega^j}$, the maps $\sum_j g_j \to \sum_j \rho_{\omega^j}\left(g_j\right) b_{\omega^j}$ are continuous at *e*; consequently, this is true at all points of *G*. Hence all ρ_{ω^j} are strongly continuous.

The condition also implies that, for all $(1+\epsilon)$-integrable simple section, the maps $\sum_j g_j \mapsto \sum_j \rho_{g_j}\left(g_j\right)\left[s^j\right]$ are continuous at *e*. By the density statement in Lemma 3.1, the uniform boundedness of the $\rho_{(1+\epsilon)}\left(g_j\right)$ then implies that $\rho_{(1+\epsilon)}$ is strongly continuous.

There is a natural way to obtain decomposable the sequence of operators on $\left\{\left(B_j\right)_{\omega^j}\right\}_{\omega^j \in \Omega}$ and, consequently, bounded operators on $\left(\int_\Omega^\oplus \sum_j \left(B_j\right)_{\omega^j} \mathrm{d}\nu^j\left(\omega^j\right)\right)_{L^{(1+\epsilon)}}$ for $0 \leq \epsilon < \infty$ from one suitable linear map on the "core" space *V*, as follows. Suppose that $\tilde{T}^j$ are linear map on the abstract vector space *V* with the property that there exist constants $M_{\omega^j}\left(\omega^j \in \Omega\right)$ and M such $\sum_j \left\|\tilde{T}^j x\right\|_{\omega^j} \leq \sum_j M_{\omega^j} \|x\|_{\omega^j}\left(x \in V, \omega^j \in \Omega\right)$ and $M_{\omega^j} \leq M$ for ν^j-almost all ω^j. Then, for all $\omega^j \in \Omega$, $\ker \sum_j \|\cdot\|_{\omega^j}$ are T^j-invariant, hence there exists a linear map on $V/\ker\|\cdot\|_{\omega^j}$, denoted by $\left(T^j\right)_{\omega^j}$, and given by $\sum_j \left(T^j\right)_{\omega^j}[x]_{\omega^j} = \sum_j \left[\tilde{T}^j x\right]_{\omega^j}(x \in V)$. Then $\left\|\left(T^j\right)_{\omega^j}[x]\right\|_{\omega^j} \leq M_{\omega^j}\left\|[x]_{\omega^j}\right\|_{\omega^j}$ for all $[x]_{\omega^j} \in V/\ker\|\cdot\|_{\omega^j}$. This sequence of operators extends to a bounded operator on $\left(B_j\right)_{\omega^j}$, still denoted by $\left(T^j\right)_{\omega^j}$, and then $\left\|\sum_j \left(T^j\right)_{\omega^j}\right\|_{\omega^j} \leq M$ for ν^j-almost all ω^j. The point is that the family $\left\{\left(T^j\right)_{\omega^j}\right\}\left(\omega^j \in \Omega\right)$ automatically leaves the space $\int_\Omega^\oplus \sum_j \left(B_j\right)_{\omega^j} \mathrm{d}\nu^j\left(\omega^j\right)$ of measurable section invariant, so that it defines a decomposable the sequence of operators T^j on $\left\{\left(B_j\right)_{\omega^j}\right\}_{\omega^j \in \Omega}$. To see this, first note that, if $s^j \in \int_\Omega^\oplus \sum_j \left(B_j\right)_{\omega^j} \mathrm{d}\nu^j\left(\omega^j\right)$ are simple section, say $\sum_j s^j\left(\omega^j\right) = \left[\sum_{k=1}^n \sum_j 1_{\left(A_j\right)_k}\left(\omega^j\right) x_k\right]_{\omega^j}\left(\omega^j \in \Omega\right)$ for some $n \in \mathbb{N}$, $x_1, \cdots, x_n \in V$, and measurable subsets $\left(A_j\right)_1, \cdots, \left(A_j\right)_n$ of Ω, then $\sum_j \left(T^j s^j\right)\left(\omega^j\right) = \sum_j \left(T^j\right)_{\omega^j}\left[\sum_{k=1}^n 1_{A_k}\left(\omega^j\right) x_k\right]_{\omega^j} = \left[\sum_{k=1}^n \sum_j 1_{A_k}\left(\omega^j\right) \tilde{T}^j x_k\right]_{\omega^j}$. Hence T^j are simple section again. If s^j are measurable section, say $s^j\left(\omega^j\right) = \lim_{n\to\infty}\left(s^j\right)_n\left(\omega^j\right)\left(\omega^j \in \Omega\right)$ for simple sections s_n, then, as a consequence of the continuity of the $\left(T^j\right)_{\omega^j}$ on $\left(B_j\right)_{\omega^j}$, see that $\sum_j \left(T^j s^j\right)\left(\omega^j\right) = \sum_j \left(T^j\right)_{\omega^j}\left(s^j\left(\omega^j\right)\right) = \lim_{n\to\infty}\left(T^j\right)_{\omega^j}\left(\left(s^j\right)_n\left(\omega^j\right)\right)$. Hence $T^j = \lim_{n\to\infty}\left(T^j\left(s^j\right)_n\right)\left(\omega^j\right)\left(\omega^j \in \Omega\right)$

are measurable section again if s^j are, as desired, and the families $\left\{\left(T^j\right)_{\omega^j}\right\}_{\omega^j\in\Omega}$ are Decomposable the sequence of operators. Conclude that, for $0\le\epsilon<\infty$, this "core" linear maps $\tilde{T}^j$ gives rise to bounded the sequence of operators $\left(T^j\right)_{(1+\epsilon)}$ on $\int_\Omega^\oplus\sum_j\left(B_j\right)_{\omega^j}\mathrm{d}\nu^j\left(\omega^j\right)$ such that $\left\|\sum_j\left(T^j\right)_{(1+\epsilon)}\right\|\le M$.

If the $\left(B_j\right)_{\omega^j}$ is Banach lattices, and $\tilde{T}^j$ are positive operator on *V*, then all $\left(T^j\right)_{\omega^j}$ and $\left(T^j\right)_{(1+\epsilon)}$ are positive the sequence of operators. If $\tilde{T}^j$ are lattice homomorphism, then all $\left(T^j\right)_{\omega^j}$ and $\left(T^j\right)_{(1+\epsilon)}$ are lattice homomorphisms.

Consequently, there is also a natural way to obtain a decomposable representation of a group G from one "core" representation $\tilde{\rho}$ of G on *V*. Say that $\tilde{\rho}$ is point wise essentially bounded if, for all $g_j\in G$, there exist constants $M_{\omega^j,g_j}\left(\omega^j\in\Omega\right)$ and M_{g_j} such that $\left\|\sum_j\tilde{\rho}\left(g_j\right)x\right\|_{\omega^j}\le\sum_j M_{\omega^j,g_j}\|x\|_{\omega^j}$ for all $x\in V$ and $\omega^j\in\Omega$, and $M_{\omega^j,g_j}\le M_{g_j}$ for ν^j-almost all ω^j. It is immediate from the above, applied to all $\tilde{\rho}\left(g_j\right)\left(g_j\in G\right)$, that there are families $\left\{\rho_{\omega^j}\right\}_{\omega^j\in\Omega}$ of representations of G as bounded the sequence of operators on the spaces $\left(B_j\right)_{\omega^j}$ that constitutes a decomposable representation ρ of *G*; these are determined by $\sum_j\rho_{\omega^j}\left(g_j\right)[x]_{\omega^j}=\sum_j\left[\tilde{\rho}\left(g_j\right)x\right]_{\omega^j}\left(g_j\in G,x\in V,\omega^j\in\Omega\right)$. Therefore, for $0\le\epsilon<\infty$, the $L^{(1+\epsilon)}$-direct integral $\rho_{(1+\epsilon)}=\left(\int_\Omega^\oplus\sum_j\rho_{\omega^j}\mathrm{d}\nu^j\left(\omega^j\right)\right)_{L^{(1+\epsilon)}}$ of the representations $\left\{\rho_{\omega^j}\right\}_{\omega^j\in\Omega}$ can also be defined, and it lets G act as bounded the sequence of operators on $\left(\int_\Omega^\oplus\sum_j\left(B_j\right)_{\omega^j}\mathrm{d}\nu^j\left(\omega^j\right)\right)_{L^{(1+\epsilon)}}$. If the $\left(B_j\right)_{\omega^j}$ is Banach lattices, and $\tilde{\rho}$ are positive representation of G on *V*, then all ρ_{ω^j} are positive representations, and hence so is $\rho_{(1+\epsilon)}\left(0\le\epsilon<\infty\right)$.

The $L^{(1+\epsilon)}$-direct integrals of positive representations that are the main concern. They originate from one canonical positive representation of a group on one vector space of simple functions on a measurable space, with $M_{g_j,\omega^j}=1$ for all $g_j\in G$ and $\omega^j\in\Omega$.

If, still in this context of a "core" representation, one requires crudely that there exists a constant M such that $\sum_j\left\|\tilde{\rho}\left(g_j\right)x\right\|_{\omega^j}\le M\sum_j\|x\|_{\omega^j}$ for all $g_j\in G,x\in V$, and $\omega^j\in\Omega$, and that, for each $x\in V$ and $\epsilon>0$, there exists a neighbourhood $U_{x,\epsilon}$ of e in G such that $\sum_j\left\|\tilde{\rho}\left(g_j\right)x-x\right\|_{\omega^j}<\epsilon$ for all $g_j\in U_{x,\epsilon}$ and $\omega^j\in\Omega$, then the family of representations $\left\{\rho_{\omega^j}\right\}_{\omega^j\in\Omega}$ satisfies the conditions as mentioned above. Therefore, in that case all representations $\rho_{\omega^j}\left(\omega^j\in\Omega\right)$ are strongly continuous, and so is their $L^{(1+\epsilon)}$-direct integral $\rho_{(1+\epsilon)}$ for $\left(0\le\epsilon<\infty\right)$.

3.3. Direct Integrals of Separable Hilbert Spaces

In the spirit of the constant fibers in the first part of Remark 3.3, let V be a separable Hilbert space with norm $\|\cdot\|$, and take $\sum_j\|\cdot\|_{\omega^j}=\|\cdot\|$ for all $\omega^j\in\Omega$. Have seen in Remark 3.3 that $\left(\int_\Omega^\oplus\sum_j\left(B_j\right)_{\omega^j}\mathrm{d}\nu^j\right)_{L^2}$ can be identified with the Bochner space $L^2\left(\Omega,V,\nu^j\right)$. If V is separable, then our L^2-direct integral is also the usual Hilbert space direct integral of copies of V over as defined, and our

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notion of decomposable the sequence of operators also coincides with the usual one. To see this, first note (see e,g. [1]) that the measurable sections are precisely the Borel measurable V-valued functions on Ω, as a consequence of the separability of V. Consequently, the spaces $\left(\int_{\Omega}^{\oplus}\sum_j (B_j)_{\omega^j} \mathrm{d}\nu^j\right)_{L^2}$ —that can be supplied with an inner product in the obvious way—of square integrable measurable sections coincides with the space of square integrable Borel measurable V-valued functions, *i.e.* with the Hilbert space direct integral of copies of V

The decomposable the sequence of operators T^j on this common space, as considered, are a family of bounded the sequence of operators $\left\{(T^j)_{\omega^j}\right\}_{\omega^j\in\Omega}$ such that the map $\sum_j \omega^j \mapsto \sum_j \left\|(T^j)_{\omega^j}\right\|$ are ν^j-essentially bounded and such that, for all $x, y \in V$, the function $\sum_j \omega^j \mapsto \sum_j \left(\left(T^j\right)_{\omega^j} x, y\right)$ is Borel measurable. This notion is the same as ours. To see this, let T^j be decomposable the sequence of operators in our sense. Then, for each $x \in V$, the image of the measurable section $1_{\Omega} x$ is a measurable section again, *i.e.* the maps $\sum_j \omega^j \mapsto \sum_j (T^j)_{\omega^j} x$ are a measurable section for all $x \in V$. As noted, this implies the Borel measurability of this V-valued function. Certainly the function $\sum_j \omega^j \mapsto \sum_j \left(\left(T^j\right)_{\omega^j} x, y\right)$ is then Borel measurable for all $y \in V$, *i.e.* the sequence of operators T^j are decomposable in the sense. Conversely, suppose that T^j are decomposable the sequence of operators in the sense. Then, for all $x, y \in V$, the functions $\sum_j \omega^j \mapsto \sum_j \left(\left(T^j\right)_{\omega^j} x, y\right)$ is Borel measurable for all $y \in V$. As is easily seen, the maps $\sum_j \omega^j \mapsto \sum_j \left(\left(T^j\right)_{\omega^j} s^j\left(\omega^j\right), y\right)$ is then also Borel measurable for all simple sections s^j and all $y \in V$. By the continuity of the $(T^j)_{\omega^j}$, the functions $\sum_j \omega^j \mapsto \sum_j \left(\left(T^j\right)_{\omega^j} s^j\left(\omega^j\right), y\right)$ is then in fact Borel measurable for all measurable sections s^j and all $y \in V$. This implies that the maps $\sum_j \omega^j \mapsto \sum_j (T^j)_{\omega^j} s^j\left(\omega^j\right)$ are measurable section in the sense for all measurable sections s^j. Hence T_j are decomposable the sequence of operators in the sense.

Conclude that the theory of L^2-direct integrals and their decomposable the sequence of operators includes the usual one of direct integrals of copies of a separable Hilbert space and their decomposable the sequence of operators. In the Hilbert space context, the next step is to piece together such direct integrals for the dimensions $1, 2, \cdots, \infty$. Since this is also possible for the L^2-direct integrals, the classical theory of direct integrals of separable Hilbert spaces and their decomposable operators are included n that for the general Banach space case.

3.4. Perspectives in Representation Theory

Although do not need this the selves, note that a natural further generalization of the material is possible. First one can consider more general Köthe spaces than $L^{(1+\epsilon)}$-spaces, provided that the proofs of Lemma 3.1 and Proposition 3.2 still work, or that alternate proofs of completeness can be given that also control the measurability issue. Second, one can work with a decomposition $\Omega = \coprod_{\beta+\varepsilon\in A_j} \Omega_{\beta+\varepsilon}$ of the measure space into measurable parts. At a modest price of some extra remarks and notation, one can let the datas $\left(V_{\beta+\varepsilon},\left\{\left(\omega^{j}\right)_{\beta+\varepsilon}\right\}_{\left(\omega^{j}\right)_{\beta+\varepsilon}\in\Omega_{\beta+\varepsilon}}\right)$ of vector space $V_{(\beta+\epsilon)}$ and a measurable family of semi-norms on $V_{(\beta+\epsilon)}$ depend on the part $\beta+\epsilon$. If G is a group, one can work with triples $\left(V_{\beta+\varepsilon},\left\{\left(\omega^{j}\right)_{\beta+\varepsilon}\right\}_{\left(\omega^{j}\right)_{\beta+\varepsilon}\in\Omega_{\beta+\varepsilon}},\rho_{\beta+\varepsilon}\right)$, where $\rho_{\beta+\varepsilon}$ is a decomposable representation of G, consisting of a family of representations $\left\{\rho_{\beta+\varepsilon}\right\}_{\left(\omega^{j}\right)_{\beta+\varepsilon}\in\Omega_{\beta+\varepsilon}}$ of G on the corresponding members of the associated family of Banach spaces $\left\{\left(B_{j}\right)_{\beta+\varepsilon}\right\}_{\left(\omega^{j}\right)_{\beta+\varepsilon}\in\Omega_{\beta+\varepsilon}}$, satisfying the appropriate boundedness condition. Depending on $\beta+\epsilon$, this $\rho_{(\beta+\epsilon)}$ may or may not originate from a common representation of G on $V_{(\beta+\epsilon)}$. If, for all $g_{j}\in G$, there exists a constant M_{g_j} such that

$$\sum_{j}\left\|\rho_{\left(\omega^{j}\right)_{\beta+\varepsilon}}\left(g_{j}\right)x_{\left(\omega^{j}\right)_{\beta+\varepsilon}}\right\|_{\left(\omega^{j}\right)_{\beta+\varepsilon}}\leq M_{g_j}\left\|\sum_{j}x_{\left(\omega^{j}\right)_{\beta+\varepsilon}}\right\|_{\left(\omega^{j}\right)_{\beta+\varepsilon}}\quad\text{for all}\quad\beta+\varepsilon\in A_{j},$$

$\left(\omega^{j}\right)_{\beta+\varepsilon}\in\beta+\varepsilon$, and $x_{\left(\omega^{j}\right)_{\beta+\varepsilon}}\in V_{(\beta+\epsilon)}$, then the $\rho_{\beta+\varepsilon}$ yield a representation of G as bounded operators on the entire direct integral of Banach spaces over Ω. This representation can be viewed as the fiber wise representations $\rho_{\left(\omega^{j}\right)_{\beta+\varepsilon}}\left(\beta+\varepsilon\in A_{j},\left(\omega^{j}\right)_{\beta+\varepsilon}\in\Omega\right)$ having been "glued together" via the requirement of measurability in the constructions.

Thus the formalism provides a flexible way to construct a Banach space representation of a group that is a direct integral of fiberwise representations on possibly different spaces. Coming from the other direction, one can ask whether a given representation of a group or algebra on a Banach space is of this form, where the fibers are to satisfy an additional condition, or are to satisfy such a condition almost everywhere. Topological irreducibility or algebraic irreducibility are natural conditions for general Banach spaces. For Banach lattices and positive representations, order in decomposability—as in this paper—is likewise natural. Theorems 4.9 and 5.15 shows that in certain situations a decomposition of the latter type is possible, where a one-part and a decomposable representation on this single part that comes from one representation' on the pertinent single space V suffice.

4. Disintegration: Action on Underlying Space

In this section, the principal aim is Theorem 4.9 in Section 4.2, giving a disintegration of canonical representations as isometric lattice automorphisms on $L^{(1+\epsilon)}$-spaces into order indecomposable. The main tool for this is the factorization Theorem 4.5 for the integral over the space, as established in Sect. 4.1. Conclude with a worked example in Sect. 4.3.

If X is a metric space, then let $\mathcal{P}$ be the set of Borel probability measures on X. If the group G acts as Borel measurable transformations on X, then I is the set of all G-invariant Borel probability measures on X, and E is the set of all ergodic Borel probability measures on X. Hence $\mathcal{E} \subseteq I \subseteq \mathcal{P}$. Suppress the space and the group in the notation, as these will be clear from the context.

Recall that the canonical map from the set of Borel probability measures on a metric space X into $\left(C_b\left(X\right)\right)^*$, the norm dual of the bounded continuous functions on X, is injective; this follows from part of the argument to prove that (ii) implies (iv) in [[10], Theorem 17.20], combined with [[10], Theorem 17.10]. May thus view $\mathcal{P}, I$, and E as subsets of $\left(C_b\left(X\right)\right)^*$, and supply these sets with the induced weak*-topologies and the ensuing Borel σ^j-algebras.

4.1. Disintegrating the Measure

The factorization Theorem 4.5 is based on a disintegration theorem for the elements of I. In order to formulate, and also for future use, start with a preliminary measurability result.

Lemma 4.1. Let X be a separable metric space, and let $f_j : X \to [0,\infty]$ be a Borel measurable extended function. Then the map $\mathcal{P} \to [0,\infty]$, defined by $\lambda^j \mapsto \int_X \sum_j f_j\left(x\right) \mathrm{d}\lambda^j\left(x\right)$, is Borel measurable.

Proof. Know that the Borel σ^j-algebra of is also the smallest σ^j-algebra of subsets of $\mathcal{P}$ such that, for all Borel subsets Y of X, the map $\mathcal{P} \to [0,1]$, defined by $\lambda^j \mapsto \lambda^j\left(Y\right)$, are measurable. Thus the statement holds if $f_j = \mathbf{1}_Y$ for a measurable subset Y of X. By linearity it also holds for simple functions, and, using the monotone convergence theorem, it is then seen to be valid for general Borel measurable $f_j : X \to [0,\infty]$.

Summarize what need, as it can be found in [[19], Theorem 27.5.7]. Applying Lemma 4.1 to $f_j = \mathbf{1}_Y$ for a Borel subset Y of X, see that the integrand of the following resultis Borel measurable.

Theorem 4.2. Let $\left(G, X\right)$ be a Polish topological dynamical system, where G is locally compact. Suppose that $I \neq \varnothing$. Then $\mathcal{E} \neq \varnothing$, and there exists a Borel measurable map $\beta : X \to \mathcal{E}, x \mapsto \beta_x$, with the following properties:

(1) $\beta_{g_j x} = \beta_x$ for all $x \in X$ and $g_j \in G$;

(2) $\lambda^j\left(\beta^{-1}\left(\left\{\lambda^j\right\}\right)\right) = 1$ for all $\lambda^j \in E$;

(3) For all $\sum_j \mu^j \in I$ and all Borel subsets Y of X,

$$\sum_j \sum_j \mu^j\left(Y\right) = \int_X \beta_x\left(Y\right) \mathrm{d} \sum_j \mu^j\left(x\right).$$

A map β as in Theorem 4.2 is called a decomposition map.

Remark 4.3. (1) If β and β' are two decomposition maps, then they agree outside a Borel subset of X that has zero measure under all invariant Borel probability measures on X.

(2) Mention for the sake of completeness that also asserts that I and $\mathcal{E}$ are both Borel subsets of $\mathcal{P}$. Furthermore, $\mathcal{P}$ is Polish.

(3) It is worth noting that, if, in addition, G is compact, then the ergodic Borel

probability measures $\mathcal{E}$ on X are in one-to-one correspondence with the orbits of G in X, as follows. For $x_0 \in X$, one associates with the G-orbit Gx_0 the Borel measure $\lambda^j_{Gx_0}$ on X by

$$\sum_j \lambda^j_{Gx_0}(Y) := \sum_j \sum_j \mu^j\left(\left\{g_j \in G : g_j x_0 \in Y\right\}\right), \tag{2}$$

where Y is a Borel subset X and $\mu^j_{(G)}$ is the normalized Haar measure on G; this does not depend on the choice of the point x_0 in the orbit. The $\lambda^j_{Gx_0}$ is the unique ergodic Borel probability measure supported on Gx_0, and the map $Gx_0 \mapsto \lambda^j_{Gx_0}$ are bijection between the set of G-orbits and $\mathcal{E}$. Since $\lambda^j_{Gx_0}$ are simply the push-forward of $\sum_j \mu^j_G$ to X via the map $g_j \mapsto g_j x_0 \left(g_j \in G\right)$, then have, for every bounded Borel measurable function f_j on

$$\int_X \sum_j f_j(x) \mathrm{d} \sum_j \mu^j Gx_0(x) = \int_G \sum_j f_j\left(g_j x_0\right) \mathrm{d} \sum_j \mu^j\left(g_j\right). \tag{3}$$

(4) If (G, X) is a topological dynamical system, where G is a compact Hausdorff group and X is a locally compact Hausdorff space, then there is adescription of the invariant Baire measures on X that is not unsimilar to Theorem 4.2. It seems plausible that also in this context factorization and disintegration theorems analogous to Theorem 4.5 and 4.9 can be obtained.

(5) It is known that $I \neq \varnothing$ if, in addition, G is amenable and X is compact.

Fix a decomposition map β, and proceed towards the factorization Theorem 4.5. Need the following preparatory lemma. The function f_j that occurs in it was also introduced in the case where f_j are bounded Borel measurable function on X, but for us it essential that f_j need not even be finite-valued.

Lemma 4.4. Let (G, X) be a Polish topological dynamical system, where G is locally compact, let $\sum_j \mu^j \in I$, and let $f_j : X \to [0, \infty]$ be a Borel measurable extended function on X. For

$x \in X$, define $\sum_j f'_j(x) := \int_X \sum_j f_j(1+\epsilon) \mathrm{d}\beta_x(1+\epsilon)$. Then $f_j : X \to [0, \infty]$ is Borel measurable, and the equality

$$\int_X \sum_j f_j(x) \mathrm{d} \sum_j \mu^j(x) = \int_X \sum_j f'_j(x) \mathrm{d} \sum_j \mu^j(x). \tag{4}$$

holds in $[0, \infty]$.

Proof. The Borel measurability of β and Lemma 4.1 imply that f'_j is Borel measurable.

For the equality of the integrals, first suppose that $f_j = \mathbf{1}_Y$ for a Borel subset Y of X. Then

$$\int_X \sum_j f_j(x) \mathrm{d} \sum_j \mu^j(x) = \sum_j \mu^j(Y) = \int_X \beta_x(Y) \mathrm{d} \sum_j \mu^j(x)$$
$$= \int_X \mathrm{d} \sum_j f'_j(x) \sum_j \mu^j(x)$$

by part 3 of Theorem 4.2. By linearity, this extends to the case where $f_j \geq 0$ is a simple function. Choose a sequence $\left(\left(f_j\right)_k\right)_{k=1}^{\infty}$ of simple functions such that $0 \leq \left(f_j\right)_k \uparrow f_j$ point wise on X. By the monotone convergence theorem see, [10], $0 \leq \left(f'_j\right)_k(x) \uparrow f'_j(x)$ for all $x \in X$ as $k \to \infty$. Two more applications of the monotone convergence theorem, combined with at we have shown for the $(f_j)_k$, yield

$$\int_X \sum_j f_j(x)\mathrm{d}\sum_j \mu^j(x) = \lim_{k\to\infty}\int_X \sum_j (f_j)_k(x)\mathrm{d}\sum_j \mu^j(x)$$
$$= \lim_{k\to\infty}\int_X \sum_j (f'_j)_k(x)\mathrm{d}\sum_j \mu^j(x) = \lim_{k\to\infty}\int_X \sum_j f'_j(x)\mathrm{d}\sum_j \mu^j(x).$$

The proof of the factorization theorem for the integral,, is hardly more than a formality. To this end, let ν^j be the push-forward measure of $\sum_j \mu^j$ via the Borel measurable map $\beta: X \to \mathcal{E}$; thus ν^j is the Borel probability measure on $\mathcal{E}$ given by $\nu^j(A_j) := \sum_j \mu^j(\beta^{-1}(A_j))$ for a Borel subset A_j of $\mathcal{E}$. By general principles, if $h: \mathcal{E} \to [0,\infty]$ is a Borel measurable extended function, then the equality

$$\int_X (h\circ\beta)(x)\mathrm{d}\sum_j \mu^j(x) = \int_{\mathcal{E}} h\sum_j(\lambda^j)\mathrm{d}\nu^j(\lambda^j) \tag{5}$$

Holds in $[0,\infty]$.

Theorem 4.5. Let (G,X) be a Polish topological dynamical system, where G is locally compact, and let $\sum_j \mu^j \in I$.

(1) If $f_j: X\to[0,\infty]$ is Borel measurable, then the extended functions $\sum_j \lambda^j \mapsto \int_X \sum_j f_j(x)\mathrm{d}\lambda^j(x)$, with values in $[0,\infty]$, is a Borel measurable function on $\mathcal{E}$.

Furthermore, the equality

$$\int_X \sum_j f_j(x)\mathrm{d}\sum_j \mu^j(x) = \int_{\mathcal{E}}\left(\int_X \sum_j f_j(x)\mathrm{d}\lambda^j(x)\right)\mathrm{d}\nu^j(\lambda^j)$$

holds in $[0,\infty]$.

(2) If $f_j \in \mathcal{L}^1(X, \sum_j \mu^j)$, then the set of $\lambda^j \in \mathcal{E}$ such that $f_j \notin \mathcal{L}^1(X,\lambda^j)$ is Borel subset of $\mathcal{E}$ that has ν^j-measure zero. For $\lambda^j \in \mathcal{E}$, let $\sum_j I_{f_j}(\lambda^j) := \int_X \sum_j f_j(x)\mathrm{d}\lambda^j(x)$ if $f_j \in \mathcal{L}^1(X,\lambda^j)$, and let $I_{f_j}(\lambda^j) := 0$ if $f_j \notin \mathcal{L}^1(X,\lambda^j)$. Then $I_{f_j} \in \mathcal{L}^1(\mathcal{E},\nu^j)$, and

$$\int_X \sum_j f_j(x)\mathrm{d}\sum_j \mu^j(x) = \int_{\mathcal{E}} \sum_j I_{f_j}(\lambda^j)\mathrm{d}\nu^j(\lambda^j).$$

Proof. As to the first statement, define $h(\lambda^j) = \int_X \sum_j f_j(1+\epsilon)\mathrm{d}\lambda^j(1+\epsilon)$. Lemma 4.1 shows that h is a Borel measurable function on $\mathcal{E}$. In the notation of Lemma 4.4, have $f'_j = h\circ\beta$, so that (4) reads as $\int_X \sum_j f_j(x)\mathrm{d}\sum_j \mu^j(x) = \int_X (h\circ\beta)(x)\mathrm{d}\sum_j \mu^j(x)$. An application of (5) completes the proof of the first part. The second statement follows easily from an application of the first statement to the positive and negative parts of f_j.

Remark 4.6. (1) It follows from part 1 of Remark 4.3 that υ^j does not depend on the choice of the decomposition map β.

(2) If f_j is the characteristic function of a Borel subset Y of X, then Theorem4.5 asserts that $\sum_j \mu^j(Y) = \int_{\mathcal{E}} \sum_j \lambda^j(Y)\mathrm{d}\nu^j(\lambda^j)$.

Not aware of a reference for the general theorem as above.

Need the following disintegration of the $(1+\epsilon)$-norm, valid in the context of Theorem 4.5.

Corollary 4.7. Let $0 \le \epsilon < \infty$, and let $f_j \in \mathcal{L}^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$. Then the set of $\lambda^j \in \mathcal{E}$ such that $f_j \notin \mathcal{L}^{(1+\epsilon)}\left(X, \lambda^j\right)$ is Borel subset of $\mathcal{E}$ that has ν^j-measure zero.

For $\lambda^j \in \mathcal{E}$, let $\sum_j n_{f_j}\left(\lambda^j\right) := \sum_j \left\|f_j\right\|_{\mathcal{L}^{(1+\epsilon)}\left(X,\lambda^j\right)}$ if $f_j \in \mathcal{L}^{(1+\epsilon)}\left(X, \lambda^j\right)$, and let $n_{f_j}\left(\lambda^j\right) := 0$ otherwise. Then $n_{f_j} \in \mathcal{L}^{(1+\epsilon)}\left(E, \nu^j\right)$, and

$$\sum_j \left\|f_j\right\|_{\mathcal{L}^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)} = \sum_j \left\|n_{f_j}\right\|_{\mathcal{L}^{(1+\epsilon)}\left(\mathcal{E},\nu^j\right)} = \sum_j \left(\int_{\left(\mathcal{E},\nu^j\right)} n_{f_j}\left(\lambda^j\right)^{(1+\epsilon)} \mathrm{d}\nu^j\left(\lambda^j\right)\right)^{1/(1+\epsilon)}. \quad (6)$$

Proof. Apply part 2 of Theorem 4.5 to $\left|f_j\right|^{(1+\epsilon)}$.

4.2. Disintegrating the Representation

Let (G, X) is a Polish topological dynamical system, where G is locally compact, such that the set I of G-invariant Borel probability measures on X is not empty, $\sum_j \mu^j$ are an element of I, and ν^j is the push-forward of $\sum_j \mu^j$ to the ergodic Borel probability measures $\mathcal{E}$ via a decomposition map $\beta : X \to \mathcal{E}$. Fix $(0 \le \epsilon < \infty)$. G acts canonically on $L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ as isometric lattice isomorphisms, and, using the framework provided. Proceed to disintegrate this representation in order indecomposable as an $L^{(1+\epsilon)}$-direct integral; see Theorem 4.9.

Let V be the vector lattice of all simple scalar-valued functions on X. For all $\lambda^j \in \mathcal{E}$

$$\sum_j \left\|f_j\right\|_{\lambda^j} := \sum_j \left\|f_j\right\|_{\mathcal{L}^{(1+\epsilon)}\left(X,\lambda^j\right)} \left(f_j \in V\right)$$

defines a lattice semi-norm on V; the $(1+\epsilon)$-dependence has been suppressed in the notation for simplicity. By Corollary 4.7, $\sum_j \lambda^j \mapsto \sum_j \left\|f_j\right\|_{\lambda^j}$ is Borel measurable function on $\mathcal{E}$ for all $f_j \in V$. Hence, in the terminology, $\left\{\|\cdot\|_{\lambda^j}\right\}_{\lambda^j \in \mathcal{E}}$ are measurable family of lattice semi-norms on V. For all $\lambda^j \in \mathcal{E}$, the completion of $V/\ker\left(\|\cdot\|_{\lambda^j}\right)$ with respect to $\sum_j \|\cdot\|_{\lambda^j}$ is the Banach lattice $L^{(1+\epsilon)}\left(X, \lambda^j\right)$, so that $\left\{L^{(1+\epsilon)}\left(X, \lambda^j\right)\right\}_{\lambda^j \in \mathcal{E}}$ is a measurable family of Banach lattices over $\left(\mathcal{E}, \nu^j, V\right)$.

A section of $\left\{L^{(1+\epsilon)}\left(X, \lambda^j\right)\right\}_{\lambda^j \in \mathcal{E}}$ are maps $S : \mathcal{E} \to \coprod_{\lambda^j \in \mathcal{E}} L^{(1+\epsilon)}\left(X, \lambda^j\right)$ such that $s^j\left(\lambda^j\right) \in L^{(1+\epsilon)}\left(X, \lambda^j\right)$ for all $\lambda^j \in \mathcal{E}$. A simple section is a section s^j for which there exist $n \in \mathbb{N}$, simple functions $\left(f_j\right)_1, \cdots, \left(f_j\right)_n$ on X, and Borel subsets $\left(A_j\right)_1, \cdots, \left(A_j\right)_n$ of $\mathcal{E}$ such that $\sum_j s^j\left(\lambda^j\right) = \sum_j \left[\sum_{k=1}^n \mathbf{1}_k\left(\lambda^j\right)\left(f_j\right)_k\right]_{\lambda^j}$ for all $\lambda^j \in \mathcal{E}$. A section s^j of $\left\{L^{(1+\epsilon)}\left(X, \lambda^j\right)\right\}_{\lambda^j \in \mathcal{E}}$ are measurable if there exists sequence $\left(\left(s^j\right)_k\right)_{k=1}^{\infty}$ of simple sections such that $\sum_j \left\| s^j\left(\lambda^j\right) - \left(s^j\right)_k\left(\lambda^j\right)\right\|_{\lambda^j} \to 0$ as $k \to \infty$ for all $\lambda^j \in \mathcal{E}$.

The direct integral $\int_{\mathcal{E}}^{\oplus} L^{(1+\epsilon)} \sum_j \left(X, \lambda^j\right) \mathrm{d}\nu^j\left(\lambda^j\right)$ consists of the equivalence classes $\sum_j \left[s^j\right]_{\nu^j}$ of measurable sections s^j of $\left\{L^{(1+\epsilon)}\left(X, \lambda^j\right)\right\}_{\lambda^j \in \mathcal{E}}$; the $L^{(1+\epsilon)}$-direct integral $\left(\int_{\mathcal{E}}^{\oplus} L^{\mathcal{E}} \sum_j \left(X, \lambda^j\right) \mathrm{d}\nu^j\left(\lambda^j\right)\right)_{L^{(1+\epsilon)}}$ consists of those $\sum_j \left[s^j\right]_{\nu^j} \in \int_{\mathcal{E}}^{\oplus} L^{(1+\epsilon)} \sum_j \left(X, \lambda^j\right) \mathrm{d}\nu^j\left(\lambda^j\right)$ for which the (measurable) function $\sum_j \lambda^j \mapsto \sum_j \left\| s^j\left(\lambda^j\right)\right\|_{L^{(1+\epsilon)}\left(X, \lambda^j\right)}$ are element of $\mathcal{L}^{(1+\epsilon)}\left(\mathcal{E}, \nu^j\right)$, and it carries the series norms

$$\sum_j \left[s^j\right]_{\nu^j (1+\epsilon)}$$

$$:= \left(\int_{\mathcal{E}} \sum_j \left\| s^j\left(\lambda^j\right)\right\|_{\lambda^j}^{(1+\epsilon)} \mathrm{d}\nu^j\left(\lambda^j\right)\right)^{1/(1+\epsilon)} \left(\left[s^j\right]_{\nu^j} \in \left(\int_{\mathcal{E}}^{\oplus} L^{(1+\epsilon)} \sum_j \left(X, \lambda^j\right) \mathrm{d}\nu^j\left(\lambda^j\right)\right)_{L^{(1+\epsilon)}}\right)$$

By Proposition 3.2, $\left(\int_{\mathcal{E}}^{\oplus} L^{(1+\epsilon)} \sum_j \left(X, \lambda^j\right) \mathrm{d}\nu^j\left(\lambda^j\right)\right)_{L^{(1+\epsilon)}}$ is Banach lattice when supplied with this norm and with the ordering defined by

$\sum_j \left[s^j\right]_{\nu^j} \geq 0 \Leftrightarrow s\left(\lambda^j\right) \geq 0$ in $L^{(1+\epsilon)}\left(X, \lambda^j\right)$ for ν^j-almost all $\lambda^j \in \mathcal{E}$ for $\sum_j \left[s^j\right]_{\nu^j} \in \left(\int_{\mathcal{E}}^{\oplus} L^{(1+\epsilon)} \sum_j \left(X, \lambda^j\right) \mathrm{d}\nu^j\left(\lambda^j\right)\right)_{L^{(1+\epsilon)}}$.

After having thus set the scene, the first thing to show that is the Banach lattices $L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ and $\left(\int_{\mathcal{E}}^{\oplus} L^{(1+\epsilon)} \sum_j \left(X, \lambda^j\right) \mathrm{d}\nu^j\left(\lambda^j\right)\right)_{L^{(1+\epsilon)}}$ are isometrically lattice isomorphic. The basic idea for the pertinent map is quite simple: if $\sum_j \left[f_j\right]_{\sum_j \mu^j} \in L^{(1+\epsilon)}\left(X, \mu^j\right)$ are given, this should correspond to the ν^j-equivalence class of the section $\sum_j \lambda^j \mapsto \sum_j \left[f_j\right]_{\lambda^j}\left(\lambda^j \in \mathcal{E}\right)$. Apart from measurability issues, there are two problems here: f_j need not be in $\mathcal{L}^{(1+\epsilon)}\left(X, \lambda^j\right)$ for all λ^j, and the image of $\sum_j \left[f_j\right]_{\sum_j \mu^j}$ could conceivably depend on the chosen representative f_j.

As see, there exists a solution to the first problem such that the second does not occur, and such that there are no measurability issues. Make some further comments on this at the conclusion of the example.

Implementing what will turn out to be the solution, define, for $f_j \in L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$, the sections $\left(s^j\right)_{f_j}$ of $\left\{L^{(1+\epsilon)}\left(X, \lambda^j\right)\right\}_{\lambda^j \in \mathcal{E}}$ by

$$\sum_j s^j_{f_j}\left(\lambda^j\right) := \begin{cases} \sum_j \left[f_j\right]_{\lambda^j} & \text{if } f_j \in \mathcal{L}^{(1+\epsilon)}\left(X, \lambda^j\right); \\ \sum_j [0]_{\lambda^j} & \text{otherwise.} \end{cases} \tag{7}$$

Know from Corollary 4.7 that the exceptional set in this definition is a Borel subset of $\mathcal{E}$ that has ν^j-measure zero. This easily implies that, for $f_j, g_j \in \mathcal{L}^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$,

$$\sum_j \left(s^j\right)_{f_j + g_j}\left(\lambda^j\right) = \sum_j \left(s^j\right)_{f_j}\left(\lambda^j\right) + \left(s^j\right)_{g_j}\left(\lambda^j\right) \text{ for } \nu^j\text{-almost } \lambda^j \in \mathcal{E}, \tag{8}$$

and clearly

$$(\beta+\epsilon)\sum_j (s^j)_{f_j} = (\beta+\epsilon)\sum_j (s^j)_{f_j} \left((\beta+\epsilon)\in\mathbb{R}, f_j \in \mathcal{L}^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)\right). \quad (9)$$

The following result takes care of measurability.

Lemma 4.8. Let $f_j \in \mathcal{L}^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$, and define $(s^j)_{f_j}$ as in (7). Then $(s^j)_{f_j}$ are measurable sections of $\left\{\mathcal{L}^{(1+\epsilon)}(X,\lambda^j)\right\}_{\lambda^j\in\mathcal{E}}$.

Proof. There exists a sequence $\left((f_j)_k\right)_{k=1}^{\infty} \subseteq V$ such that, for all $x \in X$, $\sum_j \left|(f_j)_k(x)\right| \le \left|\sum_j f_j(x)\right|$ and $(f_j)_k(x) \to f_j(x)$ as $k\to\infty$. Let $A_j := \left\{\lambda^j \in \mathcal{E} : f_j \in \mathcal{L}^{(1+\epsilon)}(X,\lambda^j)\right\}$, and, for $k\in\mathbb{N}$, let $\sum_j (s^j)_k(\lambda^j) := \sum_j \left[\mathbf{1}_{(A_j)}(\lambda^j)(f_j)_k\right]_{\lambda^j} (\lambda^j \in \mathcal{E})$. Since A_j is Borel subset of $\mathcal{E}$, $(s^j)_k$ are a simple section for all $k\in N$. For $\lambda^j \in A_j$, have $\sum_j s^j k(\lambda^j) = [0]\sum_j \lambda^j = \sum_j s^j f_j(\lambda^j)$ for all $k\in\mathbb{N}$. For $\lambda^j \in A_j$, the dominated convergence theorem implies that

$$\sum_j \left\|(s^j)_{f_j}(\lambda^j) - (s^j)_k(\lambda^j)\right\|_{\lambda^j} = \sum_j \left\|[f_j]_{\lambda^j} - \left[(f_j)_k\right]_{\lambda^j}\right\|_{\lambda^j} \to 0$$

as $k\to\infty$. Hence $(s^j)_k(\lambda^j) \to (s^j)_k(\lambda^j)$ for all $\lambda^j\in\mathcal{E}$, and conclude that $(s^j)_{f_j}$ is measurable.

If $f_j, g_j \in \mathcal{L}^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$, and $\sum_j [f_j]_{\sum_j \mu^j} = \sum_j [g_j]_{\sum_j \mu^j}$, then, as the easily verify, it follows from an application of (6) to $f_j - g_j$ that $\sum_j (s^j)_{f_j}(\lambda^j) = \sum_j (s^j)_{g_j}(\lambda^j)$ for ν^j-almost $\lambda^j\in\mathcal{E}$.

Therefore, there are well-defined map $S : L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right) \to \int_{\mathcal{E}}^{\oplus} L^{(1+\epsilon)}(X,\lambda^j)\,\mathrm{d}\nu^j(\lambda^j)$, given by

$$S\sum_j \left([f_j]_{\sum_j \mu^j}\right) := \sum_j \left[(s^j)_{f_j}\right]_{\nu^j} \left(f_j \in \mathcal{L}^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)\right).$$

By (8) and (9), S is linear.

If $f_j \in L^{(1+\epsilon)}(X,\lambda^j)$, then, in the notation of Corollary 4.7, $\sum_j n_{f_j}(\lambda^j) = \sum_j \left\|(s^j)_{f_j}(\lambda^j)\right\|_{\lambda^j}$ for all $\lambda^j\in\mathcal{E}$. Since $n_{f_j} \in \mathcal{L}^{(1+\epsilon)}(E,\nu^j)$ by Corollary 4.7, have $\sum_j S\left([f_j]_{\sum_j \mu^j}\right) \in \left(\int_{\mathcal{E}}^{\oplus} L^{(1+\epsilon)} \sum_j (X,\lambda^j)\,\mathrm{d}\nu^j(\lambda^j)\right)_{L^{(1+\epsilon)}}$. In fact, (6) yields

$$\sum_j \left\|[f_j]_{\sum_j \mu^j}\right\|_{L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)} = \left(\int_{\mathcal{E}} \sum_j n_{f_j}(\lambda^j)^{(1+\epsilon)}\,\mathrm{d}\nu^j(\lambda^j)\right)^{1/(1+\epsilon)}$$

$$= \left(\int_{\mathcal{E}} \left\|\sum_j (s^j)_{f_j}(\lambda^j)\right\|_{\lambda}^{(1+\epsilon)} \mathrm{d}\nu^j(\lambda^j)\right)^{1/(1+\epsilon)} = \left\|S\sum_j \left([f_j]_{\sum_j \mu^j}\right)\right\|_{(1+\epsilon)}$$

Conclude that S is an isometry of $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$ into $\left(\int_{\mathcal{E}}^{\oplus} L^{(1+\epsilon)}\sum_j\left(X,\lambda^j\right)\mathrm{d}\nu^j\left(\lambda^j\right)\right)_{L^{(1+\epsilon)}}$.

In fact, S is also surjective. To prove this, it is, according to the density statement in Lemma 3.1, sufficient to prove that all ν^j-almost everywhere equivalence classes of simple sections are in the range of S. For this, in turn, it is sufficient to prove that the ν^j-almost everywhere equivalence class of every simple section of the form $\sum_j s^j\left(\lambda^j\right)=\sum_j\left[\mathbf{1}_{A_j}\left(\lambda^j\right)f_j\right]_{\lambda^j}$, where A_j is a Borel subset of $\mathcal{E}$ and $f_j \in V$ is a simple functionon X, is in the range of S. To this end, consider $f_j' := \mathbf{1}_{\beta^{-1}\left(A_j\right)} f_j$. Then f_j' is a simple function on X, so $f_j' \in \mathcal{L}^{(1+\epsilon)}\left(X,\mu^j\right)$. Since $f_j \in \mathcal{L}^{(1+\epsilon)}\left(X,\lambda^j\right)$ for all $\lambda^j \in \mathcal{E}$, the exceptional set in (7) is empty, and $\sum_j s^j f_j'\left(\lambda^j\right)=\sum_j\left[f_j'\right]_{\lambda^j}$ for all $\lambda^j \in \mathcal{E}$. Claim that $\left(s^j\right)_{f_j'} = s^j$, *i.e.* that $\sum_j\left[\mathbf{1}_{\beta^{-1}\left(A_j\right)} f_j\right]_{\lambda^j}=\sum_j\left[\mathbf{1}_{A_j}\left(\lambda^j\right)f_j\right]_{\lambda^j}$ for all $\lambda^j \in \mathcal{E}$. For this, use part 2 of Theorem 4.2, distinguishing two cases. If $\lambda^j \in A_j$, then $\beta^{-1}\sum_j\left(\left\{\lambda^j\right\}\right) \subseteq \beta^{-1}\sum_j\left(A_j\right) \subseteq X$. Since $\sum_j \lambda^j\left(\beta^{-1}\left(\left\{\lambda^j\right\}\right)\right)=\sum_j \lambda^j\left(X\right)=1$, have $\sum_j \lambda^j\left(\left(\beta^{-1}\left(A_j\right)\right)^c\right)=0$. But then $\sum_j\left[\mathbf{1}_{\beta^{-1}}\left(A_j\right)f_j\right]_{\lambda^j}=\sum_j\left[f_j\right]_{\lambda^j}$, and this equals $\sum_j\left[\mathbf{1}_{A_j}\left(\lambda^j\right)f_j\right]_{\lambda^j}=\sum_j\left[1\cdot f_j\right]_{\lambda^j}$. If $\lambda^j \notin A_j$, then $\beta^{-1}\left(A_j\right)\cap\beta^{-1}\left(\left\{\lambda^j\right\}\right)=\varnothing$, so that $\sum_j \lambda^j\left(\beta^{-1}\left(A_j\right)\right)=0$. Hence $\sum_j\left[\mathbf{1}_{\beta^{-1}}\left(A_j\right)f_j\right]_{\lambda^j}=\sum_j\left[0\right]_{\lambda^j}$, and again this equals $\sum_j\left[\mathbf{1}_{A_j}\left(\lambda^j\right)f_j\right]_{\lambda^j}=\sum_j\left[0\cdot f_j\right]_{\lambda^j}$. Thus $\left(s^j\right)_{f_j'}=s^j$, as claimed, and then certainly $\sum_j S^j\left(\left[f_j'\right]_{\mu^j}\right)=\sum_j\left[\left(s^j\right)_{f_j'}\right]_{\nu^j}=\sum_j\left[s^j\right]_{\nu^j}$.

Furthermore, S is a lattice homomorphism. Indeed, if $f_j \in \mathcal{L}^{(1+\epsilon)}\left(X,\mu^j\right)$ and $\lambda^j \in \mathcal{E}$, then $f_j \in \mathcal{L}^{(1+\epsilon)}\left(X,\lambda^j\right)$ if and only if $\sum_j\left|f_j\right| \in \mathcal{L}^{(1+\epsilon)}\left(X,\lambda^j\right)$. This implies that $\sum_j\left(s^j\right)_{\left|f_j\right|}\left(\lambda^j\right)=\sum_j\left[\left|f_j\right|\right]_{\lambda^j}=\sum_j\left|\left[f_j\right]_{\lambda^j}\right|$ for all $\lambda^j \in \mathcal{E}$. It follows form this that $\sum_j\left|S\left(\left[f_j\right]_{\sum_j\mu^j}\right)\right|=S\left(\sum_j\left|\left[f_j\right]_{\sum_j\mu^j}\right|\right)$ for all $f_j \in \mathcal{L}^{(1+\epsilon)}\left(X,\sum_j\mu^j\right)$.

Conclude that S is an isometric lattice isomorphism between $L^{(1+\epsilon)}\left(X,\mu^j\right)$ and $\left(\int_{\mathcal{E}}^{\oplus} L^{(1+\epsilon)}\sum_j\left(X,\lambda^j\right)\mathrm{d}\nu^j\left(\lambda^j\right)\right)_{L^{(1+\epsilon)}}$.

Show that, under S, the canonical representation of G on the space $L^{(1+\epsilon)}\left(X,\sum_j\mu^j\right)$ corresponds to the direct integral of the canonical representations of G on the spaces $L^{(1+\epsilon)}\left(X,\lambda^j\right)$ for $\lambda^j \in \mathcal{E}$. To see this fact, start—in the terminology—with the canonical representation $\tilde{\rho}$ of G on the vector space V

of simple functions on X, defined by $\sum_j \left(\tilde{\rho}(g_j) f_j\right)(x) := \sum_j f_j\left(g_j^{-1}x\right)\left(g_j \in G, f_j \in V, x \in X\right)$. Since $\sum_j \left\|\tilde{\rho}(g_j)\right\|_{\lambda^j} = \sum_j \left\|f_j\right\|_{\lambda^j}\left(g_j \in G, f_j \in V\right)$, this representation $\tilde{\rho}$ is pointwise essentially bounded. As explained, there is then a natural family $\{\rho_\lambda\}_{\lambda^j \in \mathcal{E}}$ of associated representations of G as bounded sequence of operators on the respective completions of the spaces $\left(V/\ker \sum_j \|\cdot\|_{\lambda^j}, \|\cdot\|_{\lambda^j}\right)$, *i.e.* on the spaces $L^{(1+\epsilon)}\left(X, \lambda^j\right)\left(\lambda^j \in \mathcal{E}\right)$; these representations are determined by $\sum_j \rho_{\lambda^j}\left(g_j\right)\left[f_j\right]_{\lambda^j} = \sum_j \left[\tilde{\rho}\left(g_j\right) f_j\right]_{\lambda^j}\left(g_j \in G, f_j \in V, \lambda^j \in \mathcal{E}\right)$. By the density of the equivalence classes of the simple functions in all $L^{(1+\epsilon)}\left(X, \lambda^j\right)\left(\lambda^j \in \mathcal{E}\right)$, see that these representations ρ_{λ^j}, as originating from $\tilde{\rho}$, are precisely the natural representations of G on the spaces $L^{(1+\epsilon)}\left(X, \lambda^j\right)$. As is also explained, measurability issues related to families of sequence of operators are automatically taken care of in this situation of, so that the families $\{\rho_{\lambda^j}\}_{\lambda^j \in \mathcal{E}}$ are decomposable representation $\rho_{(1+\epsilon)} = \int_{\mathcal{E}}^{\oplus} \rho_{\lambda^j} \mathrm{d}\nu^j\left(\lambda^j\right)$ of G as bounded sequence of operators on the $L^{(1+\epsilon)}$-direct integral $\left(\int_{\mathcal{E}}^{\oplus} L^{(1+\epsilon)} \sum_j \left(X, \lambda^j\right) \mathrm{d}\nu^j\left(\lambda^j\right)\right)_{L^{(1+\epsilon)}}$. Claim that the canonical representation $\rho_{\sum_j \mu^j}$ on $L^{(1+\epsilon)}\left(X, \mu^j\right)$ and the representation $\rho_{(1+\epsilon)}$ on $\left(\int_{\mathcal{E}}^{\oplus} L^{(1+\epsilon)} \sum_j \left(X, \lambda^j\right) \mathrm{d}\nu^j\left(\lambda^j\right)\right)_{L^{(1+\epsilon)}}$ correspond under the isomorphism S between these spaces. To see this, let $f_j \in V \subseteq \mathcal{L}^{(1+\epsilon)}\left(X, \mu^j\right)$ and $g_j \in G$. Then $\tilde{\rho}\left(g_j\right) f_j \in L^{(1+\epsilon)}\left(X, \lambda^j\right)$ for all $\lambda^j \in \mathcal{E}$, so that $\sum_j \left(s^j\right)_{\tilde{\rho}}\left(g_j\right) f_j\left(\lambda^j\right) = \sum_j \left[\tilde{\rho}\left(g_j\right) f_j\right]_{\lambda^j}$ for all $\lambda^j \in \mathcal{E}$. Unwinding the definitions, then see that

$$
\begin{aligned}
& S\sum_j \left(\rho_{\sum_j \mu^j}\left(g_j\right)\left[f_j\right]_{\sum_j \mu^j}\right) \\
& = S\sum_j \left(\left[\tilde{\rho}\left(g_j\right) f_j\right]_{\sum_j \mu^j}\right) = \sum_j \left[s_{\tilde{\rho}}^j\left(g_j\right) f_j\right]_{\nu^j} \\
& = \sum_j \left[\lambda^j \mapsto \left[\tilde{\rho}\left(g_j\right) f_j\right]_{\lambda^j}\right]_{\nu^j} = \sum_j \left[\lambda^j \mapsto \rho_{\lambda^j}\left(g_j\right)\left[f_j\right]_{\lambda^j}\right]_{\nu^j} \\
& = \left(\int_{\mathcal{E}}^{\oplus} \sum_j \rho_{\lambda^j}\left(g_j\right) \mathrm{d}\nu^j\left(\lambda^j\right)\right)_{L^{(1+\epsilon)}}\left(\left[\lambda^j \mapsto \left[f_j\right]_{\lambda^j}\right]_{\nu^j}\right) \\
& = \left(\int_{\mathcal{E}}^{\oplus} \sum_j \rho_{\lambda^j}\left(g_j\right) \mathrm{d}\nu^j\left(\lambda^j\right)\right)_{L^{(1+\epsilon)}}\left(\left[s_{f_j}^j\right]_{\nu^j}\right) \\
& = \left[\left(\int_{\mathcal{E}}^{\oplus} \sum_j \rho_{\lambda^j} \mathrm{d}\nu^j\left(\lambda^j\right)\right)_{L^{(1+\epsilon)}}\left(g_j\right)\right]\left(S\left(\left[f_j\right]_{\sum_j \mu^j}\right)\right).
\end{aligned}
$$

By the density of the $\sum_j \mu^j$-equivalence classes of the simple functions in $L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$, the claim then follows.

Collect some of the main results so far in the following theorem. The added final part follows from part 4 of Proposition 2.2, and it shows that the canonical representation of G as isometric lattice automorphisms of $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$ can be disintegrate edicto order indecomposable similar representations.

Theorem 4.9. Let (G,X) be a Polish topological dynamical system, where G islocally compact. Suppose that there exists an invariant Borel probability measure $\sum_j \mu^j$ on X. Let $\mathcal{E}$ be the non-empty set of ergodic Borel probability measures on X, and supply $\mathcal{E}$ with the weak*-topology induced by $C_b(X)$.

Let $\beta: X \to \mathcal{E}$ be a decomposition map as in Theorem 4.2, and let υ^j be the push-forward measure of $\sum_j \mu^j$ via β, so that υ^j is a Borel probability measure on $\mathcal{E}$ that is independent of the choice of β.

Let $(0 \le \epsilon < \infty)$.

(1) Let V be the vector space of simple functions on X. Then $\left\{\|\cdot\|_{\lambda^j}\right\}_{\lambda^j \in \mathcal{E}}$ are measurable family of semi-series norms on V. The resulting family of completions of the spaces $\left(V/\ker \sum_j \|\cdot\|_{\lambda^j}, \sum_j \|\cdot\|_{\lambda^j}\right)$ are the family $\left\{L^{(1+\epsilon)}\left(X,\lambda^j\right)\right\}_{\lambda^j \in \mathcal{E}}$, which are measurable family of Banach lattices over $\left(\mathcal{E}, v^j, V\right)$. Therefore, the $L^{(1+\epsilon)}$-direct integral $\left(\int_{\mathcal{E}}^{\oplus} L^{(1+\epsilon)} \sum_j \left(X,\lambda^j\right) \mathrm{d}v^j\left(\lambda^j\right)\right)_{L^{(1+\epsilon)}}$ of this family with respect to v^j can be defined, and this space is a Banach lattice;

(2) Define $S: L^{(1+\epsilon)} \sum_j \left(X, \sum_j \mu^j\right) \to \left(\int_{\mathcal{E}}^{\oplus} L^{(1+\epsilon)} \sum_j \left(X,\lambda^j\right) \mathrm{d}v^j\left(\lambda^j\right)\right)$ by $\sum_j S\left(\left[f_j\right]_{\sum_j \mu^j}\right) := \sum_j \left[\left(S\right)_{f_j}\right]_{v^j}\left(f_j \in L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)\right)$, where $\left(s^j\right)_{f_j}$ are as defined in (7). Then S is an isometric lattice isomorphism between the Banach lattices $L^{(1+\epsilon)} \sum_j \left(X, \sum_j \mu^j\right)$ and $\left(\int_{\mathcal{E}}^{\oplus} L^{(1+\epsilon)} \sum_j \left(X,\lambda^j\right) \mathrm{d}v^j\left(\lambda^j\right)\right)_{L^{(1+\epsilon)}}$;

(3) S is an intertwining sequence of operators between the canonical representations $\rho_{\sum_j \mu^j}$ of G as isometric lattice automorphisms of $L^{(1+\epsilon)}\left(X,\mu^j\right)$ and the representation $\left(\int_{\mathcal{E}}^{\oplus} \sum_j \rho_{\lambda^j} \mathrm{d}v^j\left(\lambda^j\right)\right)_{L^{(1+\epsilon)}}$ on $\left(\int_{\mathcal{E}}^{\oplus} L^{(1+\epsilon)} \sum_j \left(X,\lambda^j\right) \mathrm{d}v^j\left(\lambda^j\right)\right)_{L^{(1+\epsilon)}}$, which is the $L^{(1+\epsilon)}$-direct integral of the family $\left\{\rho_{\lambda^j}\right\}_{\lambda^j \in \mathcal{E}}$ of canonical representations of G as isometric lattice automorphisms on the Banach lattices $L^{(1+\epsilon)}\left(X,\lambda^j\right)\left(\lambda^j \in \mathcal{E}\right)$. That is, for all $g_j \in G$, the following diagram commutes:

$$
\begin{tikzcd}
L^{(1+\epsilon)} \sum\limits_j \left(X, \sum\limits_j \mu^j\right) \arrow[r, "\sum\limits_j \rho_{\sum_j \mu^j}(g_j)"] \arrow[d, "S"] & L^{(1+\epsilon)} \sum\limits_j \left(X, \sum\limits_j \mu^j\right) \arrow[d, "S"] \\
\left(\int_{\mathcal{E}}^{\oplus} L^{(1+\epsilon)} \sum\limits_j \left(X,\lambda^j\right) \mathrm{d}v^j\left(\lambda^j\right)\right)_{L^{(1+\epsilon)}} \arrow[r, "\left(\int_{\mathcal{E}}^{\oplus} \sum\limits_j \rho_{\lambda^j}(g_j) E v^j(\lambda^j)\right)_{L^{(1+\epsilon)}}"] & \left(\int_{\mathcal{E}}^{\oplus} L^{(1+\epsilon)} \sum\limits_j \left(X,\lambda^j\right) \mathrm{d}v^j\left(\lambda^j\right)\right)_{L^{(1+\epsilon)}}
\end{tikzcd}
$$

(4) For all $\lambda^j \in \mathcal{E}$, the representation ρ_{λ^j} of G on the fiber $L^{(1+\epsilon)}\left(X,\lambda^j\right)$ is

order indecomposable.

Remark 4.10. The strong continuity of the $L^{(1+\epsilon)}$-direct integral of representations was briefly addressed. Although strong continuity played no role in the proofs, let us still mention that in the present context this is automatic: according to Corollary 2.8, $\rho_{\sum_j \mu^j}$ and hence $\left(\int_{\mathcal{E}}^{\oplus} \sum_j \rho_{\lambda^j} \mathrm{d}v^j\left(\lambda^j\right)\right)_{L^{(1+\epsilon)}}$ and all $\rho_{\lambda^j}\left(\lambda^j \in \mathcal{E}\right)$ are strongly continuous representations.

4.3. Worked Example

Conclude this section with a simple example of a representation that disintegrate explicitly.

Let $\mathbb{D} := \left\{z \in \mathbb{C} : |z| \le 1\right\}$ and let $T^j := \left\{z \in \mathbb{C} : |z| = 1\right\}$. Then $\left(T^j, D\right)$ is Polish topological dynamical system with compact group when supplied with the rotation action: $\left(z_1, z_2\right) \mapsto z_1 z_2 \left(z_1 \in T^j, z_2 \in \mathbb{D}\right)$. Let μ^j be the normalized restriction of the Lebesgue measure on $\mathbb{R}^2$ to the Borel σ^j-algebra of $\mathbb{D}$. Then $\sum_j \mu^j$ are $\mathbb{T}$-invariant Borel probability measure on *X*. Fix $\left(0 \le \epsilon < \infty\right)$. The aim is to exhibit an explicit disintegration of $L^{(1+\epsilon)}\left(D, \sum_j \mu^j\right)$ and the representation of $\rho_{\sum_j \mu^j}$ of $\mathbb{T}$ on this space, as provided in abstract by Theorem 4.9.

The first step is to determine the set $\mathcal{E}$ of ergodic Borel probability measures on $\mathbb{D}$. Know from part 3 of Remark 4.3 that these measures are in one-to-one correspondence with the orbits of $\mathbb{T}$, *i.e.* with the elements of the interval $[0,1]$ that parameterizes the radius of the orbits. From (2) we infer an explicit formula for the ergodic Borel probability measure $\left(\lambda^j\right)_{(1+\epsilon)}$ corresponding to an orbit of radius $\left(0 \le \epsilon \le \infty\right)$, namely

$$\sum_j \left(\lambda^j\right)_{(1+\epsilon)}(Y) = \frac{1}{2\pi} \int_{[0,2\pi]} \mathbf{1}_Y\left(\left(1+\epsilon\right) \mathrm{e}^{i\theta}\right) \mathrm{d}\theta, \tag{10}$$

where *Y* is a Borel subset of $\mathbb{D}$. More generally, if $f_j : \mathbb{D} \to \mathbb{R}$ is a bounded Borel measurable function, then (3) gives, for $\left(0 \le \epsilon \le \infty\right)$,

$$\int_{\mathbb{D}} \sum_j f_j(z) \mathrm{d}\left(\lambda^j\right)_{(1+\epsilon)}(z) = \frac{1}{2\pi} \int_{[0,2\pi]} \sum_j f_j\left(\left(1+\epsilon\right) \mathrm{e}^{i\theta}\right) \mathrm{d}\theta. \tag{11}$$

The second step is to determine $\mathcal{E}$ as a topological space; recall that $\mathcal{E}$ is endowed with the weak*-topology via the inclusion $\mathcal{E} \subseteq C_b\left(\mathbb{D}\right)^*$. Know that $\varphi_j : [0,1] \to \mathcal{E}$, given by $\varphi_j\left(\left(1+\epsilon\right)\right) = \left(\lambda^j\right)_{(1+\epsilon)}$, is a bijection; we claim that it is even a homeomorphism. Tosee this, let $\left(\left(1+\epsilon\right)_n\right)_{n \in \mathbb{N}} \subseteq [0,1]$ and let $\left(1+\epsilon\right)_n \to \left(0 \le \epsilon \le \infty\right)$ as $n \to \infty$. If $f_j \in C_b\left(\mathbb{D}\right)$, then, using (11) and the dominated convergence theorem, see that

$$\begin{aligned} &\int_{\mathbb{D}} \sum_j f_j(z) \mathrm{d}\left(\lambda^j\right)_{(1+\epsilon)_n}(z) = \frac{1}{2\pi} \int_{[0,2\pi]} \sum_j f_j\left(\left(1+\epsilon\right)_n \mathrm{e}^{i\theta}\right) \mathrm{d}\theta \\ &\to \frac{1}{2\pi} \int_{[0,2\pi]} \sum_j f_j\left(\left(1+\epsilon\right) \mathrm{e}^{i\theta}\right) \mathrm{d}\theta = \int_{\mathbb{D}} \sum_j f_j(z) \mathrm{d}\left(\lambda^j\right)_{(1+\epsilon)}(z) \end{aligned}$$

as $n \to \infty$. Hence φ_j are continuous. Since $[0,1]$ is compact and $\mathcal{E}$ is

Hausdorff, conclude that φ_j are home omorphism.

The third step is to find a decomposition map $\beta:\mathbb{D}\to\mathcal{E}$. In this case, this map is uniquely determined by parts 1 and 2 of Theorem 4.2. Indeed, let $(0\le\epsilon<\infty)$. Then part 2 shows that $\beta^{-1}\left(\left\{\sum_j(\lambda^j)_{(1+\epsilon)}\right\}\right)$ cannot be disjoint from the orbit $\mathbb{T}_{(1+\epsilon)}$ and subsequently part 1implies that this set contains the entire orbit. Since $\beta^{-1}\left(\left\{\sum_j(\lambda^j)_{(1+\epsilon)_1}\right\}\right)$ and $\beta^{-1}\left(\left\{\sum_j(\lambda^j)_{(1+\epsilon)_2}\right\}\right)$ are obviously disjoint for $(1+\epsilon)_1\ne(1+\epsilon)_2$, must have $\beta^{-1}\left(\left\{\sum_j(\lambda^j)_{(1+\epsilon)}\right\}\right)=\mathbb{T}_{(1+\epsilon)}$ .Conclude that $\beta_{(1+\epsilon)e^{i\theta}}=(\lambda^j)_{(1+\epsilon)}$ for $(0\le\epsilon\le\infty)$ and $\theta\in\mathbb{R}$ so that β is, in fact, uniquely determined. Know a priori from Theorem 4.2 that β is Borel measurable, but this can also be seen directly. In fact, β is even continuous, because $\varphi_j^{-1}\circ\beta:\mathbb{D}\to[0,1]$ is continuous it maps $(1+\epsilon)e^{i\theta}$ to $(1+\epsilon)$, and hence so is $\beta=\varphi_j\circ(\varphi_j^{-1}\circ\beta)$.

Also know a priori that part 3 of Theorem 4.2 is satisfied for the $\sum_j\mu^j$, but using (10) this can also be seen directly. Indeed, using polar coordinates have, for a Borel subset Y of $\mathbb{D}$,

$$\begin{aligned}\sum_j\mu^j(Y)&=\frac{1}{\pi}\int_{[0,1]}\int_{[0,2\pi]}(1+\epsilon)1_Y\left((1+\epsilon)e^{i\theta}\right)d\theta d(1+\epsilon)\\&=2\int_{[0,1]}(1+\epsilon)\sum_j(\lambda^j)_{(1+\epsilon)}(Y)d(1+\epsilon)\\&=\frac{1}{\pi}\int_{[0,1]}\int_{[0,2\pi]}(1+\epsilon)\beta_{(1+\epsilon)e^{i\theta}}(Y)d\theta d(1+\epsilon)\\&=\int_{\mathbb{D}}\beta_z(Y)d\sum_j\mu^j(z)\end{aligned}$$

Theorem 4.9 gives a disintegration of the action of $\mathbb{T}$ on $L^{(1+\epsilon)}\left(\mathbb{D},\sum_j\mu^j\right)$ as an $L^{(1+\epsilon)}$-direct integral of representations with $\mathcal{E}$ as underlying point set, but it is more intuitive to formulate this with $[0,1]$, which is homeomorphic to $\mathcal{E}$, as underlying point set.

Therefore, let ν^j be the push-forward measure of $\sum_j\mu^j$ via $\varphi_j^{-1}\circ\beta:\mathbb{D}\to[0,1]$. Thus, if A_j is a Borel subset of $[0,1]$, then $\sum_j\nu^j(A_j)=\sum_j\mu^j\left(\beta^{-1}\circ\varphi_j(A_j)\right)=\sum_j\mu^j\left(\left\{z\in\mathbb{C}\,|z|\in A_j\right\}\right)$.

Using polar coordinates, obtain that

$$\begin{aligned}\sum_j\nu^j(A_j)&=\sum_j\mu^j\left(\left\{(1+\epsilon)e^{i\theta}:(1+\epsilon)\in A_j\right\}\right)\\&=\frac{1}{\pi}\int_{[0,1]}\int_{[0,2\pi]}(1+\epsilon)\sum_j\mathbf{1}_{A_j}\left(\left|(1+\epsilon)e^{i\theta}\right|\right)d\theta d(1+\epsilon)\\&=\int_{[0,1]}\sum_j\mathbf{1}_{A_j}\cdot2(1+\epsilon)d(1+\epsilon)\end{aligned}$$

Conclude that ν^j is the measure $2(1+\epsilon)d(1+\epsilon)$ on the Borel subsets of $[0,1]$. For abounded Borel measurable function f_j on $\mathbb{D}$, part 2 of the factorization Theorem 4.5 then takes the form

$$\begin{aligned}&\frac{1}{\pi}\int_{\mathbb{D}}\sum_j f_j(z)\,\mathrm{d}\sum_j \mu^j(z)\\&=\int_{[0,1]}\left(\frac{1}{2\pi}\int_{[0,2\pi]}\sum_j f_j\left((1+\epsilon)\mathrm{e}^{i\theta}\right)\mathrm{d}\theta\right)2(1+\epsilon)\,\mathrm{d}(1+\epsilon)\end{aligned}\tag{12}$$

where (11) has been used. The validity of this formula in itself is, of course, clear; the point is its interpretation as an instance of the factorization in Theorem 4.5.

Let V be the vector lattice of simple functions on $\mathbb{D}$. According to Theorem 4.9, $\left\{\sum_j \|\cdot\|_{(\lambda^j)_{(1+\epsilon)}}\right\}_{(0\le\epsilon\le\infty)}$ are measurable family of semi-norms on V, so that $\left\{L^{(1+\epsilon)}\left(\mathbb{D},\sum_j(\lambda^j)_{(1+\epsilon)}\right)\right\}_{(0\le\epsilon\le\infty)}$ are measurable family of Banach lattices over $\left([0,1],\nu^j,V\right)$, and the $L^{(1+\epsilon)}$-direct integral $\left(\int_{[0,1]}^{\oplus}L^{(1+\epsilon)}\left(\mathbb{D},\sum_j(\lambda^j)_{(1+\epsilon)}\right)2(1+\epsilon)\,\mathrm{d}(1+\epsilon)\right)_{L^{(1+\epsilon)}}$ can be defined. Let $S:L^{(1+\epsilon)}\sum_j\left(\mathbb{D},\sum_j\mu^j\right)\to\left(\int_{[0,1]}^{\oplus}L^{(1+\epsilon)}\left(\mathbb{D},(\lambda^j)_{(1+\epsilon)}\right)2(1+\epsilon)\,\mathrm{d}(1+\epsilon)\right)_X$ be such that $S\sum_j\left([f_j]_{\mu^j}\right):=\sum_j\left[(s^j)_{f_j}\right]_{2(1+\epsilon)\mathrm{d}(1+\epsilon)}$, where $\sum_j(s^j)_{f_j}((1+\epsilon))=\sum_j[f_j]_{(\lambda^j)_{(1+\epsilon)}}$ if $f_j\in\mathcal{L}^{(1+\epsilon)}\left(\mathbb{D},(\lambda^j)_{(1+\epsilon)}\right)$, and $\sum_j(s^j)_{f_j}((1+\epsilon))=\sum_j[0]_{(\lambda^j)_{(1+\epsilon)}}$ if $f_j\notin\mathcal{L}^{(1+\epsilon)}\left(\mathbb{D},(\lambda^j)_{(1+\epsilon)}\right)$. The latter exceptional set is Borel measurable and has $2(1+\epsilon)\mathrm{d}(1+\epsilon)$-measure zero. Equivalently, it has $\mathrm{d}(1+\epsilon)$-measure zero, likewise, could have written $\sum_j\left[(s^j)_{f_j}\right]_{\mathrm{d}(1+\epsilon)}$ for $\sum_j\left[(s^j)_{f_j}\right]_{2(1+\epsilon)\mathrm{d}(1+\epsilon)}$. Then, according to Theorem 4.9, S is a well-defined isometric lattice isomorphism between $L^{(1+\epsilon)}\left(\mathbb{D},\sum_j\mu^j\right)$ and $\left(\int_{[0,1]}^{\oplus}L^{(1+\epsilon)}\left(\mathbb{D},\sum_j(\lambda^j)_{(1+\epsilon)}\right)2(1+\epsilon)\,\mathrm{d}(1+\epsilon)\right)_{L^{(1+\epsilon)}}$. If f_j is a bounded Bore measurable function on $\mathbb{D}$, then the exceptional set is empty, and, using (11), the isometric nature of S at the point $\sum_j[f_j]_{\sum_j\mu^j}\in L^{(1+\epsilon)}\left(X,\sum_j\mu^j\right)$ is an application of (12) to $\sum_j|f_j|^{(1+\epsilon)}$:

$$\begin{aligned}\sum_j\left\|[f_j]\right\|_{\sum_j\mu^j}^{(1+\epsilon)}&=\frac{1}{\pi}\int_{\mathbb{D}}\sum_j|f_j(z)|^{(1+\epsilon)}\,\mathrm{d}\sum_j\mu^j(z)\\&=\int_{[0,1]}\frac{1}{2\pi}\left(\int_{[0,2\pi]}\sum_j\left|f_j\left((1+\epsilon)\mathrm{e}^{i\theta}\right)\right|^{(1+\epsilon)}\mathrm{d}\theta\right)2(1+\epsilon)\,\mathrm{d}(1+\epsilon)\\&=\int_{[0,1]}\left(\int_{\mathbb{D}}\sum_j|f_j(z)|^{(1+\epsilon)}\,\mathrm{d}(\lambda^j)_{(1+\epsilon)}(z)\right)2(1+\epsilon)\,\mathrm{d}(1+\epsilon)\\&=\int_{[0,1]}\sum_j\left\|(s^j)_{f_j}((1+\epsilon))\right\|_{(\lambda^j)_{(1+\epsilon)}}^{(1+\epsilon)}\;2(1+\epsilon)\,\mathrm{d}(1+\epsilon)\\&=\sum_j\left\|S\left([f_j]_{\sum_j\mu^j}\right)\right\|_{(1+\epsilon)}^{(1+\epsilon)}.\end{aligned}$$

Furthermore, S is an intertwining operator between the canonical representations $\rho_{\sum_j \mu^j}$ of $\mathbb{T}$ on $L^{(1+\epsilon)}\left(\mathbb{D},\sum_j \mu^j\right)$ and the $L^{(1+\epsilon)}$-direct integral $\left(\int_{[0,1]}^{\oplus}\sum_j \rho_{\left(\lambda^j\right)_{(1+\epsilon)}} 2(1+\epsilon)\mathrm{d}(1+\epsilon)\right)_{L^{(1+\epsilon)}}$ of the order indecomposable representations $\rho_{\left(\lambda^j\right)_{(1+\epsilon)}}$ of $\mathbb{T}$ on the spaces $L^{(1+\epsilon)}\left(\mathbb{D},\left(\lambda^j\right)_{(1+\epsilon)}\right)$. That is, for all $z\in\mathbb{T}$, the diagram

$$
\begin{array}{ccc}
L^{(1+\epsilon)}\left(\mathbb{D},\sum_j \mu^j\right) & \xrightarrow{\sum_j \rho_{\sum_j \mu^j}(z)} & L^{(1+\epsilon)}\left(\mathbb{D},\sum_j \mu^j\right) \\
S\downarrow & & S\downarrow \\
\left(\int_{[0,1]}^{\oplus} L^{(1+\epsilon)}\left(\mathbb{D},\sum_j\left(\lambda^j\right)_{(1+\epsilon)}\right)2(1+\epsilon)\mathrm{d}(1+\epsilon)\right)_{L^{(1+\epsilon)}} & \xrightarrow{\left(\int_{[0,1]}^{\oplus}\sum_j \rho_{\left(\lambda^j\right)_{(1+\epsilon)}}(z)2(1+\epsilon)\mathrm{d}(1+\epsilon)\right)_{L^{(1+\epsilon)}}} & \left(\int_{[0,1]}^{\oplus} L^{(1+\epsilon)}\left(\mathbb{D},\sum_j\left(\lambda^j\right)_{(1+\epsilon)}\right)2(1+\epsilon)\mathrm{d}(1+\epsilon)\right)_{L^{(1+\epsilon)}}
\end{array}
\tag{13}
$$

is commutative.

Intuitively, this is certainly plausible, since "restricting a function to an orbit" is clearly a $\mathbb{T}$-equivariant operation, and the commutativity of diagram (13) merely reflects that this is what the sequence of operator S tries to do. Write "tries to do", and not "does", because "restricting" is meaningless for the elements of the actual domain of S, which are $\sum_j \mu^j$-equivalence classes of measurable functions. The "actual" intertwining statement in Theorem 4.9 is, therefore, that this intuitive observation can be modified into a form that is meaningful and that survives during the measure-theoretical constructions. In this case, it comes down to the following.

If there is an empty exceptional set in the definition of $\left(s^j\right)_{f_j}$ for $f_j\in\mathcal{L}^{(1+\epsilon)}\left(\left(\mathbb{D},\sum_j \mu^j\right)\right)$, then, for each fixed $(0\le\epsilon<\infty)$, the value $\sum_j\left(s^j\right)_{f_j}(1+\epsilon)=\left[f_j\right]_{\left(\lambda^j\right)_{(1+\epsilon)}}$ are determined by the restriction of f_j to the corresponding orbit of radius $(1+\epsilon)$. Since the characteristic function of this orbit are $\sum_j \mu^j$-almost everywhere zero, it is likewise that $\sum_j\left[f_j\right]_{\left(\lambda^j\right)_{(1+\epsilon)}}$ always depends on the choice of the representative f_j of $\sum_j\left[f_j\right]_{\sum_j \mu^j}$. Nevertheless, the $2(1+\epsilon)\mathrm{d}(1+\epsilon)$-equivalence class of the section $(1+\epsilon)\mapsto\sum_j\left(s^j\right)_{f_j}(1+\epsilon)=\sum_j\left[f_j\right]_{\left(\lambda^j\right)_{(1+\epsilon)}}$ does not depend on this choice. Moreover, the map S sending $\sum_j\left[f_j\right]_{\sum_j \mu^j}$ to this $2(1+\epsilon)-\mathrm{d}(1+\epsilon)$-equivalence class is $\mathbb{T}$-equivariant.

Furthermore, this can still be made to work when there is a non-empty exceptional set in the definition of $\sum_j\left(s^j\right)_{f_j}(1+\epsilon)$; *i.e.* when $(1+\epsilon)$-integrability of f_j are lost when f_j are restricted to certain orbits. For each fixed orbit, there are evidently $f_j \in L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ for which this is the case, but for all fixed $f_j \in \mathcal{L}^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ there are $2(1+\epsilon)\mathrm{d}(1+\epsilon)$-almost none of such orbits.

5. Disintegration: General Case

In section 4, started with a topological dynamical system (G, K) and a G invariant Borel probability measure on *X*. In that context, there existed canonically associated strongly continuous representations of G as isometric lattice automorphisms of the spaces $L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$, $(0 \le \epsilon < \infty)$ that fix the constants.

In the current section, we turn the tables. Start with an (at first) abstract group G and a probability spaces $\left(X, \sum_j \mu^j\right)$, and assume that, for some $(0 \le \epsilon < \infty)$, G acts as isometric lattice automorphisms of $L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ such that the constants are left fixed. It is then established that, in fact, G acts naturally in a similar manner on $L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ for all $(0 \le \epsilon < \infty)$ see Corollary 5.10. Furthermore, if G is a locally compact Hausdorff group and the original representation is strongly continuous, then it is shown that there is an isomorphic model in which this G-action on all $L^{(1+\epsilon)}$-spaces originates canonically from a measure preserving continuous G-action on a compact Hausdorff space; see Theorem 5.14. Under mild additional assumptions, then conclude from our disintegration Theorem 4.9 that, even though there was originally no action of G on an underlying point set, the original representation(s) of G on $L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ can still be disintegrated into order indecomposable representations as an $L^{(1+\epsilon)}$-direct integral. This leads to Theorem 5.15, which is an ordered relative of the general unitary disintegration.

Remark 5.1. It follows from the combination of [[12], Vol. I, Exercise 1.12.102], [[4], Vol. II, Example 6.5.2], and [[12], Vol. I Exercise 4.7.63] that, for $(0 \le \epsilon < \infty)$

$L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ is always separable whenever X is a separable metric space and $\sum_j \mu^j$ are Borel probability measure on *X*. Therefore, the representation spaces are all separable. Furthermore, have observed in Remark 4.10 that the representations on the relevant spaces are all strongly continuous. Neither of the seproperties has played a role in the proofs so far. Quite to the contrary, in the current section both properties will be essential in order to be able to exhibit a model in Theorem 5.14 to which the disintegration Theorem 5.15 can subsequently be applied.

Remark 5.2. With the exception of Remark 5.8, the combination of ideas, arguments and results in Lemma 5.3 up to and including Theorem 5.14 is rather similar to that in [9]. Unfortunately, we cannot directly apply results. The reason is that the so-called Markov operators on $L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ that are considered are positive operators T^j that fix the constants and satisfy $\int_X \sum_j f_j \mathrm{d}\sum_j \mu^j = \int_X \sum_j f_j \mathrm{d}\sum_j \mu^j$ for all $f_j \in L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$. The point of departure, where $\mathbb{T}$ preserves the norm rather than the integral, and is a lattice homomorphism rather than merely a positive sequence of operators, are different. This necessitates an independent, albeit similar, development; see also Remark 5.6.

Begin by showing that representations of an abstract group G as isometric lattice automorphisms that fix the constants come in families. There will be only one measure section, and happily resort to the usual practice of ignoring the distinction between equivalence classes of functions and their representatives.

Start with the following key observation.

Lemma 5.3. Let $\left(X,\sum_j \mu^j\right)$ be a probability space, and let $T^j : L^\infty\left(X,\sum_j \mu^j\right) \to L^\infty\left(X,\sum_j \mu^j\right)$ be a lattice homomorphism that fixes the constants. Then $\sum_j T^j\left(\left|f_j\right|^{(1+\epsilon)}\right) = \sum_j \left|T^j\left(f_j\right)\right|^{(1+\epsilon)}$ for all $f_j \in L^\infty\left(X,\mu^j\right)$ and all $(0 \le \epsilon < \infty)$

Lemma 5.4. Let $\left(X,\mu^j\right)$ be a probability space, and let $T^j : L^\infty\left(X,\mu^j\right) \to L^\infty\left(X,\mu^j\right)$ be a lattice homomorphism that fixes the constants. Then the following are equivalent:

(1) $\int_X \sum_j T^j\left(f_j\right) \mathrm{d}\sum_j \mu^j = \int_X \sum_j f_j \mathrm{d}\sum_j \mu^j$ for all $f_j \in L^\infty\left(X,\sum_j \mu^j\right)$;

(2) $\sum_j \left\|T^j\left(f_j\right)\right\|_1 = \sum_j \left\|f_j\right\|_1$ for all $f_j \in L^\infty\left(X,\mu^j\right)$;

(3) There exists $(0 \le \epsilon < \infty)$ such that $\sum_j \left\|T^j\left(f_j\right)\right\|_{(1+\epsilon)} = \sum_j \left\|f_j\right\|_{(1+\epsilon)}$ for all $f_j \in L^\infty\left(X,\mu^j\right)$;

(4) For all $0 \le \epsilon < \infty$, have $\sum_j \left\|T^j\left(f_j\right)\right\|_{(1+\epsilon)} = \sum_j \left\|f_j\right\|_{(1+\epsilon)}$ for all $f_j \in L^\infty\left(X,\sum_j \mu^j\right)$.

Proof. To see that (1) implies (4), we use Lemma 5.3 to note that $\sum_j \left\|T^j\left(f_j\right)\right\|_{(1+\epsilon)}^{(1+\epsilon)} = \int_X \sum_j \left|T^j\left(f_j\right)\right|_{(1+\epsilon)} \mathrm{d}\sum_j \mu^j = \int_X \sum_j T^j\left(\left|f_j\right|^{(1+\epsilon)}\right) \mathrm{d}\sum_j \mu^j = \int_X \sum_j \left|f_j\right|^{(1+\epsilon)} \mathrm{d}\sum_j \mu^j = \left\|f_j\right\|_{(1+\epsilon)}^{(1+\epsilon)}$. It is clear that (4) implies (3).

To see that (3) implies (2), invoke Lemma 5.3 to compute as follows:

$$\begin{aligned}
\sum_j \left\|T^j\left(f_j\right)\right\|_1 &= \int_X \sum_j \left|T^j\left(f_j\right)\right| \mathrm{d}\sum_j \mu^j = \int_X \sum_j T^j\left(\left|f_j\right|\right) \mathrm{d}\sum_j \mu^j \\
&= \int_X \sum_j T^j\left(\left|\left(\left|f_j\right|^{1/(1+\epsilon)}\right)\right|^{(1+\epsilon)}\right) \mathrm{d}\sum_j \mu^j \\
&= \int_X \sum_j \left|T^j\left(\left|f_j\right|^{1/(1+\epsilon)}\right)\right|^{(1+\epsilon)} \mathrm{d}\sum_j \mu^j \\
&= \sum_j \left\|T^j\left(\left|f_j\right|^{1/(1+\epsilon)}\right)\right\|_{(1+\epsilon)}^{(1+\epsilon)} \\
&= \sum_j \left\|\left|f_j\right|^{1/(1+\epsilon)}\right\|_{(1+\epsilon)}^{(1+\epsilon)} = \sum_j \left\|f_j\right\|_1
\end{aligned}$$

To see that (2) implies (1), note that the equality in (1) is just the one in (2) if $f_j \ge 0$ (note that $T^j f_j \ge 0$ then. For general f_j, then have

$$
\begin{aligned}
&\int_X \sum_j T^j\left(f_j\right)\mathrm{d}\sum_j \mu^j \\
&= \int_X \sum_j \left(T^j\left(f_j\right)\right)^+ \mathrm{d}\sum_j \mu^j - \int_X \sum_j \left(T^j\left(f_j\right)\right)^- \mathrm{d}\sum_j \mu^j \\
&= \int_X \sum_j T^j\left(f_j^+\right)\mathrm{d}\sum_j \mu^j - \int_X \sum_j T^j\left(f_j^-\right)\mathrm{d}\sum_j \mu^j \\
&= \int_X \sum_j f_j^+ \mathrm{d}\mu^j - \int_X \sum_j f_j^- \mathrm{d}\mu^j = \int_X \sum_j f_j \mathrm{d}\mu^j
\end{aligned}
$$

Fix $0 \le \epsilon < \infty$, and consider a lattice homomorphisms $\sum_j T^j : L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right) \to L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ that leaves the constants fixed. Then T^j, being a positive sequence of operators on a Banachlattice, is continuous in the $(1+\epsilon)$-norm. Furthermore, T^j leaves $L^\infty\left(X, \sum_j \mu^j\right)$ invariant. If $f_j \in L^\infty\left(X, \sum_j \mu^j\right)$, then $\sum_j \left|f_j\right| \le \left\|\sum_j f_j\right\|_\infty \mathbf{1}_X$ in the lattice $L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$. An application of T^j shows that $T^j\left(f_j\right)$ are in $L^\infty(X)$ again, and also that $\sum_j T^j : L^\infty\left(X, \sum_j \mu^j\right) \to L^\infty\left(X, \sum_j \mu^j\right)$ are contractive in the supermom-norm. For later use, let us note that the latter implies that a group of lattice automorphisms of $L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ that fixes the constants automatically acts on $L^\infty(X)$ as isometric lattice automorphisms.

Using continuity and density arguments, the following result is an easy consequence of Lemma 5.4.

Lemma 5.5. Let $\left(X, \sum_j \mu^j\right)$ be a probability space, let $(0 \le \epsilon < \infty)$, and let $\sum_j T^j : L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right) \to L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ be a lattice homomorphism that leaves $\mathbf{1}_X$ fixed. Then T^j leaves $L^\infty\left(X, \sum_j \mu^j\right)$ invariant, and the restriction of T^j to $L^\infty\left(X, \sum_j \mu^j\right)$ are contractive lattice homomorphism for the supermom-norm that leaves the constants fixed. Furthermore, the following are equivalent:

(1) $\int_X \sum_j T^j\left(f_j\right)\mathrm{d}\sum_j \mu^j = \int_X \sum_j f_j \mathrm{d}\sum_j \mu^j$ for all $f_j \in L^\infty\left(X, \sum_j \sum_j \mu^j\right)$;

(2) $\int_X \sum_j T^j\left(f_j\right)\mathrm{d}\mu^j = \int_X \sum_j f_j \mathrm{d}\mu^j$ for all $f_j \in L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$;

(3) $\sum_j \left\|T^j\left(f_j\right)\right\|_{(1+\epsilon)} = \sum_j \left\|f_j\right\|_{(1+\epsilon)}$ for all $f_j \in L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$.

Remark 5.6. In the terminology of [[9], Section 13.1], the equivalence of (2) and (3) in Lemma 5.5 implies that a lattice homomorphism $T^j : L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right) \to L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ that leaves the constants fixed is a Markov operator on $L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ precisely when it is an isometry.

Note that $(1+\epsilon)$ is absent from part 1 of Lemma 5.5, but present in parts 2 and 3.

Lemma 5.4 has similar features. Using restriction to, and extension from, the common dense subspace $L^\infty\left(X, \sum_j \mu^j\right)$ of all spaces $L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)(0 \le \epsilon < \infty)$, one readily obtains the following result.

Lemma 5.7. Let $\left(X, \sum_j \mu^j\right)$ be a probability space, let $(0 \le \epsilon < \infty)$ and let $\sum_j T^j : L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right) \to L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ be an isometric lattice homomorphism that leaves the constants fixed. Then T^j leaves $L^{(1+\epsilon)}\left(X, T^j\right) \subseteq L^{(1+\epsilon)}\left(X, \sum_j T^j\right)$ invariant, and the restriction

$\sum_j T^j : L^{(1+\epsilon)}\left(X, \sum_j T^j\right) \to L^{(1+\epsilon)}\left(X, \sum_j T^j\right)$ are isometric lattice homomorphism that leaves the constants fixed. Moreover, every isometric lattice homomorphism of $L\left(X, \sum_j T^j\right)$ that leaves the constants fixed can thus be obtained from a unique.

Remark 5.8. There is an alternative way to understand why Lemma 5.7 holds.

According to Lamperti's theorem [[20], Theorem 3.2.5], the isometries of $L^{(1+\epsilon)}\left(X, \sum_j T^j\right)$ are, for $(0 \le \epsilon < \infty)$ the composition of a multiplication sequence of operators and the sequence of operators that are induced by a regular set isomorphism. An inspection of the proof shows that the theorem actually describes all disjointness preserving isometries; this disjointness preserving property being automatic if $\epsilon \neq 1$. Consequently, if $(0 \le \epsilon < \infty)$ is fixed, and $\sum_j T^j : L^{(1+\epsilon)}\left(X, \sum_j T^j\right) \to L^{(1+\epsilon)}\left(X, \sum_j T^j\right)$ is an isometric lattice isomorphism that fixes the constants, then the description in Lamperti's theorem applies to the operator. Since $\sum_j T^j$ fixes the constants, the multiplication operator is the identity, so that T^j is actually induced by a regular set isomorphism. Since $\sum_j T^j$ is an isometry, this regular set isomorphism must be measure preserving. It is then clear why and how $\sum_j T^j$ acts as isometric lattice automorphisms on all spaces $L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$: all these actions arise from the same underlying measure preserving regular set isomorphism.

At the cost of invoking Lamperti's result, and of some technical details of a different nature, a different proof of Lemma 5.7 can thus be given.

If $\left(X, \sum_j \mu^j\right)$ are probability space, and if $(0 \le \epsilon < \infty)$, then, as is well known, the topology that is induced on $\left\{f_j \in L^\infty\left(X, \sum_j \mu^j\right) : \sum_j \left\|f_j\right\|_\infty \le 1\right\}$ by $L^{(1+\epsilon)}\left(X, \mu^j\right)$ does not depend on $(1+\epsilon)$. As a first consequence, the spaces $L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ for $(0 \le \epsilon < \infty)$ are either all separable, or all non-separable; their separability is known to be equivalent to the separability of μ^j, (see e.g. [1]). As a second consequence, when combined with Lemma 5.7 and with the observed fact that a lattice homomorphism of $L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$, that leaves the constants fixed, automatically leaves $\left\{\sum_j f_j \in L^\infty\left(X, \sum_j \mu^j\right) : \sum_j \left\|f_j\right\|_\infty \le 1\right\}$ invariant, this $(1+\epsilon)$-independence of the topology yields the statement on the strong operator topology of the following result.

Proposition 5.9. Let $\left(X, \sum_j \mu^j\right)$ be a probability space, and let $(0 \le \epsilon < \infty)$. Then the semigroup/group of isometric lattice homomorphisms/automorphisms of $L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ into/onto itself that leaves the constants fixed is, via the restriction map, isomorphic to the semigroup/group of isometric lattice homomorphisms automorphisms of $L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ into/onto itself that leaves the constants fixed. This isomorphism is ahomeomorphism for both the strong and the weak operator topologies as induced from the bounded sequence of operators on $L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ and $L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$.

Thus have the following result concerning the type of representations always occurring in families.

Corollary 5.10. Let $\left(X,\sum_j \mu^j\right)$ be a probability space, let G be a group, and let $(0 \le \epsilon < \infty)$. Suppose that G acts on $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$ as isometric lattice automorphisms that leave the constants fixed. Then G acts naturally on $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$ as isometric lattice automorphisms that leave the constants fixed for all $(0 \le \epsilon < \infty)$. These representation spaces are either all separable, or all non-separable. If G is a topological group, then these representations are either all strongly/weakly continuous, or all strongly/weakly discontinuous.

Proceed to show that, if G is a locally compact Hausdorff group, there is a model that gives an addional "explanation" of Corollary 5.10, in addition to the observation in Remark 5.8. The main ideas leading to the pertinent Theorem5.14 are those employed, where the group is compact and $\epsilon = 0$, but with a few technical modifications, so that they lead to a stronger result that is valid for non-compact groups and for all $(0 \le \epsilon < \infty)$ simultaneously. The basic tool is an application of the commutative Gelfand-Naimark theorem, and for this need some preparations.

The proof of the following result is a technically strengthened variation. For this, invariant integration over the group is needed, and this is the reason that the requirement that G be a locally compactHausdorff group becomes part of the hypotheses.

Lemma 5.11. Let $\left(X,\sum_j \mu^j\right)$ be a probability space, let $(0 \le \epsilon < \infty)$, and let ρ be a strongly continuous representation of a locally compact Hausdorff group G on $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$ as isometric lattice automorphisms that leave the constants fixed. Then there exists a G-invariant closed subalgebra $\sum_j \left(A_j\right)_{(1+\epsilon)}$ of $\left(L^{\infty}\left(X,\sum_j \mu^j\right),\|\cdot\|_{\infty}\right)$ that contains $\mathbf{1}_X$, is dense in $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$, and is such that the restricted representation of G on $\left(\left(A_j\right)_{(1+\epsilon)},\|\cdot\|_{\infty}\right)$ are strongly continuous. If $\sum_j \mu^j$ are separable, and G is σ^j-compact, then $\left(A_j\right)_{(1+\epsilon)}$ can be taken to be a separable subalgebra of $\left(L^{\infty}\left(X,\sum_j \mu^j\right),\|\cdot\|_{\infty}\right)$.

Proof. If $\sum_j f_j \in L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$, and $\phi \in C_c(G)$, then, since the integrand is continuous and compactly supported, the $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$-valued Bochner integral $\rho(\phi) f_j = \int_X \phi(g_j)\rho(g_j) f_j \mathrm{d}\sum_j \mu^j G(g_j)$ exists; here μ_G^j is a left-invariant Haar measure on G. If $f_j \in L^{\infty}\left(X,\sum_j \mu^j\right)$, then $\rho(\phi)$ is, in fact, an element of $L^{\infty}\left(X,\sum_j \mu^j\right)$. To see this, choose, for $n = 1,2,\cdots$, a disjoint partition $\operatorname{supp}\phi = \bigcup_{i=1}^{N_n} E_i$ of the compact set $\operatorname{supp}\phi$ into measurable subsets E_i ,and $\left(g_j\right)_i \in E_i$, such that

$$\sum_j \left\|\phi(g_j)\rho(g_j) f_j - \phi\left(\left(g_j\right)_i\right)\rho\left(\left(g_j\right)_i\right) f_j\right\|_{(1+\epsilon)} \le \frac{1}{n}$$

and $\sum_j \left|\phi(g_j) - \phi\left(\left(g_j\right)_i\right)\right| \le 1/n$ for all $g_j \in E_i$. It is easy to see that

$\sum_{i=1}^{N_n}\sum_j \mu^j(E_i)\phi\left(\left(g_j\right)_i\right)\rho\left(\left(g_j\right)_i\right)f_j \to \rho(\phi)f_j$ in $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$ as $n\to\infty$. Passing to a subsequence, may assume that this convergence is pointwise almost everywhere. On the other hand, know that G acts as isometrieson $\left(L^\infty\left(X,\sum_j \mu^j\right),\|\cdot\|_\infty\right)$, so that

$\sum_j\left\|\sum_{i=1}^{N_n}\mu^j(E_i)\phi\left(\left(g_j\right)_i\right)\rho\left(\left(g_j\right)_i\right)f_j\right\|_\infty \le \sum_j \mu^j(\operatorname{supp}\phi)\|\phi\|_\infty \sum_j\left\|f_j\right\|_\infty$ for all n. Conclude that $\rho(\phi)f_j$ is an element of $L^\infty\left(X,\sum_j \mu^j\right)$, as claimed. Moreover, since

$\sum_j\left\|\sum_{i=1}^{N_n}\mu^j(E_i)\phi\left(\left(g_j\right)_i\right)\rho\left(\left(g_j\right)_i\right)f_j\right\|_\infty \le \sum_{i=1}^{N_n}\left|\phi\sum_j\left(\left(g_j\right)_i\right)\right|\sum_j \mu^j(E_i)\|\phi\|_\infty$, let $n\to\infty$ and conclude that $\sum_j\left\|\rho(\phi)f_j\right\|_\infty \le \|\phi\|_1\|\phi\|_\infty\left\|\sum_j f_j\right\|_\infty$. It follows easily from the latter inequality and the strong continuity of the left regular representation of G on $L^1(G)$ that the map $\sum_j g_j \mapsto \rho\sum_j\left(g_j\right)\rho(\phi)f_j$ from G into $\left(L^\infty\left(X,\sum_j \mu^j\right),\|\cdot\|_\infty\right)$ is continuous.

After these preparations, let

$\sum_j\left(\acute{A}_j\right)_{(1+\epsilon)} = \left\{\sum_j f_j \in L^\infty\left(X,\sum_j \mu^j\right): g_j \mapsto \rho\left(g_j\right)f_j\right.$ are continuous from G into $\left.\left(L^\infty\left(X,\sum_j \mu^j\right),\|\cdot\|_\infty\right)\right\}$.

Using that G acts as isomerizeson $\left(L^\infty\left(X,\sum_j \mu^j\right),\|\cdot\|_\infty\right)$, one sees that $\sum_j\left(\acute{A}_j\right)_{(1+\epsilon)}$ are closed G-invariant subalgebra of $\left(L^\infty\left(X,\sum_j \mu^j\right),\|\cdot\|_\infty\right)$ that contains $\mathbf{1}_X$. It follows from the preparations that $\sum_j\left(\acute{A}_j\right)_{(1+\epsilon)}$ are dense in $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$.

For the general case, one can take $\sum_j\left(A_j\right)_{(1+\epsilon)} = \sum_j\left(\acute{A}_j\right)_{(1+\epsilon)}$. If $\sum_j \mu^j$ are separable, and G is σ^j-compact, select a countable subset S of $\sum_j\left(\acute{A}_j\right)_{(1+\epsilon)}$ containing $\mathbf{1}_X$ that is dense in $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$. If $f_j \in S$, then $\rho(G)f_j$ are σ^j-compact, and hence a separable, subset of $\left(\left(\acute{A}_j\right)_{\frac{1+\epsilon}{\epsilon}}\|\cdot\|_\infty\right)$.

Therefore there exists a countable subset G_{f_j} of G, containing the identity element, such that $\rho(G)f_j \subseteq \overline{\left\{\rho\left(g_j\right)f_j : g_j \in G_{f_j}\right\}}^{\|\cdot\|_\infty} \subseteq \left(\acute{A}_j\right)_{(1+\epsilon)}$. One take $\left(\acute{A}_j\right)_{(1+\epsilon)}$ to be the closed subalgebra of $\left(L^\infty\left(X,\mu^j\right),\|\cdot\|_\infty\right)$ that is generated by the $\rho\left(g_j\right)f_j$ for $f_j \in S$ and $g_j \in G_{f_j}$.

Note. Deduce that:

$$\sum_j\left\|\rho(\phi)f_j\right\|_\infty \le \|\phi\|_1\|\phi\|_\infty\left\|\sum_j f_j\right\|_\infty$$

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Proof. From proof of lemma 5.11 we can get

$$\sum_{k=1}\sum_{j}\left\|\rho(\phi_k)\left(f_j\right)_k\right\|_\infty$$

$$\leq \|\phi\|_1\left\|\rho\left(\sum_{k=1}^{n}(\phi_k)^2\right)^{\frac{1}{2}}\right\|_\infty\left\|\left(\sum_{k=1}^{n}\sum_{j}\left(f_j\right)_k^2\right)^{\frac{1}{2}}\right\|_\infty = \|\phi\|_1\|\phi\|_\infty\left\|\sum_j f_j\right\|_\infty$$

Remark 5.12. It is worth noting that every separable locally compact Hausdorff group G is σ^j-compact. Indeed, there exists an open neighbourhood of V of e in G that is σ^j-compact, and if $S\subset G$ is a countable dense subset, then $G=\bigcup_{s^j\in S}s^j\left(V\cap V^{-1}\right)$ is σ^j-compact.

The following result has no counterpart It is needed when one wants to transfer the "whole" picture in Theorem 5.14, *i.e.* for all $1\leq\epsilon<\infty$ simultaneously.

Proposition 5.13. Let $\left(X,\sum_j\mu^j\right)$ be a probability space, let $(0\leq\epsilon<\infty)$ and let ρ be a strongly continuous representation of a locally compact Hausdorff group G on $L^{(1+\epsilon)}\left(X,\sum_j\mu^j\right)$ as isometric lattice automorphisms that leave the constants fixed, so that G acts naturally in a similar fashion on $L^{(1+\epsilon)}\left(X,\sum_j\mu^j\right)$ for all $(0\leq\epsilon<\infty)$. Then there exists a G-invariant closed subalgebra A_j of $\left(L^\infty\left(X,\sum_j\mu^j\right),\|\cdot\|_\infty\right)$ that contains $\mathbf{1}_X$, is dense in $L^{(1+\epsilon)}\left(X,\sum_j\mu^j\right)$ for all $(0\leq\epsilon<\infty)$, and is such that the restricted representation of G on $\left(A_j,\|\cdot\|_\infty\right)$ are strongly continuous. If μ^j are separable, and G is σ^j-compact, then A_j can be taken to be a separable subalgebra of $\left(L^\infty\left(X,\sum_j\mu^j\right),\|\cdot\|_\infty\right)$.

Proof. For $n=1,2,\cdots$, choose an algebra $\left(A_j\right)_n$ as in Lemma 5.11 that is dense in $L^n\left(X,\sum_j\mu^j\right)$, and let A_j be the closed subalgebra of $\left(L^\infty\left(X,\sum_j\mu^j\right),\|\cdot\|_\infty\right)$ that is generated by the $\left(A_j\right)_n$.

The following "transfer theorem" is a stronger version. Include the short proof for the convenience, but hasten to add that it is a modest variation on that of [[9], Theorem 15.27], where only $\epsilon=0$ is considered and where the group is compact.

Theorem 5.14. Let $\left(X,\sum_j\mu^j\right)$ be a probability space, let $(0\leq\epsilon<\infty)$, and let $\rho^{(1+\epsilon)}$ be a strongly continuous representation of a locally compact Hausdorff group G on $L^{(1+\epsilon)}\left(X,\sum_j\mu^j\right)$ as isometric lattice automorphisms that leave the constants fixed, so that G acts naturally in a similar fashion on $L^{(1+\epsilon)}\left(X,\sum_j\mu^j\right)$ for all $(0\leq\epsilon<\infty)$.

Then there exist

(1) a topological dynamical system (G,K), where K is a compact Hausdorff space;

(2) a G-invariant regular Borel probability measure $\sum_j\tilde{\mu}^j$ on K with $\operatorname{supp}\sum_j\tilde{\mu}^j=K$;

(3) a family $\left\{\Phi_{(1+\epsilon)}\right\}_{(0\leq\epsilon<\infty)}$ of isometric lattice isomorphisms

$\Phi_{(1+\epsilon)}:L^{(1+\epsilon)}\left(X,\sum_j\mu^j\right)\to L^{(1+\epsilon)}\left(K,\sum_j\tilde{\mu}^j\right)$ that

(a) send $\mathbf{1}_X$ to $\mathbf{1}_K$;

(b) are compatible with the inclusions between $L^{(1+\epsilon)}$-spaces;

(c) Intertwine the strongly continuous representations of G on the spaces $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$ with the canonical strongly continuous representations of G on the spaces $L^{(1+\epsilon)}\left(K,\sum_j \tilde{\mu}^j\right)$.

If $\sum_j \mu^j$ is separable, and G is σ^j-compact, then K can be taken to be amortizable.

Proof. Choose an algebra A_j as in Proposition 5.13. By the commutative Gelfand-Naimark theorem, there exist a compact Hausdorff space K and a unital isometric algebra isomorphism $\Phi:\left(A_j,\|\cdot\|_\infty\right)\to\left(C(K),\|\cdot\|_\infty\right)$. If A_j is separable, then K is amortizable. Know that Φ is a lattice isomorphism, that $\left|f_j\right|^{(1+\epsilon)}\in A_j$ for all $f_j\in A_j$ and $0\le\epsilon<\infty$, and that

$$\Phi\left(\left|f_j\right|^{(1+\epsilon)}\right)=\left|\Phi\left(f_j\right)\right|^{(1+\epsilon)}\left(f_j\in A_j,\left(0\le\epsilon<\infty\right)\right). \tag{14}$$

Transfer the strongly continuous action of G on $\left(A_j,\|\cdot\|_\infty\right)$ to $\left(C(K),\|\cdot\|_\infty\right)$ via Φ. As is well known, this transferred action necessarily originates from a topological dynamical system (G,K).

The Riesz representation theorem furnishes a regular Borel probability measure $\sum_j \mu^j$ on K, easily seen to be of full support, such that

$$\int_K \Phi\left(f_j\right)\mathrm{d}\sum_j \tilde{\mu}^j=\int_X f_j\mathrm{d}\sum_j \mu^j\left(f_j\in A_j\right). \tag{15}$$

Since Φ intertwines the G-actions on $C(K)$ and A_j by construction, it is immediate from (15) and part 1 of Lemma 5.5 that $\sum_j \check{\mu}^j$ are G-invariant. Furthermore, combination of (14) and (15) shows that

$$\begin{aligned}&\int_K\left|\Phi\left(f_j\right)\right|^{(1+\epsilon)}\mathrm{d}\sum_j \tilde{\mu}^j\\&=\int_K\Phi\left(\left|f_j\right|^{(1+\epsilon)}\right)\mathrm{d}\sum_j \tilde{\mu}^j=\int_X\left|f_j\right|^{(1+\epsilon)}\mathrm{d}\sum_j \mu^j\left(f_j\in A_j,1\le\epsilon<\infty\right)\end{aligned}$$

Since, for all $(0\le\epsilon<\infty)$, A_j are dense in $L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)$, and $C(K)$ is dense in $L^{(1+\epsilon)}\left(K,\sum_j \mu^j\right)$, by extension obtain a family of isomerizes $\Phi_{(1+\epsilon)}:L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)\to L^{(1+\epsilon)}\left(K,\sum_j \tilde{\mu}^j\right)(0\le\epsilon<\infty)$.

Since Φ is a lattice isomorphism, so are the $\Phi_{(1+\epsilon)}$. The statements in parts 3b and 3c are routinely verified.

It is clear that Theorems 5.14 can still be used to disintegrate representations even when there is no initial action on the underlying point set, since—under mild conditions—the latter is furnished by Theorem 5.14. The result is the following, which should be compared with the general unitary disintegration [21]. Note the separability assumption on the probability space, needed to ensure that the compact Hausdorff space from Theorem 5.14 is Polish.

Theorem 5.15. Let G be a locally compact Polish group, let $(0\le\epsilon<\infty)$, and let $\left(X,\sum_j \mu^j\right)$ be a separable probability space. Let $\rho^{(1+\epsilon)}:G\to L\left(L^{(1+\epsilon)}\left(X,\sum_j \mu^j\right)\right)$ be a strongly continuous representation of G as isometric lattice automorphisms that leave the constants fixed. Then, for all

$(0 \le \epsilon < \infty)$ there exists a representation $\rho^{(1+\epsilon)}$ of G on $L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right)$ with the same properties, that is obtained from $\rho^{(1+\epsilon)}$ via restriction to, and extension from, $L^{\infty}\left(X, \sum_j \mu^j\right)$. Furthermore, there exist a Borel probability space $\left(\Omega, \nu^j\right)$ and a vector space V such that, for all $(0 \le \epsilon < \infty)$, there exist

(1) A measurable family $\left\{\sum_j \left(B_j\right)_{\omega^j}^{(1+\epsilon)}\right\}_{\omega^j \in \Omega}$ of Banach lattices over $\left(\Omega, \nu^j, V\right)$;

(2) A family of strongly continuous and order indecomposable representations $\sum_j \rho_{\omega^j}^{(1+\epsilon)} : G \to L\left(\sum_j \left(B_j\right)_{\omega^j}^{(1+\epsilon)}\right)\left(\omega^j \in \Omega\right)$ of G as isometric lattice isomorphisms of $\left(B_j\right)_{\omega^j}^{(1+\epsilon)}$;

(3) An isometric lattice isomorphism $S^{(1+\epsilon)} : L^{(1+\epsilon)}\left(X, \sum_j \mu^j\right) \to \left(\int_\Omega^{\oplus} \sum_j \left(B_j\right)_{\omega^j}^{(1+\epsilon)} \mathrm{d}\nu^j\left(\omega^j\right)\right)_{L^{(1+\epsilon)}}$ such that the following diagram commutes for all $g_j \in G$:

$$\begin{tikzcd}
L^{(1+\epsilon)}\left(X, \sum\limits_j \mu^j\right) \arrow[r, "\sum_j \rho^{(1+\epsilon)}(g_j)"] \arrow[d, "S"'] & L^{(1+\epsilon)}\left(X, \sum\limits_j \mu^j\right) \arrow[d, "S"] \\
\left(\int_\Omega^{\oplus} \sum\limits_j \left(B_j\right)_{\omega^j}^{(1+\epsilon)} \mathrm{d}\nu^j\left(\omega^j\right)\right)_{L^{(1+\epsilon)}} \arrow[r, "\left(\int_\Omega^{\oplus} \sum_j \rho_{\omega^j}^{(1+\epsilon)}(g_j)\mathrm{d}\nu^j(\omega^j)\right)_{L^{(1+\epsilon)}}"] & \left(\int_\Omega^{\oplus} \sum\limits_j \left(B_j\right)_{\omega^j}^{(1+\epsilon)} \mathrm{d}\nu^j\left(\omega^j\right)\right)_{L^{(1+\epsilon)}}
\end{tikzcd}$$

Inspection of the proofs shows that there is some more information available. V can be taken to be the vector lattice of all simple functions on the compact metric space K that is furnished by Theorem 5.14, Ω is then the set of all ergodic Borel probability measures on K, and ν^j is then the push-forward of the measure $\sum_j \tilde{\mu}^j$ on K in Theorem 5.14 to the set of ergodic Borel probability measures, using a decomposition map for $\sum_j \tilde{\mu}^j$. The families $\left\{\sum_j \left(B_j\right)_{\omega^j}^{(1+\epsilon)}\right\}_{\omega^j \in \Omega}$ of Banach lattices is then the families $\left\{L^{(1+\epsilon)}\left(K, \omega^j\right)\right\}_{\omega^j \in \Omega}$ of $L^{(1+\epsilon)}$-spaces corresponding to the ergodic Borel probability measures on K, and the representations $\rho_{\omega^j}^{(1+\epsilon)}$ are then the canonical representations of G on these spaces.

6. Perspective

Put forward the task of disintegrating strongly continuous representations of a locally compact group as isometric lattice automorphisms of Banach lattices into similar representations that are order indecomposable. This would be the analogue of what is known to be possible for strongly continuous unitary representations of separable groups on separable Hilbert spaces. The $L^{(1+\epsilon)}$-spaces for finite $(1+\epsilon)$ are arguably the prime examples of Banach lattices that can serve as representation spaces, and in that cases, the goal was achieved in Theorem 5.15 for a certain class of such representations. As explained, this class includes e.g. all natural representations on $L^{(1+\epsilon)}$-spaces corresponding to topological actions of Lie groups on compact manifolds with an invariant Borel probability measure.

Consequently, do not only know that the ensuing natural unitary representation of the group on the pertinent L^2-space is a direct integral of irreducible (*i.e.* indecomposable) unitary representations, but also that the natural representations of the group as isometric lattice automorphisms of the pertinent real $L^{(1+\epsilon)}$-spaces for finite $(1+\epsilon)$ are direct integrals of similar representations that are order indecomposable.

Still, it is clear that Theorem 5.15 is only a first step in the study of the disintegration of general strongly continuous group representations as isometric lattice automorphisms of $L^{(1+\epsilon)}$-spaces. At a conceptual level, the main insight seems to be that this is, in fact, possible for the representation, and that (a modification of) the direct integral theory provides the language to formalize such a disintegration. This is not so clear at the outset.

It is hoped that further steps can be taken. One possible development, still for a probability measure $\sum_j \mu^j$ and a strongly continuous representation as isometric lattice automorphisms that leave the constants fixed, would be to attempt to relax the conditions in Theorem 5.15 that G be Polish and/or that the probability space be separable. As is indicated in Remark 5.8, if the constants are fixed, then one is "actually" looking at a measure preserving action of G on the measure algebra $(A_j)_{\sum_j \mu^j}$. It is conceivable can then be used to improve on the technical hypotheses in Theorem 5.15, since the main basic results (Theorems 1, 2a, and 2b) do not involve any topology. They can be applied in the context of any measure preserving abstract group action on a measure algebra, and yield a decomposition of $(A_j)_{\sum_j \mu^j}$ with respect to the sub-algebra of the fixed points of G in $(A_j)_{\sum_j \mu^j}$. It is shown that this can be used to yield anergodic decompositon of the group action at the level of measure spaces if G equals the integers or the real numbers, and it is mentioned that a similar theorem holds in more general cases.

It is open to investigation whether such a decomposition at the level of measure spaces—once actually established for more general G—can be pushed still further to the G-action on the $L^{(1+\epsilon)}$-spaces themselves, while at the same time incorporating the direct integral formalism of [13]. There are definitely some measurability issues to be taken care of, and perhaps the assumptions on G and $\sum_j \mu^j$ in Theorem 5.15 are not only not too restrictive from a practical point of view, but also not so easy to avoid when needing to ensure measurability in the proofs. After all, for the disintegration of a strongly continuous unitary group representation both the group and the Hilbert space are also required to be separable. On a positive note, since our main sources for the ergodic decomposition, but concentrate on Borel spaces and group actions.

Another possible development is the bold leap to consider the most general case of strongly continuous representations as isometric lattice automorphisms of $L^{(1+\epsilon)}$ spaces—for possibly infinite measure $\sum_j \mu^j$—that do not necessarily arise from an underlying measure preserving action, such operators are—this is true for σ-finite measures—always a composition of a multiplication operator and an operator that arises from, an action on $\left(A_j\right)_{\sum_j \mu^j}$; see also Remark 5.8 for $\epsilon = 1$. With this factorization available, or else attempt a route via measure algebras by generalizing the material

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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